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Research Article

Estimates of Some Operators on One-Sided Weighted Morrey Spaces

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A version of one-sided weighted Morrey space is introduced. The boundedness of some classical one-sided operators in harmonic analysis and PDE on these spaces are discussed, including the Riemann-Liouville fractional integral.

1. Introduction

The reasons to study one-sided operators involve not only the generalization of the theory of two-sided operators but also the requirements of ergodic theory [1]. The well-known Riemann-Liouville fractional integral can be viewed as the one-sided version of Riesz potential (the solution of Laplace equation) in harmonic analysis and PDE [2, 3]. The study of weighted theory for one-sided operators was first introduced by Sawyer [4] and many authors thereafter [5–10]. Many of their results show that, for a class of smaller operators (one-sided operators) and a class of wider weights (one-sided weights), many of the famous findings of harmonic analysis still hold.

The study of one-sided spaces emerged naturally alongside the study of one-sided operators. In one previous study, the authors studied one-sided BMO spaces associated with one-sided sharp functions and their relationship to good weights for the one-sided Hardy-Littlewood maximal functions [9]. Aimar and Crescimbeni [11] further investigated the structures of these one-sided regular functions and their basic properties. Other classical works regarding one-sided spaces have also been published [12–14].

A version of one-sided weighted Morrey spaces and Campanato spaces is introduced in this paper. The boundedness of some one-sided operators and its effects on these spaces are investigated. First recall some definitions of the classical Campanato spaces and Morrey spaces.

Let $-1/p \leq \beta < 1$ and $1 \leq p < \infty$. Then the Campanato space $\mathscr{C}^{p,\beta}(\mathbb{R})$ can be defined using the following norm

$$\begin{aligned}\|f\|_{\mathscr{C}^{p,\beta}(\mathbb{R})} &= \sup_{\mathscr{I}}\|f\|_{\mathscr{C}^{p,\beta}(\mathscr{I})} \\ &=: \sup_{\mathscr{I}}\frac{1}{|\mathscr{I}|^{\beta}}\left(\frac{1}{|\mathscr{I}|}\int_{\mathscr{I}}|f-f_{\mathscr{I}}|^{p}dx\right)^{1/p},\end{aligned} \tag{1}$$

where $f_{\mathscr{I}} = (1/|\mathscr{I}|)\int_{\mathscr{I}} f(x)dx$, $\mathscr{I}$ denotes an interval contained in $\mathbb{R}$, and $|\mathscr{I}|$ is the Lebesgue measure of $\mathscr{I}$. The excellent structures of Campanato spaces render them useful in the study of the regularity theory of PDEs. They allow the user to determine an integral characterization of the spaces of Hölder continuous functions. This allows generalization of the classical Sobolev embedding theorems; see [15–17], for example. It is also well known that $\mathscr{C}^{1,1/p-1}(\mathbb{R})$ is the dual space of Hardy space $H^{p}(\mathbb{R})$ when $0 < p < 1$ [18]. There has been also a recent account of the theory on Campanato spaces [19–21]. The original form of classical Morrey space was first introduced by Morrey Jr. [22] to investigate the local behavior of solutions to the second order elliptic PDEs,

$$\begin{aligned}\|f\|_{\mathscr{M}^{p,\beta}(\mathbb{R})} &= \sup_{\mathscr{I}}\|f\|_{\mathscr{M}^{p,\beta}(\mathscr{I})} \\ &= \sup_{\mathscr{I}}\frac{1}{|\mathscr{I}|^{\beta}}\left(\frac{1}{|\mathscr{I}|}\int_{\mathscr{I}}|f(x)|^{p}dx\right)^{1/p}.\end{aligned} \tag{2}$$

It is obvious that $\mathscr{M}^{p,-1/p}(\mathbb{R}) = L^p(\mathbb{R})$. Many properties of solutions to PDEs are concerned with the boundedness of some operators on Morrey type spaces. In fact, the better inclusion between the Morrey and the Hölder spaces permits obtaining higher regularity of the solutions to different elliptic and parabolic boundary problems. In recent years, there has been an explosion of interest in the study of the boundedness of operators on Morrey type spaces [23–25].

The study of weighted estimates and their effects on these spaces is important to harmonic analysis. Weighted inequalities arise naturally in Fourier analysis, but their use is best justified by the variety of applications in which they appear. For example, the theory of weights plays an important role in the study of boundary value problems inherent in Laplace's equations on Lipschitz domains. Many authors are interested in the study of the events that occur when the weight function belongs to one of the Muckenhoupt classes. Let $1 < p < \infty$. The Muckenhoupt class A_p [26] consists of all positive locally integrable functions w for which

$$\sup_{\mathcal{J}} \left(\frac{1}{|\mathcal{J}|} \int_{\mathcal{J}} w(x)\,dx \right) \left(\frac{1}{|\mathcal{J}|} \int_{\mathcal{J}} w(x)^{1-p'} dx \right)^{p-1} < \infty, \tag{3}$$

where $1/p + 1/p' = 1$. We say that $w \in A_1$ if there is a constant $C > 0$ such that $Mw(x) \le Cw(x)$. Here M is the Hardy-Littlewood maximal operator

$$Mf(x) = \sup_{h>0} \frac{1}{h} \int_{x-h}^{x+h} |f(y)|\,dy. \tag{4}$$

The study of weights for one-sided operators is motivated by their natural emergence in harmonic analysis. For example, certain measures are required when the one-sided Hardy-Littlewood maximal operators [4]

$$\begin{aligned} M^+ f(x) &= \sup_{h>0} \frac{1}{h} \int_{x}^{x+h} |f(y)|\,dy, \\ M^- f(x) &= \sup_{h>0} \frac{1}{h} \int_{x-h}^{x} |f(y)|\,dy \end{aligned} \tag{5}$$

arising in the ergodic maximal function are treated. The classical Dunford-Schwartz ergodic theorem can be considered the first result regarding weights for these operators. In [4], Sawyer introduced the one-sided A_p classes A_p^+ and A_p^-; they are defined by the following conditions:

