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Review Article

Compactness Conditions in the Study of Functional, Differential, and Integral Equations

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We discuss some existence results for various types of functional, differential, and integral equations which can be obtained with the help of argumentations based on compactness conditions. We restrict ourselves to some classical compactness conditions appearing in fixed point theorems due to Schauder, Krasnosel'skii-Burton, and Schaefer. We present also the technique associated with measures of noncompactness and we illustrate its applicability in proving the solvability of some functional integral equations. Apart from this, we discuss the application of the mentioned technique to the theory of ordinary differential equations in Banach spaces.

1. Introduction

The concept of the compactness plays a fundamental role in several branches of mathematics such as topology, mathematical analysis, functional analysis, optimization theory, and nonlinear analysis [1–5]. Numerous mathematical reasoning processes depend on the application of the concept of compactness or relative compactness. Let us indicate only such fundamental and classical theorems as the Weierstrass theorem on attaining supremum by a continuous function on a compact set, the Fredholm theory of linear integral equations, and its generalization involving compact operators as well as a lot of fixed point theorems depending on compactness argumentations [6, 7]. It is also worthwhile mentioning such an important property saying that a continuous mapping transforms a compact set onto compact one.

Let us pay a special attention to the fact that several reasoning processes and constructions applied in nonlinear analysis depend on the use of the concept of the compactness [6]. Since theorems and argumentations of nonlinear analysis are used very frequently in the theories of functional, differential, and integral equations, we focus in this paper on the presentation of some results located in these theories which can be obtained with the help of various compactness conditions.

We restrict ourselves to present and describe some results obtained in the last four decades which are related to some problems considered in the theories of differential, integral, and functional integral equations. Several results using compactness conditions were obtained with the help of the theory of measures of noncompactness. Therefore, we devote one section of the paper to present briefly some basic background of that theory.

Nevertheless, there are also successfully used argumentations not depending of the concept of a measure on noncompactness such as Schauder fixed point principle, Krasnosel'skii-Burton fixed point theorem, and Schaefer fixed point theorem.

Let us notice that our presentation is far to be complete. The reader is advised to follow the most expository monographs in which numerous topics connected with compactness conditions are broadly discussed [6, 8–10].

Finally, let us mention that the presented paper has a review form. It discusses some results described in details in the papers which will be cited in due course.

2. Selected Results of Nonlinear Analysis Involving Compactness Conditions

In order to solve an equation having the form

$$x = Fx, \tag{1}$$

where F is an operator being a self-mapping of a Banach space E, we apply frequently an approach through fixed point theorems. Such an approach is rather natural and, in general, very efficient. Obviously, there exists a huge number of miscellaneous fixed point theorems [6, 7, 11] depending both on order, metric, and topological argumentations.

The most efficient and useful theorems seem to be fixed point theorems involving topological argumentations, especially those based on the concept of compactness. The reader can make an acquaintance with the large theory of fixed point-theorems involving compactness conditions in the above, mentioned monographs, but it seems that the most important and expository fixed point theorem in this fashion is the famous Schauder fixed point principle [12]. Obviously, that theorem was generalized in several directions but till now it is very frequently used in application to the theories of differential, integral, and functional equations.

At the beginning of our considerations we recall two well-known versions of the mentioned Schauder fixed point principle (cf. [13]). To this end assume that $(E, \|\cdot\|)$ is a given Banach space.

Theorem 1. *Let Ω be a nonempty, bounded, closed, and convex subset of E and let $F : \Omega \to \Omega$ be a completely continuous operator (i.e., F is continuous and the image $F\Omega$ is relatively compact). Then F has at least one fixed point in the set Ω (this means that the equation $x = Fx$ has at least one solution in the set Ω).*

Theorem 2. *If Ω is a nonempty, convex, and compact subset of E and $F : \Omega \to \Omega$ is continuous on the set Ω, then the operator F has at least one fixed point in the set Ω.*

Observe that Theorem 1 can be treated as a particular case of Theorem 2 if we apply the well-known Mazur theorem asserting that the closed convex hull of a compact subset of a Banach space E is compact [14]. The basic problem arising in applying the Schauder theorem in the version presented in Theorem 2 depends on finding a convex and compact subset of E which is transformed into itself by operator F corresponding to an investigated operator equation.

In numerous situations, we are able to overcome the above-indicated difficulty and to obtain an interesting result on the existence of solutions of the investigated equations (cf. [7, 15–17]). Below we provide an example justifying our opinion [18].

To do this, let us denote by $\mathbb{R}$ the real line and put $\mathbb{R}_+ = [0, +\infty)$. Further, let $\Delta = \{(t, s) \in \mathbb{R}^2 : 0 \le s \le t\}$.

Next, fix a function $p = p(t)$ defined and continuous on $\mathbb{R}_+$ with positive real values. Denote by $C_p = C(\mathbb{R}_+, p(t))$ the space consisting of all real functions defined and continuous on $\mathbb{R}_+$ and such that

$$\sup \{|x(t)|\, p(t) : t \ge 0\} < \infty. \tag{2}$$

It can be shown that C_p forms the Banach space with respect to the norm

$$\|x\| = \sup \{|x(t)|\, p(t) : t \ge 0\}. \tag{3}$$

For our further purposes, we recall the following criterion for relative compactness in the space C_p [10, 15].

Theorem 3. *Let X be a bounded set in the space C_p. If all functions belonging to X are locally equicontinuous on the interval $\mathbb{R}_+$ and if $\lim_{T\to\infty}\{\sup\{|x(t)|p(t) : t \ge T\}\} = 0$ uniformly with respect to X, then X is relatively compact in C_p.*

In what follows, if x is an arbitrarily fixed function from the space C_p and if $T > 0$ is a fixed number, we will denote by $\nu^T(x, \varepsilon)$ the modulus of continuity of x on the interval $[0, T]$; that is,

$$\nu^T(x, \varepsilon) = \sup \{|x(t) - x(s)| : t, s \in [0, T], |t - s| \le \varepsilon\}. \tag{4}$$

Further on, we will investigate the solvability of the nonlinear Volterra integral equation with deviated argument having the form

$$x(t) = d(t) + \int_0^t v(t, s, x(\varphi(s)))\, ds, \tag{5}$$

where $t \in \mathbb{R}_+$. Equation (5) will be investigated under the following formulated assumptions.

(i) $v : \Delta \times \mathbb{R} \to \mathbb{R}$ is a continuous function and there exist continuous functions $n : \Delta \to \mathbb{R}_+$, $a : \mathbb{R}_+ \to (0, \infty)$, $b : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|v(t, s, x)| \le n(t, s) + a(t)\, b(s)\, |x| \tag{6}$$

for all $(t, s) \in \Delta$ and $x \in \mathbb{R}$.

In order to formulate other assumptions, let us put

$$L(t) = \int_0^t a(s)\, b(s)\, ds, \quad t \ge 0. \tag{7}$$

Next, take an arbitrary number $M > 0$ and consider the space C_p, where $p(t) = [a(t)\exp(ML(t) + t)]^{-1}$. Then, we can present other assumptions.

(ii) The function $d : \mathbb{R}_+ \to \mathbb{R}$ is continuous and there exists a nonnegative constant D such that $|d(t)| \le Da(t)\exp(ML(t))$ for $t \ge 0$.

(iii) There exists a constant $N \ge 0$ such that

$$\int_0^t n(t, s)\, ds \le Na(t)\exp(ML(t)) \tag{8}$$

for $t \ge 0$.

(iv) $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function satisfying the condition

$$L(\varphi(t)) - L(t) \le K \tag{9}$$

for $t \in \mathbb{R}_+$, where $K \ge 0$ is a constant.

(v) $D+N < 1$ and $a(\varphi(t))/a(t) \le M(1-D-N)\exp(-MK)$ for all $t \ge 0$.

Now, we can formulate the announced result.

Theorem 4. *Under assumptions* (i)–(v), (5) *has at least one solution x in the space C_p such that $|x(t)| \le a(t)\exp(ML(t))$ for $t \in \mathbb{R}_+$.*

We give the sketch of the proof (cf. [18]). First, let us define the transformation F on the space C_p by putting

$$(Fx)(t) = d(t) + \int_0^t v(t,s,x(\varphi(s)))\,ds, \quad t \ge 0. \tag{10}$$

In view of our assumptions, the function $(Fx)(t)$ is continuous on $\mathbb{R}_+$.