$$\begin{aligned} A_p^+ : A_p^+(w) &:= \sup_{a<b<c} \frac{1}{(c-a)^p} \\ &\quad \times \int_a^b w(x)\,dx \left(\int_b^c w(x)^{1-p'} dx \right)^{p-1} \\ &< \infty, \\ A_p^- : A_p^-(w) &:= \sup_{a<b<c} \frac{1}{(c-a)^p} \\ &\quad \times \int_b^c w(x)\,dx \left(\int_a^b w(x)^{1-p'} dx \right)^{p-1} \\ &< \infty, \end{aligned} \tag{6}$$

when $1 < p < \infty$; also, for $p = 1$,

$$A_1^+ : M^- w \le Cw, \qquad A_1^- : M^+ w \le Cw \tag{7}$$

for some constant C. The smallest constant C was denoted by $A_1^+(w)(A_1^-(w))$. $A_p^+(w)(A_p^-(w))$, $p \ge 1$, will be called the $A_p^+(A_p^-)$ constant of w.

Theorem 1 (see [4]). *Let $1 < p < \infty$. Then there exists $C > 0$ such that the inequality*

$$\|M^+ f\|_{L^p(w)} \le C \|f\|_{L^p(w)} \tag{8}$$

holds for all $f \in L^p(w)$ if and only if $w \in A_p^+$.

Remark 2. Similar results can be obtained for the left-hand-side operator by changing the condition A_p^+ to A_p^-.

A function K is called a one-sided Calderón-Zygmund kernel (OCZK) if K satisfies

$$\left| \int_{a<|x|<b} K(x)\,dx \right| \le C, \quad 0 < a < b, \tag{9}$$

$$|K(x)| \le \frac{C}{|x|}, \quad x \ne 0, \tag{10}$$

$$|K(x-y) - K(x)| \le \frac{C|y|}{|x|^2}, \quad |x| > 2|y| > 0 \tag{11}$$

with support in $\mathbb{R}^- = (-\infty, 0)$ or $\mathbb{R}^+ = (0, +\infty)$. Equation (10) is also called the size condition for K and (11) is the continuous condition for K. An example of such a kernel is

$$K(x) = \frac{\sin(\log|x|)}{(x \log|x|)} \chi_{(-\infty,0)}(x), \tag{12}$$

where χ_E denotes the characteristic function of a set E. Aimar et al. [5] studied the one-sided Calderón-Zygmund singular integrals which were defined by

$$\begin{aligned} T^+ f(x) &= \lim_{\varepsilon \to 0^+} \int_{x+\varepsilon}^{\infty} K(x-y) f(y)\,dy \\ T^- f(x) &= \lim_{\varepsilon \to 0^+} \int_{-\infty}^{x-\varepsilon} K(x-y) f(y)\,dy, \end{aligned} \tag{13}$$

where the kernels K are OCZKs.

The one-sided A_p classes not only control the boundedness of one-sided Hardy-Littlewood maximal operators, but also serve as the right weight classes for one-sided singular integral operators. They also appear in PDEs [27].

Theorem 3 (see [5]). *Let $1 < p < \infty$, and let K be an OCZK with support in $\mathbb{R}^- = (-\infty, 0)$. Then T^+ is bounded on $L^p(w)$ if $w \in A_p^+$.*

Also, a result concerning the converse of Theorem 3 is given in [5].

In addition to singular integral operators, fractional integral operators also play an important role in harmonic analysis. The problem of fractional derivation was an early impetus to study fractional integrals [6]. In addition to their contributions to harmonic analysis, fractional integrals also play an essential role in many fields. The Hardy-Littlewood-Sobolev inequality of fractional integral is still an indispensable tool in the establishment of time-space estimates for the heat semigroup of nonlinear evolution equations. Let $0 < \alpha < 1$; the one-sided fractional maximal operator and the one-sided fractional integrals were defined by

$$\begin{gathered} M_\alpha^+ f(x) = \sup_{h>0} \frac{1}{h^{1-\alpha}} \int_x^{x+h} |f(y)|\, dy, \\ M_\alpha^- f(x) = \sup_{h>0} \frac{1}{h^{1-\alpha}} \int_{x-h}^{x} |f(y)|\, dy, \\ I_\alpha^+ f(x) = \int_x^{\infty} \frac{f(y)}{(y-x)^{1-\alpha}}\, dy, \\ I_\alpha^- f(x) = \int_{-\infty}^{x} \frac{f(y)}{(y-x)^{1-\alpha}}\, dy, \end{gathered} \tag{14}$$

respectively. I_α^+ and I_α^- are also called the Rieman-Liouville and the Weyl fractional integral operators. The boundedness of M_α^+ was determined by Andersen and Sawyer [6].

Theorem 4 (see [6]). *Let $1 < p < q < \infty$, $1/p - 1/q = \alpha$, and $w \in A^+_{(p,q)}$. Then there exists $C > 0$ such that*

$$\begin{gathered} (a)\ \ \|M_\alpha^+ fw\|_{L^q} \le C \|fw\|_{L^p}; \\ (b)\ \ \|I_\alpha^+ fw\|_{L^q} \le C \|fw\|_{L^p}. \end{gathered} \tag{15}$$

The weight conditions $A^+_{(p,q)}$ and $A^-_{(p,q)}$ are denoted by

$$\begin{gathered} A^+_{(p,q)} : \frac{1}{(c-a)^{1-\alpha}} \left(\int_a^b w^q \right)^{1/q} \left(\int_b^c w^{-p'} \right)^{1/p'} \le C, \\ A^-_{(p,q)} : \frac{1}{(c-a)^{1-\alpha}} \left(\int_b^c w^q \right)^{1/q} \left(\int_a^b w^{-p'} \right)^{1/p'} \le C \end{gathered} \tag{16}$$

for all $a < b < c \in \mathbb{R}$, $1 < p < q$ and $1/p - 1/q = \alpha$.

The one-sided Campanato space and one-sided Morrey space can now be introduced.

Definition 5. Let $-1/p \le \beta < 1$ and $1 \le p < \infty$. A locally integrable function f is said to belong to the one-sided weighted Campanato space $\mathscr{C}^+_{p,\beta}(w)$ if

$$\begin{aligned} \|f\|_{\mathscr{C}^+_{p,\beta}(w)} = {} & \sup_{x_0} \sup_{h>0} \frac{1}{h^\beta} \\ & \times \left(\frac{1}{w(x_0 - h, x_0)} \right. \\ & \left. \times \int_{x_0}^{x_0+h} \left| f(y) - f_{(x_0, x_0+h)} \right|^p dy \right)^{1/p} \\ & < \infty, \end{aligned} \tag{17}$$

where $w(x_0 - h, x_0) = \int_{x_0-h}^{x_0} w(x)\,dx$, x_0 is a real number.

When $\beta = 0$ and $p = 1$, $\mathscr{C}^+_{1,0}(w)$ coincides with the dual space of the one-sided weighted Hardy space $H^1_+(w)$ [28], which consists of certain classes of one-sided weighted BMO functions, see also [11, 13].

When $0 < \beta < 1$, and $p = 1$, $\mathscr{C}^+_{1,\beta}(w)$ consists of all functions satisfying a weighted Lipschitz condition [13].

Case $\beta < 0$ is addressed in the present work.

Definition 6. Let $-1/p \le \beta < 0$ and $1 \le p < \infty$. The one-sided weighted Morrey space is defined by the norm

$$\begin{aligned} \|f\|_{\mathscr{M}^+_{p,\beta}(w)} = {} & \sup_{x_0} \sup_{h>0} \frac{1}{h^\beta} \left(\frac{1}{w(x_0 - h, x_0)} \int_{x_0}^{x_0+h} |f(y)|^p dy \right)^{1/p} \\ & < \infty. \end{aligned} \tag{18}$$

A standard calculation shows that $\mathscr{M}^+_{p,\beta}(w) \subseteq \mathscr{C}^+_{p,\beta}(w)$ in the sense that the following is true:

$$\begin{aligned} & \|f\|_{\mathscr{C}^+_{p,\beta}(w)} \\ & \quad \le C \sup_{h>0} \frac{1}{h^\beta} \left(\frac{1}{w(x_0 - h, x_0)} \int_{x_0}^{x_0+h} |f(y) - a|^p dy \right)^{1/p}, \\ & \qquad \forall a \in \mathbb{R},\ \beta < 0. \end{aligned} \tag{19}$$

Section 2 outlines proof of the boundedness of some one-sided operators mentioned above on $\mathscr{M}^+_{p,\beta}(w)$. In Section 3, the results in Section 1 are extended to a one-sided sublinear operator under specific size conditions, which were satisfied by many one-sided operators, including M^+, T^+, M^+_α, and I^+_α.

Throughout this paper, for $x_0 \in \mathbb{R}$ and $h, \lambda > 0$, unless otherwise stated, we will always denote that $I = (x_0, x_0 + h)$, $I^+ = (x_0 + h, x_0 + 2h)$, $I^- = (x_0 - h, x_0)$, and $\lambda I = (x_0, x_0 + \lambda h)$. C is a constant which may change from line to line.

2. Main Results

In this section, the boundedness of the one-sided operators mentioned in Section 1 and its effects on one-sided Morrey spaces are described. The primary results are formulated as follows.

Theorem 7. *Let $-1/p \le \beta < 0$ and $w \in A^+_p$. Then*

- (a) *M^+ is a bounded operator from $\mathscr{M}^+_{p,\beta}(w)$ to $\mathscr{C}^+_{p,\beta}(w)$ for $1 < p < 1/(1+\beta)$;*
- (b) *T^+ is a bounded operator from $\mathscr{M}^+_{p,\beta}(w)$ to $\mathscr{C}^+_{p,\beta}(w)$ for $1 < p < \infty$.*

Theorem 7(a) is also true when $\mathscr{C}^+_{p,\beta}(w)$ is replaced by $\mathscr{M}^+_{p,\beta}(w)$ (see proof of Theorem 7). A corresponding substitution for Theorem 7(b) under certain assumption with respect to p is given in Section 3.

For the fractional case, the following is true.

Theorem 8. *Let* $0 < \alpha < 1$, $1/q = 1/p - \alpha$, $-1/p \le \beta < 0$, *and* $w \in A^{+}_{(p,q)}$. *Then*

(a) M^{+}_{α} *is a bounded operator from* $\mathscr{M}^{+}_{p,\beta}(w^{p})$ *to* $\mathscr{C}^{+}_{q,\beta}(w^{q})$ *for* $1 < p < q < 1/(1+\beta)$;

(b) I^{+}_{α} *is a bounded operator from* $\mathscr{M}^{+}_{p,\beta}(w^{p})$ *to* $\mathscr{C}^{+}_{q,\beta}(w^{q})$ *for* $1 < p < 1/\alpha$.

First, some basic propositions of one-sided weight classes are selected for use in the analysis.

Lemma 9 (see [4]). *(a)If* $w \in A^{+}_{p}$, *then* $w \in A^{+}_{p-\varepsilon}$ *for some* $\varepsilon > 0$.

(b)$w \in A^{+}_{p}$ *for* $1 < p < \infty$ *if and only if there exists* $w_1 \in A^{+}_{1}$ *and* $w_2 \in A^{-}_{1}$ *such that* $w = w_1(w_2)^{1-p}$.

(c)If $1 \le p < \infty$, *then* $A_p = A^{+}_{p} \bigcap A^{-}_{p}$, $A_p \subset A^{+}_{p}$, *and* $A_p \subset A^{-}_{p}$.

(d)$A^{+}_{p} \subset A^{+}_{r}$, $A^{-}_{p} \subset A^{-}_{r}$ *if* $1 \le p \le r$.

According to the definitions of A^{+}_{p} and $A^{+}_{(p,q)}$, the following relationship between these two classes can be assessed easily.

Proposition 10. *Suppose* $0 < \alpha < 1$, $1 < p < q < \infty$, *and* $1/p - 1/q = \alpha$; *then the following statements are equivalent.*

(a) $w \in A^{+}_{(p,q)}$.