Further, consider the subset G of the space C_p consisting of all functions x such that $|x(t)| \le a(t)\exp(ML(t))$ for $t \in \mathbb{R}_+$. Obviously, G is nonempty, bounded, closed, and convex in the space C_p. Taking into account our assumptions, for an arbitrary fixed $x \in G$ and $t \in \mathbb{R}_+$, we get

$$\begin{aligned}
|(Fx)(t)| \le{}& |d(t)| + \int_0^t |v(t,s,x(\varphi(s)))|\,ds \\
\le{}& Da(t)\exp(ML(t)) \\
&+ \int_o^t [n(t,s) + a(t)b(s)|x(\varphi(s))|]\,ds \\
\le{}& Da(t)\exp(ML(t)) + Na(t)\exp(ML(t)) \\
&+ a(t)\int_0^t b(s)a(\varphi(s))\exp(ML(\varphi(s)))\,ds \\
\le{}& (D+N)a(t)\exp(ML(t)) \\
&+ (1-D-N)a(t)\int_0^t Ma(s)b(s) \\
&\times \exp(ML(s))\exp(-MK)\exp(MK)\,ds \\
\le{}& (D+N)a(t)\exp(ML(t)) \\
&+ (1-D-N)a(t)\exp(ML(t)) \\
={}& a(t)\exp(ML(t)).
\end{aligned} \tag{11}$$

This shows that F transforms the set G into itself.

Next we show that F is continuous on the set G.

To this end, fix $\varepsilon > 0$ and take $x, y \in G$ such that $\|x - y\| \le \varepsilon$. Next, choose arbitrary $T > 0$. Using the fact that the function $v(t,s,x)$ is uniformly continuous on the set $[0,T]^2 \times [-\alpha(T), \alpha(T)]$, where $\alpha(T) = \max\{a(\varphi(t))\exp(ML(\varphi(t))) : t \in [0,T]\}$, for $t \in [0,T]$, we obtain

$$\begin{aligned}
&|(Fx)(t) - (Fy)(t)| \\
&\quad\le \int_0^t |v(t,s,x(\varphi(s))) - v(t,s,y(\varphi(s)))|\,ds \\
&\quad\le \beta(\varepsilon),
\end{aligned} \tag{12}$$

where $\beta(\varepsilon)$ is a continuous function with the property $\lim_{\varepsilon\to 0}\beta(\varepsilon) = 0$.

Now, take $t \ge T$. Then we get

$$\begin{aligned}
&|(Fx)(t) - (Fy)(t)|\,[a(t)\exp(ML(t) + t)]^{-1} \\
&\quad\le \{|(Fx)(t)| + |(Fy)(t)|\} \\
&\quad\quad\times [a(t)\exp(ML(t))]^{-1} \cdot e^{-t} \le 2e^{-t}.
\end{aligned} \tag{13}$$

Hence, for T sufficiently large, we have

$$|(Fx)(t) - (Fy)(t)|\,p(t) \le \varepsilon \tag{14}$$

for $t \ge T$. Linking (12) and (14), we deduce that F is continuous on the set G.

The next essential step in our proof which enables us to apply the Schauder fixed point theorem (Theorem 1) is to show that the set FG is relatively compact in the space C_p. To this end let us first observe that the inclusion $FG \subset G$ and the description of G imply the following estimate:

$$|(Fx)(t)|\,p(t) \le e^{-t}. \tag{15}$$

This yields that

$$\lim_{T\to\infty}\{\sup\{|(Fx)(t)|\,p(t) : t \ge T\}\} = 0 \tag{16}$$

uniformly with respect to the set G.

On the other hand, for fixed $\varepsilon > 0$, $T > 0$ and for $t, s \in [0,T]$ such that $|t - s| \le \varepsilon$, in view of our assumptions, for $x \in G$, we derive the following estimate:

$$\begin{aligned}
&|(Fx)(t) - (Fx)(s)| \\
&\quad\le |d(t) - d(s)| \\
&\quad\quad+ \left|\int_0^t v(t,\tau,x(\varphi(\tau)))\,d\tau - \int_0^s v(t,\tau,x(\varphi(\tau)))\,d\tau\right| \\
&\quad\quad+ \left|\int_0^s v(t,\tau,x(\varphi(\tau)))\,d\tau - \int_0^s v(s,\tau,x(\varphi(\tau)))\,d\tau\right| \\
&\quad\le \nu^T(d,\varepsilon) + \varepsilon\max\{n(t,\tau) + a(\tau)b(\tau) \\
&\quad\quad\quad\times p(\varphi(\tau)) : 0 \le \tau \le t \le T\} \\
&\quad\quad+ T\nu^T(v(\varepsilon,T,\alpha(T))),
\end{aligned} \tag{17}$$

where we denoted

$$\begin{aligned}
&\nu^T(v(\varepsilon,T,\alpha(T))) \\
&\quad= \sup\{|v(t,u,v) - v(s,u,v)| : t,s \in [0,T], \\
&\quad\quad |t-s| \le \varepsilon, u \in [0,T], |v| \le \alpha(T)\}.
\end{aligned} \tag{18}$$

Taking into account the fact that $\lim_{\varepsilon\to 0}\nu^T(d,\varepsilon) = \lim_{\varepsilon\to 0}\nu^T(\upsilon(\varepsilon,T,\alpha(T))) = 0$, we infer that functions belonging to the set FG are equicontinuous on each interval $[0,T]$. Combining this fact with (16), in view of Theorem 3, we conclude that the set FG is relatively compact. Applying Theorem 1, we complete the proof.

Another very useful fixed point theorem using the compactness conditions is the well-known Krasnosel'skii fixed point theorem [19]. That theorem was frequently modified by researchers working in the fixed point theory (cf. [6, 7, 20]), but it seems that the version due to Burton [21] is the most appropriate to be used in applications.

Below we formulate that version.

Theorem 5. *Let S be a nonempty, closed, convex, and bounded subset of the Banach space X and let $A : X \to X$ and $B : S \to X$ be two operators such that*

(a) *A is a contraction, that is, there exists a constant $k \in [0,1)$ such that $\|Ax - Ay\| \le k\|x - y\|$ for $x, y \in X$,*

(b) *B is completely continuous,*

(c) *$x = Ax + By \Rightarrow x \in S$ for all $y \in S$.*

Then the equation $Ax + Bx = x$ has a solution in S.

In order to show the applicability of Theorem 5, we will consider the following nonlinear functional integral equation [22]:

$$x(t) = f\left(t, x\left(\alpha_1(t)\right), \dots, x\left(\alpha_n(t)\right)\right) + \int_0^{\beta(t)} g\left(t, s, x\left(\gamma_1(s)\right), \dots, x\left(\gamma_m(s)\right)\right) ds, \tag{19}$$

where $t \in \mathbb{R}_+$. Here we assume that f and g are given functions.

The above equation will be studied in the space $BC(\mathbb{R}_+)$ consisting of all real functions defined, continuous, and bounded on the interval $\mathbb{R}_+$ and equipped with the usual supremum norm

$$\|x\| = \sup\left\{|x(t)| : t \in \mathbb{R}_+\right\}. \tag{20}$$

Observe that the space $BC(\mathbb{R}_+)$ is a special case of the previously considered space C_p with $p(t) = 1$ for $t \in \mathbb{R}_+$. This fact enables us to adapt the relative compactness criterion contained in Theorem 3 for our further purposes.

In what follows we will impose the following requirements concerning the components involved in (19):

(i) The function $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ is continuous and there exist constants $k_i \in [0,1)$ $(i = 1,2,\dots,n)$ such that

$$\left|f\left(t, x_1, x_2, \dots, x_n\right) - f\left(t, y_1, y_2, \dots, y_n\right)\right| \le \sum_{i=1}^{n} k_i \left|x_i - y_i\right| \tag{21}$$

for all $t \in \mathbb{R}_+$ and for all $(x_1, x_2, \dots, x_n)$, $(y_1, y_2, \dots, y_n) \in \mathbb{R}^n$.

(ii) The function $t \to f(t,0,\dots,0)$ is bounded on $\mathbb{R}_+$ with $F_0 = \sup\{|f(t,0,\dots,0)| : t \in \mathbb{R}_+\}$.