(b) $w^{q} \in A^{+}_{q(1-\alpha)}$.

(c) $w^{q} \in A^{+}_{q}$, $w^{p} \in A^{+}_{p}$.

(d) $w^{-p'} \in A^{+}_{1+(q/p')}$.

Proof. Proof is given only for $(a) \Leftrightarrow (b)$. Other cases are straightforward and can be described using Lemma 9 and the definitions of A^{+}_{p} and $A^{+}_{(p,q)}$.

(a) ⇒ (b). If $w \in A^{+}_{(p,q)}$, we have

$$\frac{1}{(c-a)^{1-\alpha}}\left(\int_a^b w^{q}\right)^{1/q}\left(\int_b^c w^{-p'}\right)^{1/p'} \le C, \tag{20}$$

which if combined $1/p - 1/q = \alpha$ implies

$$\begin{aligned}
&\frac{1}{(c-a)^{q(1-\alpha)}}\int_a^b w^{q}\left(\int_b^c w^{q(1-(q(1-\alpha))')}\right)^{q(1-\alpha)-1}\\
&\quad= \frac{1}{(c-a)^{q(1-\alpha)}}\\
&\qquad\times \int_a^b w^{q}\left(\int_b^c w^{q((1-q')/(1-q'\alpha))}\right)^{q(1-\alpha)-1}\\
&\quad= \frac{1}{(c-a)^{q(1-\alpha)}}\int_a^b w^{q}\left(\int_b^c w^{-p'}\right)^{(p-1)/(1-p\alpha)}\\
&\quad< \infty.
\end{aligned} \tag{21}$$

Therefore $w^{q} \in A^{+}_{q(1-\alpha)}$.

(b) ⇒ (a). It is obvious by the reverse argument of (a) ⇒ (b). □

If $w(x) \in A_p$; then it is a doubling weight, that is, there exists $C > 0$ such that

$$\int_{a-2h}^{a+2h} w \le C \int_{a-h}^{a+h} w \tag{22}$$

for all $a \in \mathbb{R}$ and $h > 0$. However, one-sided A_p weights do not satisfy this property. But the weights A^{+}_{p} satisfy a one-sided doubling condition.

Lemma 11 (see [29]). *Let* $w(x) \in A^{+}_{p}$ $(p \ge 1)$. *Then there exists a constant* $C > 0$ *such that*

$$\int_{a-h}^{a+h} w \le C \int_{a}^{a+h} w \tag{23}$$

for all $a \in \mathbb{R}$ *and* $h > 0$.

Like the one-sided doubling condition, the following proposition also plays an important role in the present arguments.

Proposition 12. *Let* $\lambda > 0$ *and* $p, q \ge 1$. *Then*

(a) *if* $w \in A^{+}_{p}$, *we have*

$$w\left((\lambda I)^{-}\right) \le C\lambda^{p} w(I); \tag{24}$$

(b) *if* $w \in A^{+}_{(p,q)}$, *we have*

$$w\left((\lambda I)^{-}\right) \le C\lambda w(I). \tag{25}$$

Proof. For the proof of (a), we first claim that

$$(f_I)^{p} \le CA^{+}_{p}(w)\left(\frac{1}{w(I^{-})}\int_I |f(x)|^{p} w(x)\,dx\right). \tag{26}$$

In fact, we can apply Hölder's inequality with exponents p and p' to get

$$\begin{aligned}
&\left(\frac{1}{|I|}\int_I |f(x)|\,dx\right)^{p}\\
&\quad= \left(\frac{1}{|I|}\int_I |f(x)| w(x)^{1/p} w(x)^{-1/p} dx\right)^{p}\\
&\quad\le \frac{1}{|I|^{p}}\left(\int_I |f(x)|^{p} w(x)\,dx\right)\\
&\qquad\times\left(\int_I w(x)^{-p'/p} dx\right)^{p/p'}\\
&\quad= \left(\frac{1}{w(I^{-})}\int_I |f(x)|^{p} w(x)\,dx\right)\\
&\qquad\times\left(\frac{1}{|I|}\int_{I^{-}} w(x)\,dx\right)\\
&\qquad\times\left(\frac{1}{|I|}\int_I w(x)^{-p'/p} dx\right)^{p-1}\\
&\quad\le CA^{+}_{p}(w)\left(\frac{1}{w(I^{-})}\int_I |f(x)|^{p} w(x)\,dx\right).
\end{aligned} \tag{27}$$

Applying (26) to the function $f = \chi_I$ and putting λI in the place of I in (26), we obtain

$$w\left((\lambda I)^{-}\right) \leq A_p^{+}(w)\,\lambda^p w(I) \leq C\lambda^p w(I). \tag{28}$$

The proof of (b) is a byproduct of (a) and the fact that $w^p \in A_p^{+}$ if $w \in A_{(p,q)}^{+}$. □

Proof of Theorem 7. The proof of (a) is given first. Because $\mathscr{M}_{p,\beta}^{+}(w) \subseteq \mathscr{C}_{p,\beta}^{+}(w)$ when $\beta < 0$, it is sufficient to prove that there exists $C > 0$ such that

$$\frac{1}{h^{\beta}}\left(\frac{1}{w(x_0-h,x_0)}\int_{x_0}^{x_0+h}\left|M^{+}f(x)\right|^p dx\right)^{1/p} \leq C\|f\|_{\mathscr{M}_{p,\beta}^{+}(w)}. \tag{29}$$

Decompose $f = f_1 + f_2 = f\chi_{2I} + f\chi_{(2I)^c}$ to obtain

$$\begin{aligned} &\frac{1}{h^{\beta}}\left(\frac{1}{w(x_0-h,x_0)}\int_{x_0}^{x_0+h}\left|M^{+}f(x)\right|^p dx\right)^{1/p} \\ &\quad\leq \frac{1}{h^{\beta}}\left(\frac{1}{w(x_0-h,x_0)}\int_{x_0}^{x_0+h}\left|M^{+}f_1(x)\right|^p dx\right)^{1/p} \\ &\qquad + \frac{1}{h^{\beta}}\left(\frac{1}{w(x_0-h,x_0)}\int_{x_0}^{x_0+h}\left|M^{+}f_2(x)\right|^p dx\right)^{1/p} \\ &\quad =: \widetilde{I} + \widetilde{II}. \end{aligned} \tag{30}$$

Using Theorem 1 and Lemma 11, the following is true:

$$\begin{aligned} \widetilde{I} &= \frac{1}{h^{\beta}}\left(\frac{1}{w(x_0-h,x_0)}\int_{x_0}^{x_0+h}\left|M^{+}f_1(x)\right|^p dx\right)^{1/p} \\ &\leq \frac{1}{h^{\beta}}\left(\frac{1}{w(x_0-h,x_0)}\int_{x_0}^{x_0+2h}\left|f(y)\right|^p dy\right)^{1/p} \\ &\leq \left(\frac{w(x_0-2h,x_0)}{w(x_0-h,x_0)}\right)^{1/p}\|f\|_{\mathscr{M}_{p,\beta}^{+}(w)} \\ &\leq C\|f\|_{\mathscr{M}_{p,\beta}^{+}(w)}. \end{aligned} \tag{31}$$

Hölder's inequality and Proposition 12(a) allow us to estimate $\widetilde{II}$ as

$$\begin{aligned} &\frac{1}{h^{\beta}}\left(\frac{1}{w(x_0-h,x_0)}\int_{x_0}^{x_0+h}\left|M^{+}f_2(x)\right|^p dx\right)^{1/p} \\ &\quad\leq \frac{1}{h^{\beta}}\frac{1}{w(x_0-h,x_0)^{1/p}} \\ &\qquad\times\sum_{j=1}^{\infty}\left(\frac{1}{2^j}\int_{x_0-2h}^{x_0+2^{j+1}h}|f(y)|^p dy\right)^{1/p} \\ &\quad\leq \sum_{j=1}^{\infty}\frac{1}{2^{j(1/p-\beta-1)}}\|f\|_{\mathscr{M}_{p,\beta}^{+}(w)} \\ &\quad\leq C\|f\|_{\mathscr{M}_{p,\beta}^{+}(w)}. \end{aligned} \tag{32}$$

Theorem 7(b) can now be proven. Decomposing $f = f_1 + f_2 = f\chi_{2I} + f\chi_{(2I)^c}$ shows that

$$\begin{aligned} &\frac{1}{h^{\beta}}\left(\frac{1}{w(x_0-h,x_0)}\int_{x_0}^{x_0+h}\left|T^{+}f(y) - \left(T^{+}f\right)_{(x_0,x_0+h)}\right|^p dy\right)^{1/p} \\ &\quad\leq \frac{2}{h^{\beta}}\left(\frac{1}{w(x_0-h,x_0)}\right. \\ &\qquad\times\left.\int_{x_0}^{x_0+h}\left|T^{+}f(y) - T^{+}f_2(x_0+2h)\right|^p dy\right)^{1/p} \\ &\quad\leq \frac{2}{h^{\beta}}\left(\frac{1}{w(x_0-h,x_0)}\int_{x_0}^{x_0+h}\left|T^{+}f_1(y)\right|^p dy\right)^{1/p} \\ &\qquad+ \frac{2}{h^{\beta}}\left(\frac{1}{w(x_0-h,x_0)}\right. \\ &\qquad\times\left.\int_{x_0}^{x_0+h}\left|T^{+}f_2(y) - T^{+}f_2(x_0+2h)\right|^p dy\right)^{1/p} \\ &\quad=: \overline{I} + \overline{II}. \end{aligned} \tag{33}$$

The fact that, if $w \in A_p^{+}$, then T^{+} is bounded on $L^p(w)$ allows the following to be shown:

$$\begin{aligned} \overline{I} &\leq \frac{C}{h^{\beta}}\left(\frac{1}{w(x_0-h,x_0)}\int_{x_0}^{x_0+2h}\left|f(y)\right|^p dy\right)^{1/p} \\ &\leq C\left(\frac{w(x_0-2h,x_0)}{w(x_0-h,x_0)}\right)^{1/p}\|f\|_{\mathscr{M}_{p,\beta}^{+}(w)} \\ &\leq C\|f\|_{\mathscr{M}_{p,\beta}^{+}(w)}. \end{aligned} \tag{34}$$

Here, Lemma 11 is used in the last inequality.

For the term $\overline{II}$, by (11) and Proposition 12(a), we can derive the following:

$$\begin{aligned} \overline{II} &\leq \frac{C}{h^{\beta}w(x_0-h,x_0)^{1/p}} \\ &\quad\times\left(\int_{x_0}^{x_0+h}\left|\int_{x_0+2h}^{\infty}\left(K(y-z)\right.\right.\right. \\ &\qquad\qquad\left.\left.\left.-K(x_0+2h-z)\right)f(z)\,dz\right|^p dy\right)^{1/p} \\ &\leq \frac{C}{h^{\beta}w(x_0-h,x_0)^{1/p}} \\ &\quad\times\left(\int_{x_0}^{x_0+h}\left(\int_{x_0+2h}^{\infty}\left|\frac{x_0+2h-y}{\left(z-(x_0+h)\right)^2}f(z)\right|dz\right)^p dy\right)^{1/p} \end{aligned}$$

$$\begin{aligned}
&\le C\frac{h^{1-\beta}}{w(x_0-h,x_0)^{1/p}}\\
&\quad\times\left(\int_{x_0}^{x_0+h}\left(\sum_{j=1}^{\infty}\int_{x_0+2^jh}^{x_0+2^{j+1}h}\left|\frac{f(z)}{(z-(x_0+h))^2}\right|dz\right)^p dy\right)^{1/p}\\
&\le C\frac{h^{1+1/p-\beta}}{w(x_0-h,x_0)^{1/p}}\\
&\quad\times\sum_{j=1}^{\infty}\frac{1}{(h(2^j-1))^2}\int_{x_0+2^jh}^{x_0+2^{j+1}h}|f(z)|\,dz\\
&\le C\frac{h^{1/p-1-\beta}}{w(x_0-h,x_0)^{1/p}}\\
&\quad\times\sum_{j=1}^{\infty}\frac{1}{(2^j-1)^2}\left(\int_{x_0-h}^{x_0+2^{j+1}h}|f(z)|^p dz\right)^{1/p}\left(2^jh\right)^{1/p'}\\
&\le C\|f\|_{\mathscr{M}^+_{p,\beta}(w)}\sum_{j=1}^{\infty}\frac{1}{2^{j(1/p-\beta)}}\\
&\le C\|f\|_{\mathscr{M}^+_{p,\beta}(w)}.
\end{aligned}\tag{35}$$