(iii) The functions $\alpha_i, \gamma_j : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous and $\alpha_i(t) \to \infty$ as $t \to \infty$ $(i = 1,2,\dots,n;\ j = 1,2,\dots,m)$.

(iv) The function $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous.

(v) The function $g : \mathbb{R}_+^2 \times \mathbb{R}^m \to \mathbb{R}$ is continuous and there exist continuous functions $q : \mathbb{R}_+^2 \to \mathbb{R}_+$ and $a, b : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\left|g\left(t, s, x_1, \dots, x_m\right)\right| \le q(t,s) + a(t)\, b(s) \sum_{i=1}^{m} \left|x_i\right| \tag{22}$$

for all $t, s \in \mathbb{R}_+$ and $(x_1, x_2, \dots, x_m) \in \mathbb{R}^m$. Moreover, we assume that

$$\lim_{t\to\infty} \int_0^{\beta(t)} q(t,s)\, ds = 0, \qquad \lim_{t\to\infty} a(t) \int_0^{\beta(t)} b(s)\, ds = 0. \tag{23}$$

Now, let us observe that based on assumption (v), we conclude that the functions $\upsilon_1, \upsilon_2 : \mathbb{R}_+ \to \mathbb{R}_+$ defined by the formulas

$$\upsilon_1(t) = \int_0^{\beta(t)} q(t,s)\, ds, \qquad \upsilon_2(t) = a(t) \int_0^{\beta(t)} b(s)\, ds \tag{24}$$

are continuous and bounded on $\mathbb{R}_+$. Obviously this implies that the constants M_1, M_2 defined as:

$$M_i = \sup\left\{\upsilon_i(t) : t \in \mathbb{R}_+\right\} \quad (i = 1,2) \tag{25}$$

are finite.

In order to formulate our last assumption, let us denote $k = \sum_{i=1}^{n} k_i$, where the constants k_i $(i = 1,2,\dots,n)$ appear in assumption (i).

(vi) $k + mM_2 < 1$.

Then we have the following result [22] which was announced above.

Theorem 6. *Under assumptions* (i)–(vi), (19) *has at least one solution in the space $BC(\mathbb{R}_+)$. Moreover, solutions of* (19) *are globally attractive.*

Remark 7. In order to recall the concept of the global attractivity mentioned in the above theorem (cf. [22]), suppose that Ω is a nonempty subset of the space $BC(\mathbb{R}_+)$ and $Q : \Omega \to BC(\mathbb{R}_+)$ is an operator. Consider the operator equation

$$x(t) = (Qx)(t), \quad t \in \mathbb{R}_+. \tag{26}$$

We say that solutions of (26) are *globally attractive* if for arbitrary solutions x, y of this equation, we have that

$$\lim_{t\to\infty} \left(x(t) - y(t)\right) = 0. \tag{27}$$

Let us mention that the above-defined concept was introduced in [23, 24].

Proof of Theorem 6. We provide only the sketch of the proof. Consider the ball B_r in the space $X = BC(\mathbb{R}_+)$ centered at the zero function θ and with radius $r = (F_0+M_1)/[1-(k+mM_2)]$. Next, define two mappings $A : X \to X$ and $B : B_r \to X$ by putting

$$\begin{aligned}(Ax)(t) &= f(t, x(\alpha_1(t)), \dots, x(\alpha_n(t))),\\ (Bx)(t) &= \int_0^{\beta(t)} g(t, s, x(\gamma_1(s)), \dots, x(\gamma_m(s)))\, ds,\end{aligned} \tag{28}$$

for $t \in \mathbb{R}_+$. Then (19) can be written in the form

$$x(t) = (Ax)(t) + (Bx)(t), \quad t \in \mathbb{R}_+. \tag{29}$$

Notice that in view of assumptions (i)–(iii), the mapping A is well defined, and for arbitrarily fixed function $x \in X$, the function Ax is continuous and bounded on $\mathbb{R}_+$. Thus A transforms X into itself. Similarly, applying assumptions (iii)–(v), we deduce that the function Bx is continuous and bounded on $\mathbb{R}_+$. This means that B transforms the ball B_r into X.

Now, we check that operators A and B satisfy assumptions imposed in Theorem 5. To this end take $x, y \in X$. Then, in view of assumption (i), for a fixed $t \in \mathbb{R}_+$, we get

$$\begin{aligned}|(Ax)(t) - (Ay)(t)| &\le \sum_{i=1}^{n} k_i |x(\alpha_i(t)) - y(\alpha_i(t))|\\ &\le k\|x-y\|.\end{aligned} \tag{30}$$

This implies that $||Ax - Ay|| \le k||x - y||$, and in view of assumption (vi), we infer that A is a contraction on X.

Next, we prove that B is completely continuous on the ball B_r. In order to show the indicated property of B, we fix $\varepsilon > 0$ and we take $x, y \in B_r$ with $||x - y|| \le \varepsilon$. Then, taking into account our assumptions, we obtain

$$\begin{aligned}&|(Bx)(t) - (By)(t)|\\ &\quad\le \int_0^{\beta(t)} [|g(t, s, x(\gamma_1(s)), \dots, x(\gamma_m(s)))|\\ &\qquad + |g(t, s, y(\gamma_1(s)), \dots, y(\gamma_m(s)))|]\, ds\\ &\quad\le 2(v_1(t) + rmv_2(t)).\end{aligned} \tag{31}$$

Hence, keeping in mind assumption (v), we deduce that there exists $T > 0$ such that $v_1(t) + rmv_2(t) \le \varepsilon/2$ for $t \ge T$. Combining this fact with (31), we get

$$|(Bx)(t) - (By)(t)| \le \varepsilon \tag{32}$$

for $t \ge T$.

Further, for an arbitrary $t \in [0, T]$, in the similar way, we obtain

$$|(Bx)(t) - (By)(t)| \le \int_0^{\beta(t)} \omega_r^T(g, \varepsilon)\, ds \le \beta_T \omega_r^T(g, \varepsilon), \tag{33}$$

where we denoted $\beta_T = \sup\{\beta(t) : t \in [0, T]\}$ and

$$\begin{aligned}\omega_r^T(g, \varepsilon) = \sup\{&|g(t, s, x_1, x_2, \dots, x_m)\\ &- g(t, s, y_1, y_2, \dots, y_m)| : t \in [0, T],\\ &s \in [0, \beta_T], x_i, y_i \in [-r, r],\\ &|x_i - y_i| \le \varepsilon\ (i = 1, 2, \dots, m)\}.\end{aligned} \tag{34}$$

Keeping in mind the uniform continuity of the function g on the set $[0, T]\times[0, \beta_T]\times[-r, r]^m$, we deduce that $\omega_r^T(g, \varepsilon) \to 0$ as $\varepsilon \to 0$. Hence, in view of (32) and (33), we conclude that the operator B is continuous on the ball B_r.

The boundedness of the operator B is a consequence of the inequality

$$|(Bx)(t)| \le v_1(t) + rmv_2(t) \tag{35}$$

for $t \in \mathbb{R}_+$. To verify that the operator B satisfies assumptions of Theorem 3 adapted to the case of the space $BC(\mathbb{R}_+)$, fix arbitrarily $\varepsilon > 0$. In view of assumption (v), we can choose $T > 0$ such that $v_1(t) + rmv_2(t) \le \varepsilon$ for $t \ge T$. Further, take an arbitrary function $x \in B_r$. Then, keeping in mind (35), for $t \ge T$, we infer that

$$|(Bx)(t)| \le \varepsilon. \tag{36}$$

Next, take arbitrary numbers $t, \tau \in [0, T]$ with $|t - \tau| \le \varepsilon$. Then we obtain the following estimate:

$$\begin{aligned}&|(Bx)(t) - (Bx)(\tau)|\\ &\quad\le \left|\int_{\beta(\tau)}^{\beta(t)} |g(t, s, x(\gamma_1(s)), \dots, x(\gamma_m(s)))|\, ds\right|\\ &\qquad + \int_0^{\beta(\tau)} |g(t, s, x(\gamma_1(s)), \dots, x(\gamma_m(s)))\\ &\qquad\qquad - g(\tau, s, x(\gamma_1(s)), \dots, x(\gamma_m(s)))|\, ds\\ &\quad\le \left|\int_{\beta(\tau)}^{\beta(t)} \left[q(t, s) + a(t) b(s) \sum_{i=1}^{m} |x(\gamma_i(s))|\right] ds\right|\\ &\qquad + \int_0^{\beta(\tau)} \omega_1^T(g, \varepsilon; r)\, ds\\ &\quad\le \left|\int_{\beta(\tau)}^{\beta(t)} q(t, s)\, ds\right|\\ &\qquad + a(t) mr \left|\int_{\beta(\tau)}^{\beta(t)} b(s)\, ds\right| + \beta_T \omega_1^T(g, \varepsilon; r),\end{aligned} \tag{37}$$

where we denoted

$$\begin{aligned}\omega_1^T(g, \varepsilon; r) = \sup\{&|g(t, s, x_1, \dots, x_m)\\ &- g(\tau, s, x_1, \dots, x_m)| : t, \tau \in [0, T],\\ &|t - s| \le \varepsilon, s \in [0, \beta_T],\\ &|x_1| \le r, \dots, |x_m| \le r\}.\end{aligned} \tag{38}$$