On account of the estimates for $\overline{I}$ and $\overline{II}$ given above, the following can be proved:

$$\|T^+f\|_{\mathscr{C}^+_{p,\beta}(w)}\le C\|f\|_{\mathscr{M}^+_{p,\beta}(w)}.\tag{36}$$

□

Proof of Theorem 8. We begin with the proof for M^+_α, which is similar to that of Theorem 7(a). It is sufficient to show that there exists constant $C>0$ such that

$$\begin{aligned}
&\frac{1}{h^\beta}\left(\frac{1}{w(x_0-h,x_0)^q}\int_{x_0}^{x_0+h}|M^+_\alpha f(y)|^q dy\right)^{1/q}\\
&\qquad\le C\|f\|_{\mathscr{M}^+_{p,\beta}(w^p)}.
\end{aligned}\tag{37}$$

Decompose $f=f_1+f_2=f\chi_{2I}+f\chi_{(2I)^c}$ to obtain

$$\begin{aligned}
&\frac{1}{h^\beta}\left(\frac{1}{w(x_0-h,x_0)^q}\int_{x_0}^{x_0+h}|M^+_\alpha f(x)|^q dx\right)^{1/q}\\
&\quad\le\frac{1}{h^\beta}\left(\frac{1}{w(x_0-h,x_0)^q}\int_{x_0}^{x_0+h}|M^+_\alpha f_1(x)|^q dx\right)^{1/q}\\
&\qquad+\frac{1}{h^\beta}\left(\frac{1}{w(x_0-h,x_0)^q}\int_{x_0}^{x_0+h}|M^+_\alpha f_2(x)|^q dx\right)^{1/q}\\
&\quad=:\widetilde{J}+\widetilde{JJ}.
\end{aligned}\tag{38}$$

By Theorem 4 and Lemma 11,

$$\begin{aligned}
\widetilde{J}&=\frac{1}{h^\beta}\left(\frac{1}{w(x_0-h,x_0)^q}\int_{x_0}^{x_0+h}|M^+_\alpha f(x)|^q dx\right)^{1/q}\\
&\le C\|f\|_{\mathscr{M}^+_{p,\beta}(w^p)}.
\end{aligned}\tag{39}$$

By the same arguments as those of $\widetilde{II}$, $\widetilde{JJ}$ can be estimated as

$$\begin{aligned}
&\frac{1}{h^\beta}\left(\frac{1}{w(x_0-h,x_0)^q}\int_{x_0}^{x_0+h}|M^+_\alpha f_2(x)|^q dx\right)^{1/q}\\
&\qquad\le\sum_{j=1}^{\infty}\frac{1}{2^{j(1/q-\beta-1)}}\|f\|_{\mathscr{M}^+_{p,\beta}(w)}\\
&\qquad\le C\|f\|_{\mathscr{M}^+_{p,\beta}(w^p)}.
\end{aligned}\tag{40}$$

The proof of (b) is a reprise of the argument given in the proof of Theorem 7(b). Set $f=f_1+f_2=f\chi_{2I}+f\chi_{(2I)^c}$ to obtain

$$\begin{aligned}
&\frac{1}{h^\beta}\left(\frac{1}{w(x_0-h,x_0)^q}\right.\\
&\qquad\left.\times\int_{x_0}^{x_0+h}\left|I^+_\alpha f(y)-(I^+_\alpha f)_{(x_0,x_0+h)}\right|^q dy\right)^{1/q}\\
&\quad\le\frac{2}{h^\beta}\left(\frac{1}{w(x_0-h,x_0)^q}\right.\\
&\qquad\qquad\left.\times\int_{x_0}^{x_0+h}\left|I^+_\alpha f(y)-I^+_\alpha f_2(x_0+2h)\right|^q dy\right)^{1/q}\\
&\quad\le\frac{2}{h^\beta}\left(\frac{1}{w(x_0-h,x_0)^q}\int_{x_0}^{x_0+h}\left|I^+_\alpha f_1(y)\right|^q dy\right)^{1/q}\\
&\qquad+\frac{2}{h^\beta}\left(\frac{1}{w(x_0-h,x_0)^q}\right.\\
&\qquad\qquad\left.\times\int_{x_0}^{x_0+h}\left|I^+_\alpha f_2(y)-I^+_\alpha f_2(x_0+2h)\right|^q dy\right)^{1/q}\\
&\quad=:J+JJ.
\end{aligned}\tag{41}$$

Theorem 4 and Lemma 11 allow us to estimate J as

$$\begin{aligned}
J&\le\frac{C}{h^\beta}\frac{1}{w(x_0-h,x_0)}\left(\int_{x_0}^{x_0+2h}|f(y)|^p dy\right)^{1/p}\\
&\le C\left(\frac{w(x_0-2h,x_0)}{w(x_0-h,x_0)}\right)\|f\|_{\mathscr{M}^+_{p,\beta}(w)}\\
&\le C\|f\|_{\mathscr{M}^+_{p,\beta}(w^p)}.
\end{aligned}\tag{42}$$