From estimate (37), we get

$$|(Bx)(t) - (Bx)(\tau)| \leq q_T \nu^T(\beta, \varepsilon) + rma_T b_T \nu^T(\beta, \varepsilon) + \beta_T \omega_1^T(g, \varepsilon; r), \tag{39}$$

where $q_T = \max\{q(t,s) : t \in [0,T], s \in [0,\beta_T]\}$, $a_T = \max\{a(t) : t \in [0,T]\}$, $b_T = \max\{b(t) : t \in [0,\beta_T]\}$, and $\nu^T(\beta,\varepsilon)$ denotes the usual modulus of continuity of the function β on the interval $[0,T]$.

Now, let us observe that in view of the standard properties of the functions $\beta = \beta(t)$, $g = g(t,s,x_1,\ldots,x_m)$, we infer that $\nu^T(\beta,\varepsilon) \to 0$ and $\omega_1^T(g,\varepsilon;r) \to 0$ as $\varepsilon \to 0$. Hence, taking into account the boundedness of the image BB_r and estimates (36) and (39), in view of Theorem 3, we conclude that the set BB_r is relatively compact in the space $BC(\mathbb{R}_+)$; that is, B is completely continuous on the ball B_r.

In what follows fix arbitrary $x \in BC(\mathbb{R}_+)$ and assume that the equality $x = Ax + By$ holds for some $y \in B_r$. Then, utilizing our assumptions, for a fixed $t \in \mathbb{R}_+$, we get

$$\begin{aligned} |x(t)| \leq\ & |(Ax)(t)| + |(By)(t)| \\ \leq\ & |f(t, x(\alpha_1(t)), \ldots, x(\alpha_n(t))) - f(t,0,\ldots,0)| \\ & + |f(t,0,\ldots,0)| \\ & + \int_0^{\beta(t)} q(t,s)\,ds + m\|y\| a(t) \int_0^{\beta(t)} b(s)\,ds \\ \leq\ & k\|x\| + F_0 + M_1 + mrM_2. \end{aligned} \tag{40}$$

Hence we obtain

$$\|x\| \leq \frac{F_0 + M_1 + mrM_2}{1-k}. \tag{41}$$

On the other hand, we have that $r = (F_0 + M_1 + mrM_2)/(1-k)$. Thus $||x|| \leq r$ or, equivalently, $x \in B_r$. This shows that assumption (c) of Theorem 5 is satisfied.

Finally, combining all of the above-established facts and applying Theorem 5, we infer that there exists at least one solution $x = x(t)$ of (19).

The proof of the global attractivity of solutions of (19) is a consequence of the estimate

$$\begin{aligned} |x(t) - y(t)| \leq\ & \sum_{i=1}^{n} k_i \max\{|x(\alpha_i(t)) - y(\alpha_i(t))| : i = 1,2,\ldots,n\} \\ & + 2\int_0^{\beta(t)} q(t,s)\,ds \\ & + a(t)\int_0^{\beta(t)} \left(b(s)\sum_{i=1}^{m}|x(\gamma_i(s))|\right) ds \\ & + a(t)\int_0^{\beta(t)} \left(b(s)\sum_{i=1}^{m}|y(\gamma_i(s))|\right) ds \\ \leq\ & k\max\{|x(\alpha_i(t)) - y(\alpha_i(t))| : i = 1,2,\ldots,n\} \\ & + 2v_1(t) + m(\|x\| + \|y\|)v_2(t), \end{aligned} \tag{42}$$

which is satisfied for arbitrary solutions $x = x(t)$, $y = y(t)$ of (19).

Hence we get

$$\begin{aligned} \limsup_{t\to\infty} |x(t) - y(t)| \leq\ & k\max_{1\leq i\leq n}\left\{\limsup_{t\to\infty} |x(\alpha_i(t) - y(\alpha_i(t)))|\right\} \\ & + 2\limsup_{t\to\infty} v_1(t) + m(\|x\| + \|y\|)\limsup_{t\to\infty} v_2(t) \\ =\ & k\limsup_{t\to\infty}|x(t) - y(t)| \end{aligned} \tag{43}$$

which implies that $\limsup_{t\to\infty}|x(t) - y(t)| = \lim_{t\to\infty}|x(t) - y(t)| = 0$. This means that the solutions of (19) are globally attractive (cf. Remark 7). □

It is worthwhile mentioning that in the literature one can encounter other formulations of the Krasnosel'skii-Burton fixed point theorem (cf. [6, 7, 20, 21]). In some of those formulations and generalizations, there is used the concept of a measure of noncompactness (both in strong and in weak sense) and, simultaneously, the requirement of continuity is replaced by the assumption of weak continuity or weak sequential continuity of operators involved (cf. [6, 20], for instance).

In what follows we pay our attention to another fixed point theorem which uses the compactness argumentation. Namely, that theorem was obtained by Schaefer in [25].

Subsequently that theorem was formulated in other ways and we are going here to present two versions of that theorem (cf. [11, 26]).

Theorem 8. *Let $(E, ||\cdot||)$ be a normed space and let $T : E \to E$ be a continuous mapping which transforms bounded subsets of E onto relatively compact ones. Then either*

(i) *the equation $x = Tx$ has a solution*

or

(ii) *the set $\bigcup_{0\leq\lambda\leq1}\{x \in E : x = \lambda Tx\}$ is unbounded.*

The below presented version of Schaefer fixed point theorem seems to be more convenient in applications.

Theorem 9. *Let $E, ||\cdot||$ be a Banach space and let $T : E \to E$ be a continuous compact mapping (i.e., T is continuous and T maps bounded subsets of E onto relatively compact ones). Moreover, one assumes that the set*

$$\bigcup_{0\leq\lambda\leq1}\{x \in E : x = \lambda Tx\} \tag{44}$$

is bounded. Then T has a fixed point.

It is easily seen that Theorem 9 is a particular case of Theorem 8.

Observe additionally, that Schaefer fixed point theorem seems to be less convenient in applications than Schauder fixed point theorem (cf. Theorems 1 and 2). Indeed, Schaefer theorem requires a priori bound on utterly unknown solutions of the operator equation $x = \lambda Tx$ for $\lambda \in [0,1]$. On the other hand, the proof of Schaefer theorem requires the use of the Schauder fixed point principle (cf. [27], for details).

It is worthwhile mentioning that an interesting result on the existence of periodic solutions of an integral equation, based on a generalization of Schaefer fixed point theorem, may be found in [26].

3. Measures of Noncompactness and Their Applications

Let us observe that in order to apply the fundamental fixed point theorem based on compactness conditions, that is, the Schauder fixed point theorem, say, the version of Theorem 2, we are forced to find a convex and compact subset of a Banach space E which is transformed into itself by an operator F. In general, it is a hard task to encounter a set of such a type [16, 28]. On the other hand, if we apply Schauder fixed point theorem formulated as Theorem 1, we have to prove that an operator F is completely continuous. This causes, in general, that we have to impose rather strong assumptions on terms involved in a considered operator equation.

In view of the above-mentioned difficulties starting from seventies of the past century mathematicians working on fixed point theory created the concept of the so-called measure of noncompactness which allowed to overcome the above-indicated troubles. Moreover, it turned out that the use of the technique of measures of noncompactness allows us also to obtain a certain characterization of solutions of the investigated operator equations (functional, differential, integral, etc.). Such a characterization is possible provided we use the concept of a measure of noncompactness defined in an appropriate axiomatic way.

It is worthwhile noticing that up to now there have appeared a lot of various axiomatic definitions of the concept of a measure of noncompactness. Some of those definitions are very general and not convenient in practice. More precisely, in order to apply such a definition, we are often forced to impose some additional conditions on a measure of noncompactness involved (cf. [8, 29]).

By these reasons it seems that the axiomatic definition of the concept of a measure of noncompactness should be not very general and should require satisfying such conditions which enable the convenience of their use in concrete applications.

Below we present axiomatics which seems to satisfy the above-indicated requirements. That axiomatics was introduced by Banaś and Goebel in 1980 [10].

In order to recall that axiomatics, let us denote by $\mathfrak{M}_E$ the family of all nonempty and bounded subsets of a Banach space E and by $\mathfrak{N}_E$ its subfamily consisting of relatively compact sets. Moreover, let $B(x,r)$ stand for the ball with the center at x and with radius r. We write B_r to denote the ball $B(\theta,r)$, where θ is the zero element in E. If X is a subset of E, we write $\overline{X}$, $\operatorname{Conv} X$ to denote the closure and the convex closure of X, respectively. The standard algebraic operations on sets will be denoted by $X+Y$ and λX, for $\lambda \in \mathbb{R}$.

As we announced above, we accept the following definition of the concept of a measure of noncompactness [10].

Definition 10. A function $\mu : \mathfrak{M}_E \to \mathbb{R}_+ = [0,\infty)$ is said to be a *measure of noncompactness* in the space E if the following conditions are satisfied.

(1^o) The family $\ker \mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker \mu \subset \mathfrak{N}$.

(2^o) $X \subset Y \Rightarrow \mu(X) \le \mu(Y)$.

(3^o) $\mu(\overline{X}) = \mu(\operatorname{Conv} X) = \mu(X)$.

(4^o) $\mu(\lambda X + (1-\lambda)Y) \le \lambda\mu(X) + (1-\lambda)\mu(X)$ for $\lambda \in [0,1]$.

(5^o) If (X_n) is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ for $n = 1,2,\ldots$ and if $\lim_{n\to\infty}\mu(X_n) = 0$, then the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

Let us pay attention to the fact that from axiom (5^o) we infer that $\mu(X_\infty) \le \mu(X_n)$ for any $n = 1,2,\ldots$. This implies that $\mu(X_\infty) = 0$. Thus X_∞ belongs to the family $\ker\mu$ described in axiom (1^o). The family $\ker\mu$ is called the *kernel of the measure of noncompactness* μ.

The property of the measure of noncompactness μ mentioned above plays a very important role in applications.

With the help of the concept of a measure of noncompactness, we can formulate the following useful fixed point theorem [10] which is called the fixed point theorem of Darbo type.

Theorem 11. *Let Ω be a nonempty, bounded, closed, and convex subset of a Banach space E and let $F : \Omega \to \Omega$ be a continuous operator which is a contraction with respect to a measure of noncompactness μ; that is, there exists a constant k,* $k \in [0,1)$, *such that $\mu(FX) \le k\mu(X)$ for any nonempty subset X of the set Ω. Then the operator F has at least one fixed point in the set Ω.*

In the sequel we show an example of the applicability of the technique of measures of noncompactness expressed by Theorem 11 in proving the existence of solutions of the operator equations.

Namely, we will work on the Banach space $C[a,b]$ consisting of real functions defined and continuous on the interval $[a,b]$ and equipped with the standard maximum norm. For sake of simplicity, we will assume that $[a,b] = [0,1] = I$, so the space on question can be denoted by $C(I)$.

One of the most important and handy measures of noncompactness in the space $C(I)$ can be defined in the way presented below [10].

In order to present this definition, take an arbitrary set $X \in \mathfrak{M}_{C(I)}$. For $x \in X$ and for a given $\varepsilon > 0$, let us put

$$\omega(x,\varepsilon) = \sup\{|x(t) - x(s)| : t,s \in I, |t-s| \le \varepsilon\}. \quad (45)$$

Next, let us define

$$\omega(X,\varepsilon)=\sup\{\omega(x,\varepsilon):x\in X\},$$
$$\omega_0(X)=\lim_{\varepsilon\to 0}\omega(X,\varepsilon). \tag{46}$$

It may be shown that the function $\omega_0(X)$ is the measure of noncompactness in the space $C(I)$ (cf. [10]). This measure has also some additional properties. For example, $\omega_0(\lambda X)=|\lambda|\omega_0(X)$ and $\omega_0(X+Y)\le\omega_0(X)+\omega_0(Y)$ provided $X,Y\in\mathfrak{M}_{C(I)}$ and $\lambda\in\mathbb{R}$ [10].

In what follows we will consider the nonlinear Volterra singular integral equation having the form

$$x(t)=f_1(t,x(t),x(a(t)))+(Gx)(t)\int_0^t f_2(t,s)(Qx)(s)\,ds, \tag{47}$$

where $t\in I=[0,1]$, $a:I\to I$ is a continuous function, and G, Q are operators acting continuously from the space $C(I)$ into itself. Apart from this, we assume that the function f_2 has the form

$$f_2(t,s)=k(t,s)\,g(t,s), \tag{48}$$

where $k:\Delta\to\mathbb{R}$ is continuous and g is monotonic with respect to the first variable and may be discontinuous on the triangle $\Delta=\{(t,s):0\le s\le t\le 1\}$.

Equation (47) will be considered in the space $C(I)$ under the following assumptions (cf. [30]).

(i) $a:I\to I$ is a continuous function.

(ii) The function $f_1:I\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous and there exists a nonnegative constant p such that

$$\left|f_1(t,x_1,y_1)-f_1(t,x_2,y_2)\right|\le p\max\{|x_1-x_2|,|y_1-y_2|\} \tag{49}$$

for any $t\in I$ and for all $x_1,x_2,y_1,y_2\in\mathbb{R}$.

(iii) The operator G transforms continuously the space $C(I)$ into itself and there exists a nonnegative constant q such that $\omega_0(GX)\le q\omega_0(X)$ for any set $X\in\mathfrak{M}_{C(I)}$, where ω_0 is the measure of noncompactness defined by (46).

(iv) There exists a nondecreasing function $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ such that $\|Gx\|\le\varphi(\|x\|)$ for any $x\in C(I)$.

(v) The operator Q acts continuously from the space $C(I)$ into itself and there exists a nondecreasing function $\Psi:\mathbb{R}_+\to\mathbb{R}_+$ such that $\|Qx\|\le\Psi(\|x\|)$ for any $x\in C(I)$.

(vi) $f_2:\Delta\to\mathbb{R}$ has the form (48), where the function $k:\Delta\to\mathbb{R}$ is continuous.

(vii) The function $g(t,s)=g:\Delta\to\mathbb{R}_+$ occurring in the decomposition (48) is monotonic with respect to t (on the interval $[s,1]$), and for any fixed $t\in I$, the function $s\to g(t,s)$ is Lebesgue integrable over the interval $[0,t]$. Moreover, for every $\varepsilon>0$, there exists $\delta>0$ such that for all $t_1,t_2\in I$ with $t_1<t_2$ and $t_2-t_1\le\delta$ the following inequalities are satisfied:

$$\left|\int_0^{t_1}[g(t_2,s)-g(t_1,s)]\,ds\right|\le\varepsilon,$$
$$\int_{t_1}^{t_2}g(t_2,s)\,ds\le\varepsilon. \tag{50}$$

The main result concerning (47), which we are going to present now, will be preceded by a few remarks and lemmas (cf. [30]). In order to present these remarks and lemmas, let us consider the function $h:I\to\mathbb{R}_+$ defined by the formula

$$h(t)=\int_0^t g(t,s)\,ds. \tag{51}$$

In view of assumption (vii), this function is well defined.

Lemma 12. *Under assumption (vii), the function h is continuous on the interval I.*

For proof, we refer to [30].