In view of

$$\begin{aligned} &\left|I_\alpha^+ f_2(y) - I_\alpha^+ f_2(x_0+2h)\right| \\ &\quad\le \int_{x_0+2h}^{\infty}\left|\frac{1}{|z-y|^{1-\alpha}} - \frac{1}{|z-(x_0+2h)|^{1-\alpha}}\right| \\ &\qquad\times |f(z)|\,dz \\ &\quad\le C\int_{x_0+2h}^{\infty}\frac{|x_0+2h-y|}{|z-(x_0+2h)|^{2-\alpha}}|f(z)|\,dz, \end{aligned} \tag{43}$$

we obtain by Hölder's inequality and Proposition 12(b) that

$$\begin{aligned} JJ &\le C\frac{h^{1+1/q-\beta}}{w(x_0-h,x_0)}\sum_{j=1}^{\infty}\int_{x_0+2^jh}^{x_0+2^{j+1}h}\frac{|f(z)|}{(z-(x_0+h))^{2-\alpha}}dz \\ &\le C\frac{h^{1+1/q-\beta}}{w(x_0-h,x_0)}\sum_{j=1}^{\infty}\frac{1}{(h(2^j-1))^{2-\alpha}} \\ &\quad\times\int_{x_0-h}^{x_0+2^{j+1}h}|f(z)|\,dz \\ &\le C\frac{h^{\alpha+1/q-1-\beta}}{w(x_0-h,x_0)} \\ &\quad\times\sum_{j=1}^{\infty}\frac{1}{(2^j-1)^2}\left(\int_{x_0-h}^{x_0+2^{j+1}h}|f(z)|^p dz\right)^{1/p}\left(2^jh\right)^{1/p'} \\ &\le C\|f\|_{\mathscr{M}^+_{p,\beta}(w^p)}\sum_{j=1}^{\infty}\frac{1}{2^{j(1/p-\alpha-\beta)}} \\ &\le C\|f\|_{\mathscr{M}^+_{p,\beta}(w^p)}. \end{aligned} \tag{44}$$

We have thus proved Theorem 8. □

3. Boundedness of One-Sided Sublinear Operators

The method used in the proof of Theorem 7(b) depends heavily on the convolution of the kernel function. However, there are some other one-sided operators with nonconvoluted kernels. Such operators appear in many places and PDEs. The one-sided oscillatory singular integral operators that were first introduced by the authors of this paper in a previous study are one such example [7].

A class of more general one-sided operators that do not necessarily have convolution kernels can now be studied. Let $D_k = 2^kI$ and $A_k = D_k \setminus D_{k-1}$ for $k \in Z$. In this section, a definition made in a previous study [30] can be adopted to introduce a one-sided sublinear operator satisfying the following size condition:

$$\left|\mathscr{T}^+ f(x)\right| \le \frac{C}{2^kh}\|f\|_{L^1(A_k)}, \tag{45}$$

where supp $f \subseteq A_k$ and $0 \le x_0 < x \le x_0 + 2^{k-1}h$ with $k \in Z$.

For the fractional case, the corresponding size condition can be introduced:

$$\left|\mathscr{T}_\alpha^+ f(x)\right| \le \frac{C}{(2^kh)^{1-\alpha}}\|f\|_{L^1(A_k)}, \quad 0<\alpha<1, \tag{46}$$

where supp $f \subseteq A_k$ and $0 \le x_0 < x \le x_0 + 2^{k-1}h$ with $k \in Z$.

It is easy to confirm that the condition (45) is satisfied by M^+, T^+ and the one-sided oscillatory singular integral operators and both M_α^+ and I_α^+ satisfy (46).

Theorem 13. *Let $-1/p \le \beta < 0$, $1 < p < 1/(1+\beta)$, $w \in A_p^+$, and the one-sided sublinear operator $\mathscr{T}^+$ satisfy (45). Then if $\mathscr{T}^+$ is bounded on $L^p(w)$, $\mathscr{T}^+$ is bounded on $\mathscr{M}^+_{p,\beta}(w)$.*

Theorem 14. *Let $0 < \alpha < 1$, $-1/q \le \beta < 0$, $1/q = 1/p - \alpha$, $p < q < 1/(1+\beta)$, $w \in A^+_{(p,q)}$, and the one-sided sublinear operator $\mathscr{T}_\alpha^+$ satisfy (46). If $\mathscr{T}_\alpha^+$ is bounded from $L^p(w^p)$ to $L^q(w^q)$, then $\mathscr{T}_\alpha^+$ is bounded from $\mathscr{M}^+_{p,\beta}(w^p)$ to $\mathscr{M}^+_{q,\beta}(w^q)$.*

Theorems 13 and 14 agree with Theorems 7(a) and 8(a) but are different from Theorems 7(b) and 8(b). The conditions of the kernel functions in Theorems 13 and 14 are weaker than those of Theorems 7(b) and 8(b), respectively, in that only the size conditions are used there. For this reason, Theorems 13 and 14 can be seen as an extension of Theorems 7 and 8, respectively. However, the present study was conducted under the assumptions that $1 < p < 1/(1+\beta)$ and $p < q < 1/(1+\beta)$. These conditions are stronger than those of Theorems 7(b) and 8(b).

Proof of Theorem 13. It is sufficient to show that there exists $C > 0$ such that

$$\frac{1}{h^\beta}\left(\frac{1}{w(x_0-h,x_0)}\int_{x_0}^{x_0+h}\left|\mathscr{T}^+ f(x)\right|^p dx\right)^{1/p} \le C\|f\|_{\mathscr{M}^+_{p,\beta}(w)}. \tag{47}$$

$f = f_1 + f_2 = f\chi_{2I} + f\chi_{(2I)^c}$ is decomposed to produce

$$\begin{aligned} &\frac{1}{h^\beta}\left(\frac{1}{w(x_0-h,x_0)}\int_{x_0}^{x_0+h}\left|\mathscr{T}^+ f(x)\right|^p dx\right)^{1/p} \\ &\quad\le \frac{1}{h^\beta}\left(\frac{1}{w(x_0-h,x_0)}\int_{x_0}^{x_0+h}\left|\mathscr{T}^+ f_1(x)\right|^p dx\right)^{1/p} \\ &\qquad+ \frac{1}{h^\beta}\left(\frac{1}{w(x_0-h,x_0)}\int_{x_0}^{x_0+h}\left|\mathscr{T}^+ f_2(x)\right|^p dx\right)^{1/p} \\ &\quad=: K + KK. \end{aligned} \tag{48}$$

Using the fact that $\mathscr{T}^+$ is bounded on $L^p(w)$,

$$\begin{aligned} K &\le \frac{C}{h^{\beta}}\left(\frac{1}{w(x_0-h,x_0)}\int_{x_0}^{x_0+2h}|f(y)|^p\,dy\right)^{1/p} \\ &\le C\left(\frac{w(x_0-2h,x_0)}{w(x_0-h,x_0)}\right)^{1/p}\|f\|_{\mathscr{M}^+_{p,\beta}(w)} \\ &\le C\|f\|_{\mathscr{M}^+_{p,\beta}(w)} \end{aligned} \tag{49}$$

can be found easily.