In order to present the last assumptions needed further on, let us define the constants $\overline{k}$, $\overline{f_1}$, $\overline{h}$ by putting

$$\overline{k}=\sup\{|k(t,s)|:(t,s)\in\Delta\},$$
$$\overline{f_1}=\sup\{|f_1(t,0,0)|:t\in I\}, \tag{52}$$
$$\overline{h}=\sup\{h(t):t\in I\}.$$

The constants $\overline{k}$ and $\overline{f_1}$ are finite in view of assumptions (vi) and (ii), while the fact that $\overline{h}<\infty$ is a consequence of assumption (vii) and Lemma 12.

Now, we formulate the announced assumption.

(viii) There exists a positive solution r_0 of the inequality

$$pr+\overline{f_1}+\overline{kh}\varphi(r)\Psi(r)\le r \tag{53}$$

such that $p+\overline{kh}q\Psi(r_0)<1$.

For further purposes, we definite operators corresponding to (47) and defined on the space $C(I)$ in the following way:

$$(F_1x)(t)=f_1(t,x(t),x(\alpha(t))),$$
$$(F_2x)(t)=\int_0^t f_2(t,s)(Qx)(s)\,ds, \tag{54}$$
$$(Fx)(t)=(F_1x)(t)+(Gx)(t)(F_2x)(t),$$

for $t \in I$. Apart from this, we introduce two functions M and N defined on $\mathbb{R}_+$ by the formulas

$$M(\varepsilon) = \sup\left\{\left|\int_0^{t_1} [g(t_2,s) - g(t_1,g)]\,ds\right| : t_1,t_2 \in I,\right.$$
$$\left. t_1 < t_2, t_2 - t_1 \le \varepsilon\right\}, \tag{55}$$

$$N(\varepsilon) = \sup\left\{\int_{t_1}^{t_2} g(t_2,s)\,ds : t_1,t_2 \in I, t_1 < t_2, t_2 - t_1 \le \varepsilon\right\}. \tag{56}$$

Notice that in view of assumption (vii), we have that $M(\varepsilon) \to 0$ and $N(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Now, we can state the following result.

Lemma 13. *Under assumptions* (i)–(vii), *the operator F transforms continuously the space $C(I)$ into itself.*

Proof. Fix a function $x \in C(I)$. Then $F_1x \in C(I)$, which is a consequence of the properties of the so-called superposition operator [9]. Further, for arbitrary functions $x, y \in C(I)$, in virtue of assumption (ii), for a fixed $t \in I$, we obtain

$$\begin{aligned} &|(F_1x)(t) - (F_1y)(t)| \\ &\quad \le p\max\{|x(t) - y(t)|, |x(a(t)) - y(a(t))|\}. \end{aligned} \tag{57}$$

This estimate in combination with assumption (i) yields

$$\|F_1x - F_1y\| \le p\|x - y\|. \tag{58}$$

Hence we conclude that F_1 acts continuously from the space $C(I)$ into itself.

Next, fix $x \in C(I)$ and $\varepsilon > 0$. Take $t_1, t_2 \in I$ such that $|t_2 - t_1| \le \varepsilon$. Without loss of generality, we may assume that $t_1 \le t_2$. Then, based on the imposed assumptions, we derive the following estimate:

$$\begin{aligned} &|(F_2x)(t_2) - (F_2x)(t_1)| \\ &\quad \le \left|\int_0^{t_2} k(t_2,s)\,g(t_2,s)(Qx)(s)\,ds \right.\\ &\qquad \left. - \int_0^{t_1} k(t_2,s)\,g(t_2,s)(Qx)(s)\,ds\right| \\ &\qquad + \left|\int_0^{t_1} k(t_2,s)\,g(t_2,s)(Qx)(s)\,ds \right.\\ &\qquad \left. - \int_0^{t_1} k(t_1,s)\,g(t_2,s)(Qx)(s)\,ds\right| \\ &\qquad + \left|\int_0^{t_1} k(t_1,s)\,g(t_2,s)(Qx)(s)\,ds \right.\\ &\qquad \left. - \int_0^{t_1} k(t_1,s)\,g(t_1,s)(Qx)(s)\,ds\right| \\ &\quad \le \int_{t_1}^{t_2} |k(t_2,s)|\,g(t_2,s)\,|(Qx)(s)|\,ds \\ &\qquad + \int_0^{t_1} |k(t_2,s) - k(t_1,s)|\,g(t_2,s)\,|(Qx)(s)|\,ds \\ &\qquad + \int_0^{t_1} |k(t_1,s)|\,|g(t_2,s) - g(t_1,s)|\,|(Qx)(s)|\,ds \\ &\quad \le \overline{k}\Psi(\|x\|)\,N(\varepsilon) + \omega_1(k,\varepsilon)\,\Psi(\|x\|)\int_0^{t_2} g(t_2,s)\,ds \\ &\qquad + \overline{k}\Psi(\|x\|)\int_0^{t_1} |g(t_2,s) - g(t_1,s)|\,ds, \end{aligned} \tag{59}$$

where we denoted

$$\begin{aligned} \omega_1(k,\varepsilon) = \sup\{&|k(t_2,s) - k(t_1,s)| : \\ &(t_1,s),(t_2,s) \in \Delta, |t_2 - t_1| \le \varepsilon\}. \end{aligned} \tag{60}$$

Since the function $t \to g(t,s)$ is assumed to be monotonic, by assumption (vii) we get

$$\int_0^{t_1} |g(t_2,s) - g(t_1,s)|\,ds = \left|\int_0^{t_1} [g(t_2,s) - g(t_1,s)]\,ds\right|. \tag{61}$$

This fact in conjunction with the above-obtained estimate yields

$$\begin{aligned} \omega(F_2x,\varepsilon) &\le \overline{k}\Psi(\|x\|)\,N(\varepsilon) + \overline{h}\Psi(\|x\|)\,\omega_1(k,\varepsilon) \\ &\quad + \overline{k}\Psi(\|x\|)\,M(\varepsilon), \end{aligned} \tag{62}$$

where the symbol $\omega(y,\varepsilon)$ denotes the modulus of continuity of a function $y \in C(I)$.

Further observe that $\omega_1(k,\varepsilon) \to 0$ as $\varepsilon \to 0$, which is an immediate consequence of the uniform continuity of the function k on the triangle Δ. Combining this fact with the properties of the functions $M(\varepsilon)$ and $N(\varepsilon)$ and taking into account (62), we infer that $F_2x \in C(I)$. Consequently, keeping in mind that $F_1 : C(I) \to C(I)$ and assumption (iii), we conclude that the operator F is a self-mapping of the space $C(I)$.

In order to show that F is continuous on $C(I)$, fix arbitrarily $x_0 \in C(I)$ and $\varepsilon > 0$. Next, take $x \in C(I)$ such that $\|x - x_0\| \le \varepsilon$. Then, for a fixed $t \in I$, we get

$$\begin{aligned} &|(Fx)(t) - (Fx_0)(t)| \\ &\quad \le |(F_1x)(t) - (F_1x_0)(t)| \\ &\qquad + |(Gx)(t)(F_2x)(t) - (Gx_0)(t)(F_2x_0)(t)| \\ &\quad \le \|F_1x - F_1x_0\| + \|Gx\|\,|(F_2x)(t) - (F_2x_0)(t)| \\ &\qquad + \|F_2x_0\|\,\|Gx - Gx_0\|. \end{aligned} \tag{63}$$

On the other hand, we have the following estimate:

$$\begin{aligned}&\left|(F_2x)(t)-(F_2x_0)(t)\right|\\&\quad\le\int_0^t|k(t,s)|\,g(t,s)\left|(Qx)(s)-(Qx_0)(s)\right|ds\\&\quad\le\bar{k}\left(\int_0^t g(t,s)\,ds\right)\|Qx-Qx_0\|\\&\quad\le\overline{kh}\,\|Qx-Qx_0\|.\end{aligned}\tag{64}$$

Similarly, we obtain

$$\left|(F_2x_0)(t)\right|\le\bar{k}\left(\int_0^t g(t,s)\,ds\right)\|Qx_0\|\le\overline{kh}\,\|Qx_0\|,\tag{65}$$

and consequently

$$\|F_2x_0\|\le\overline{kh}\,\|Qx_0\|.\tag{66}$$

Next, linking (63)–(66) and (58), we obtain the following estimate:

$$\begin{aligned}\|Fx-Fx_0\|\le{}&p\,\|x-x_0\|+\|Gx\|\,\overline{kh}\,\|Qx-Qx_0\|\\&+\overline{kh}\,\|Gx-Gx_0\|\,\|Qx_0\|.\end{aligned}\tag{67}$$

Further, taking into account assumptions (iv) and (v), we derive the following inequality

$$\begin{aligned}\|Fx-Fx_0\|\le{}&p\varepsilon+\varphi\left(\|x_0\|+\varepsilon\right)\overline{kh}\,\|Qx-Qx_0\|\\&+\overline{kh}\Psi\left(\|x_0\|\right)\|Gx-Gx_0\|.\end{aligned}\tag{68}$$

Finally, in view of the continuity of the operators G and Q (cf. assumptions (iii) and (v)), we deduce that operator F is continuous on the space $C(I)$. The proof is complete. □

Now, we can formulate the last result concerning (47) (cf. [30]).

Theorem 14. *Under assumptions* (i)–(viii), (47) *has at least one solution in the space* $C(I)$.

Proof. Fix $x\in C(I)$ and $t\in I$. Then, evaluating similarly as in the proof of Lemma 13, we get

$$\begin{aligned}|(Fx)(t)|\le{}&\left|f_1(t,x(t),x(a(t)))-f_1(t,0,0)\right|\\&+\left|f_1(t,0,0)\right|\\&+|Gx(t)|\left|\int_0^t k(t,s)\,g(t,s)\,(Qx)(s)\,ds\right|\\\le{}&\overline{f_1}+p\,\|x\|+\overline{kh}\varphi(\|x\|)\,\Psi(\|x\|).\end{aligned}\tag{69}$$

Hence, we obtain

$$\|Fx\|\le\overline{f_1}+p\,\|x\|+\overline{kh}\varphi(\|x\|)\,\Psi(\|x\|).\tag{70}$$

From the above inequality and assumption (viii), we infer that there exists a number $r_0>0$ such that the operator F maps the ball B_{r_0} into itself and $p+\overline{kh}q\Psi(r_0)<1$. Moreover, by Lemma 13, we have that F is continuous on the ball B_{r_0}.

Further on, take a nonempty subset X of the ball B_{r_0} and a number $\varepsilon>0$. Then, for an arbitrary $x\in X$ and $t_1,t_2\in I$ with $|t_2-t_1|\le\varepsilon$, in view of (66) and the imposed assumptions, we obtain

$$\begin{aligned}&\left|(Fx)(t_2)-(Fx)(t_1)\right|\\&\quad\le\left|f_1(t_2,x(t_2),x(a(t_2)))\right.\\&\qquad\left.-f_1(t_1,x(t_2),x(a(t_2)))\right|\\&\qquad+\left|f_1(t_1,x(t_2),x(a(t_2)))\right.\\&\qquad\left.-f_1(t_1,x(t_1),x(a(t_1)))\right|\\&\qquad+\left|(F_2x)(t_2)\right|\left|(Gx)(t_2)-(Gx)(t_1)\right|\\&\qquad+\left|(Gx)(t_1)\right|\left|(F_2x)(t_2)-(F_2x)(t_1)\right|\\&\quad\le\omega_{r_0}(f_1,\varepsilon)+p\max\left\{\left|x(t_2)-x(t_1)\right|,\right.\\&\qquad\qquad\left.\left|x(a(t_2))-x(a(t_1))\right|\right\}\\&\qquad+\overline{kh}\Psi(r_0)\,\omega(Gx,\varepsilon)+\varphi(r_0)\,\omega(F_2x,\varepsilon),\end{aligned}\tag{71}$$

where we denoted

$$\begin{aligned}\omega_{r_0}(f_1,\varepsilon)={}&\sup\left\{\left|f_1(t_2,x,y)-f_1(t_1,x,y)\right|:t_1,t_2\in I,\right.\\&\left.\left|t_2-t_1\right|\le\varepsilon,x,y\in[-r_0,r_0]\right\}.\end{aligned}\tag{72}$$

Hence, in virtue of (62), we deduce the estimate

$$\begin{aligned}\omega(Fx,\varepsilon)\le{}&\omega_{r_0}(f_1,\varepsilon)+p\max\{\omega(x,\varepsilon),\omega(x,\omega(a,\varepsilon))\}\\&+\overline{kh}\Psi(r_0)\,\omega(Gx,\varepsilon)\\&+\varphi(r_0)\,\Psi(r_0)\left[\bar{k}N(\varepsilon)+\bar{h}\omega_1(k,\varepsilon)+\bar{k}M(\varepsilon)\right].\end{aligned}\tag{73}$$

Finally, taking into account the uniform continuity of the function f_1 on the set $I\times[-r_0,r_0]^2$ and the properties of the functions $k(t,s)$, $a(t)$, $M(\varepsilon)$, and $N(\varepsilon)$ and keeping in mind (46), we obtain

$$\omega_0(FX)\le p\omega_0(X)+\overline{kh}\Psi(r_0)\,\omega_0(GX).\tag{74}$$

Linking this estimate with assumption (iii), we get

$$\omega_0(FX)\le\left(p+\overline{kh}q\Psi(r_0)\right)\omega_0(X).\tag{75}$$

The use of Theorem 11 completes the proof. □

4. Existence Results Concerning the Theory of Differential Equations in Banach Spaces

In this section, we are going to present some classical results concerning the theory of ordinary differential equations in Banach space. We focus on this part of that theory in a which the technique associated with measures of noncompactness is used as the main tool in proving results on the existence of solutions of the initial value problems for ordinary differential equations. Our presentation is based mainly on the papers [31, 32] and the monograph [33].

The theory of ordinary differential equations in Banach spaces was initiated by the famous example of Dieudonné [34], who showed that in an infinite-dimensional Banach space, the classical Peano existence theorem is no longer true. More precisely, Dieudonné showed that if we consider the ordinary differential equation

$$x' = f(t, x) \tag{76}$$

with the initial condition

$$x(0) = x_0, \tag{77}$$

where $f : [0, T] \times B(x_0, r) \rightarrow E$ and E is an infinite-dimensional Banach space, then the continuity of f (and even uniform continuity) does not guarantee the existence of solutions of problem (76)-(77).

In light of the example of Dieudonné, it is clear that in order to ensure the existence of solutions of (76)-(77), it is necessary to add some extra conditions. The first results in this direction were obtained by Kisyński [35], Olech [36], and Ważewski [37] in the years 1959-1960. In order to formulate those results, we need to introduce the concept of the so-called Kamke comparison function (cf. [38, 39]).

To this end, assume that T is a fixed number and denote $J = [0, T]$, $J_0 = (0, T]$. Further, assume that Ω is a nonempty open subset of $\mathbb{R}^n$ and x_0 is a fixed element of Ω. Let $f : J \times \Omega \rightarrow \mathbb{R}^n$ be a given function.

Definition 15. A function $w : J \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ (or $w : J_0 \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$) is called a *Kamke comparison function* provided the inequality

$$\|f(t, x) - f(t, y)\| \le w(t, \|x - y\|) \tag{78}$$

for $x, y \in \Omega$ and $t \in J$ (or $t \in J_0$), together with some additional assumptions concerning the function w, guarantees that problem (76)-(77) has at most one local solution.

In the literature, one can encounter miscellaneous classes of Kamke comparison functions (cf. [33, 39]). We will not describe those classes, but let us only mention that they are mostly associated with the differential equation $u' = w(t, u)$ with initial condition $u(0) = 0$ or the integral inequality $u(t) \le \int_0^t w(s, u(s))ds$ for $t \in J_0$ with initial condition $\lim_{t\rightarrow 0} u(t)/t = \lim_{t\rightarrow 0} u(t) = 0$. It is also worthwhile recalling that the classical Lipschitz or Nagumo conditions may serve as Kamke comparison functions [39].

The above-mentioned results due to Kisyński et al. [35–37] assert that if $f : J \times B(x_0, r) \rightarrow E$ is a continuous function satisfying condition (78) with an appropriate Kamke comparison function, then problem (76)-(77) has exactly one local solution.