In view of (45), the following is true:

$$\begin{aligned} |\mathscr{T}^+ f_2(x)| &\le C\sum_{k=1}^{\infty}\frac{1}{2^k h}\int_{x_0+2^k h}^{x_0+2^{k+1}h}|f(y)|\,dy \\ &\le C\sum_{k=1}^{\infty}\frac{1}{2^k h}\left(\int_{x_0-h}^{x_0+2^{k+1}h}|f(y)|^p\,dy\right)^{1/p} \\ &\qquad\times\left(2^k h\right)^{1/p'} \\ &\le C\sum_{k=1}^{\infty}\frac{w\left(x_0-2^{k+1}h,x_0\right)^{1/p}}{\left(2^k h\right)^{1/p-\beta}}\|f\|_{\mathscr{M}^+_{p,\beta}(w)}. \end{aligned} \tag{50}$$

Using Proposition 12, KK can be estimated as

$$\begin{aligned} KK &\le C\|f\|_{\mathscr{M}^+_{p,\beta}(w)} \\ &\quad\times\sum_{k=1}^{\infty}\frac{h^{1/p-\beta}}{\left(2^k h\right)^{1/p-\beta}}\left(\frac{w(x_0-2^{k+1}h,x_0-h)}{w(x_0-h,x_0)}\right)^{1/p} \\ &\le C\|f\|_{\mathscr{M}^+_{p,\beta}(w)}\sum_{k=1}^{\infty}\frac{1}{2^{k(1/p-\beta-1)}} \\ &\le C\|f\|_{\mathscr{M}^+_{p,\beta}(w)}. \end{aligned} \tag{51}$$

□

Proof of Theorem 14. An argument similar to that used in the proof of Theorem 13 can be used to produce

$$\begin{aligned} &\frac{1}{h^{\beta}}\left(\frac{1}{w(x_0-h,x_0)^q}\int_{x_0}^{x_0+h}|\mathscr{T}^+_{\alpha}f(x)|^q dx\right)^{1/q} \\ &\quad\le\frac{1}{h^{\beta}}\left(\frac{1}{w(x_0-h,x_0)^q}\right. \\ &\qquad\left.\times\int_{x_0}^{x_0+h}|\mathscr{T}^+_{\alpha}f_1(x)|^q dx\right)^{1/q} \\ &\qquad+\frac{1}{h^{\beta}}\left(\frac{1}{w(x_0-h,x_0)^q}\right. \\ &\qquad\left.\times\int_{x_0}^{x_0+h}|\mathscr{T}^+_{\alpha}f_2(x)|^q dx\right)^{1/q} \\ &\quad=: L+LL. \end{aligned} \tag{52}$$

Estimating the term L, using Lemma 11 and the fact that $\mathscr{T}^+_{\alpha}$ is bounded from $L^p(w^p)$ to $L^q(w^q)$ produce

$$\begin{aligned} L &\le \frac{C}{h^{\beta}}\left(\frac{1}{w(x_0-h,x_0)^p}\int_{x_0}^{x_0+2h}|f(y)|^p dy\right)^{1/p} \\ &\le C\left(\frac{w(x_0-2h,x_0)}{w(x_0-h,x_0)}\right)\|f\|_{\mathscr{M}^+_{p,\beta}(w^p)} \\ &\le C\|f\|_{\mathscr{M}^+_{p,\beta}(w^p)}. \end{aligned} \tag{53}$$

The term LL (46) facilitates the production of

$$\begin{aligned} |\mathscr{T}^+_{\alpha}f_2(x)| &\le C\sum_{k=1}^{\infty}\frac{1}{\left(2^k h\right)^{1-\alpha}}\int_{x_0+2^k h}^{x_0+2^{k+1}h}|f(y)|\,dy \\ &\le C\sum_{k=1}^{\infty}\frac{1}{\left(2^k h\right)^{1-\alpha}} \\ &\quad\times\left(\int_{x_0-h}^{x_0+2^{k+1}h}|f(y)|^p dy\right)^{1/p} \\ &\quad\times\left(2^k h\right)^{1/p'} \\ &\le C\sum_{k=1}^{\infty}\frac{w\left(x_0-2^{k+1}h,x_0-h\right)}{\left(2^k h\right)^{1/p-\beta-\alpha}} \\ &\quad\times\|f\|_{\mathscr{M}^+_{p,\beta}(w^p)}. \end{aligned} \tag{54}$$

Therefore,

$$\begin{aligned} LL &\le C\|f\|_{\mathscr{M}^+_{p,\beta}(w^p)} \\ &\quad\times\sum_{k=1}^{\infty}\frac{h^{1/q-\beta}}{\left(2^k h\right)^{1/q-\beta}}\left(\frac{w\left(x_0-2^{k+1}h,x_0-h\right)}{w\left(x_0-h,x_0\right)}\right) \\ &\le C\|f\|_{\mathscr{M}^+_{p,\beta}(w^p)}\sum_{k=1}^{\infty}\frac{1}{2^{k(1/q-\beta-1)}} \\ &\le C\|f\|_{\mathscr{M}^+_{p,\beta}(w^p)}, \end{aligned} \tag{55}$$

where we have used Propositions 10 and 12. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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