Observe that the natural translation of inequality (78) in terms of measures of noncompactness has the form

$$\mu(f(t, X)) \le w(t, \mu(X)), \tag{79}$$

where X denotes an arbitrary nonempty subset of the ball $B(x_0, r)$. The first result with the use of condition (79) for $w(t, u) = Cu$ (C is a constant) was obtained by Ambrosetti [40]. After the result of Ambrosetti, there have appeared a lot of papers containing existence results concerning problem (76)-(77) (cf. [41–44]) with the use of condition (79) and involving various types of Kamke comparison functions. It turned out that generalizations of existence results concerning problem (76)-(77) with the use of more and more general Kamke comparison functions are, in fact, only apparent generalizations [45], since the so-called Bompiani comparison function is sufficient to give the most general result in the mentioned direction.

On the other hand, we can generalize existence results involving a condition like (79) taking general measures of noncompactness [31, 32]. Below we present a result coming from [32] which seems to be the most general with respect to taking the most general measure of noncompactness.

In the beginning, let us assume that μ is a measure of noncompactness defined on a Banach space E. Denote by E_μ the set defined by the equality

$$E_\mu = \{x \in E : \{x\} \in \ker \mu\}. \tag{80}$$

The set E_μ will be called the *kernel set* of a measure μ. Taking into account Definition 10 and some properties of a measure of noncompactness (cf. [10]), it is easily seen that E_μ is a closed and convex subset of the space E.

In the case when we consider the so-called *sublinear measure of noncompactness* [10], that is, a measure of noncompactness μ which additionally satisfies the following two conditions:

(6^o) $\mu(X + Y) \le \mu(X) + \mu(Y)$,

(7^o) $\mu(\lambda X) = |\lambda|\mu(X)$ for $\lambda \in \mathbb{R}$,

then the kernel set E_μ forms a closed linear subspace of the space E.

Further on, assume that μ is an arbitrary measure of noncompactness in the Banach space E. Let $r > 0$ be a fixed number and let us fix $x_0 \in E_\mu$. Next, assume that $f : J \times B(x_0, r) \rightarrow E$ (where $J = [0, T]$) is a given uniformly continuous and bounded function; say, $\|f(t, x)\| \le A$.

Moreover, assume that f satisfies the following comparison condition of Kamke type:

$$\mu(x_0 + f(t, X)) \le w(t, \mu(X)) \tag{81}$$

for any nonempty subset X of the ball $B(x_0, r)$ and for almost all $t \in J$.

Here we will assume that the function $w(t, u) = w : J_0 \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ ($J_0 = (0, T]$) is continuous with respect to u for any t and measurable with respect to t for each u. Apart from this, $w(t, 0) = 0$ and the unique solution of the integral inequality

$$u(t) \le \int_0^t w(s, u(s))\,ds \quad (t \in J_0) \tag{82}$$

such that $\lim_{t\rightarrow 0} u(t)/t = \lim_{t\rightarrow 0} u(t) = 0$, is $u \equiv 0$.

The following formulated result comes from [32].

Theorem 16. *Under the above assumptions, if additionally* $\sup\{t+a(t) : t \in J\} \leq 1$, *where* $a(t) = sup\{\|f(0,x_0)-f(s,x)\| : s \leq t,\ \|x-x_0\| \leq As\}$ *and* A *is a positive constant such that* $AT \leq r$, *the initial value problem* (76)-(77) *has at least one local solution* $x = x(t)$ *such that* $x(t) \in E_\mu$ *for* $t \in J$.

The proof of the above theorem is very involved and is therefore omitted (cf. [32, 33]). We restrict ourselves to give a few remarks.

At first, let us notice that in the case when μ is a sublinear measure of noncompactness, condition (81) is reduced to the classical one expressed by (79) provided we assume that $x_0 \in E_\mu$. In such a case, Theorem 16 was proved in [10].

The most frequently used type of a comparison function w is this having the form $w(t,u) = p(t)u$, where $p(t)$ is assumed to be Lebesgue integrable over the interval J. In such a case comparison, condition (81) has the form

$$\mu(x_0 + f(t,X)) \leq p(t)\,\mu(X). \tag{83}$$

An example illustrating Theorem 16 under condition (83) will be given later.

Further, observe that in the case when μ is a sublinear measure of noncompactness such that $x_0 \in E_\mu$, condition (83) can be written in the form

$$\mu(f(t,X)) \leq p(t)\,\mu(X). \tag{84}$$

Condition (84) under additional assumption $\|f(t,x)\| \leq P + Q\|x\|$, with some nonnegative constants P and Q, is used frequently in considerations associated with infinite systems of ordinary differential equations [46–48].

Now, we present the above-announced example coming from [33].

Example 17. Consider the infinite system of differential equations having the form

$$x_n' = a_n(t)\,x_n + f_n(x_n, x_{n+1}, \ldots), \tag{85}$$

where $n = 1, 2, \ldots$ and $t \in J = [0,T]$. System (85) will be considered together with the system of initial conditions

$$x_n(0) = x_0^n \tag{86}$$

for $n = 1, 2, \ldots$. We will assume that there exists the limit $\lim_{n\to\infty} x_0^n = a (a \in \mathbb{R})$.

Problem (85)-(86) will be considered under the following conditions.

(i) $a_n : J \to \mathbb{R}$ $(n = 1, 2, \ldots)$ are continuous functions such that the sequence $(a_n(t))$ converges uniformly on the interval J to the function which vanishes identically on J.

(ii) There exists a sequence of real nonnegative numbers (α_n) such that $\lim_{n\to\infty}\alpha_n = 0$ and $|f_n(x_n, x_{n+1}, \ldots)| \leq \alpha_n$ for $n = 1, 2, \ldots$ and for all $x = (x_1, x_2, x_3, \ldots) \in l^\infty$.

(iii) The function $f = (f_1, f_2, f_3, \ldots)$ transforms the space l^∞ into itself and is uniformly continuous.

Let us mention that the symbol l^∞ used above denotes the classical Banach sequence space consisting of all real bounded sequences (x_n) with the supremum norm; that is, $\|(x_n)\| = \sup\{|x_n| : n = 1, 2, \ldots\}$.

Under the above hypotheses, the initial value problem (85)-(86) has at least one solution $x(t) = (x_1(t), x_2(t), \ldots)$ such that $x(t) \in l^\infty$ for $t \in J$ and $\lim_{n\to\infty} x_n(t) = a$ uniformly with respect to $t \in J$, provided $T \leq 1 (J = [0,T])$.

As a proof, let us take into account the measure of noncompactness in the space l^∞ defined in the following way:

$$\mu(X) = \limsup_{n\to\infty}\left\{\sup_{x\in X}|x_n - a|\right\} \tag{87}$$

for $X \in \mathfrak{M}_{l^\infty}$ (cf. [10]). The kernel $\ker\mu$ of this measure is the family of all bounded subsets of the space l^∞ consisting of sequences converging to the limit equal to a with the same rate.

Further, take an arbitrary set $X \in \mathfrak{M}_{l^\infty}$. Then we have

$$\begin{aligned}
\mu(x_0 + f(t,X)) &= \limsup_{n\to\infty}\left\{\sup_{x\in X}\left|x_0^n + a_n(t)\,x_n + f_n(x_n, x_{n+1}, \ldots) - a\right|\right\} \\
&\leq \limsup_{n\to\infty}\left\{\sup_{x\in X}\left[|a_n(t)|\,|x_n| + \left|f_n(x_n, x_{n+1}, \ldots) + x_0^n - a\right|\right]\right\} \\
&\leq \limsup_{n\to\infty}\left\{\sup_{x\in X} p(t)\,|x_n - a|\right\} + \limsup_{n\to\infty}\left\{\sup_{x\in X}|a_n(t)|\,|a|\right\} \\
&\quad + \limsup_{n\to\infty}\left\{\sup_{x\in X}|f_n(x_n, x_{n+1}, \ldots)|\right\} + \limsup_{n\to\infty}\left\{\sup_{x\in X}|x_0^n - a|\right\},
\end{aligned} \tag{88}$$

where we denoted $p(t) = \sup\{|a_n(t)| : n = 1, 2, \ldots\}$ for $t \in J$. Hence we get

$$\mu(x_0 + f(t,X)) \leq p(t)\,\mu(X), \tag{89}$$

which means that the condition (84) is satisfied.

Combining this fact with assumption (iii) and taking into account Theorem 16, we complete the proof.

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