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Research Article

The Split Feasibility Problems for Countable Families of Asymptotically Strict Pseudocontractions

Wei-Qi Deng

College of Statistics and Mathematics, Yunnan University of Finance and Economics, Kunming, Yunnan 650221, China

Correspondence should be addressed to Wei-Qi Deng; dwq1273@126.com

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An up-to-date algorithm for solving the split feasibility problem for countable families of asymptotically strict pseudocontractions is introduced in the framework of Hilbert spaces. Our results greatly improve and extend those of other authors whose related research studies are restricted to the situation of at most finitely many such mappings.

1. Introduction

The split feasibility problem (SFP) in finite dimensional spaces was first introduced by Censor and Elfving [1] for modeling inverse problems which arise from phase retrievals and in medical image reconstruction [2]. Recently, it has been found that the SFP can also be used in various disciplines such as image restoration, computer tomograph, and radiation therapy treatment planning [3–5].

The split feasibility problem in an infinite dimensional Hilbert space can be found in [2, 4, 6–8].

Let H_1 and H_2 be two real Hilbert spaces with inner product $\langle\cdot,\cdot\rangle$ and the corresponding norm $\|\cdot\|$. Let C and Q be nonempty closed convex subsets of H_1 and H_2, respectively. The purpose of this paper is to introduce and study the following multiple-set split feasibility problem for an infinite family of asymptotically strict pseudocontractions (MSSFP) in the framework of infinite-dimensional Hilbert spaces. Find $x^* \in C$ such that

$$Ax^* \in Q, \tag{1}$$

where $A : H_1 \to H_2$ is a bounded linear operator.

In the sequel, we use Γ to denote the set of solutions of the problem (MSSFP), that is,

$$\Gamma = \{x \in C, Ax \in Q\}. \tag{2}$$

2. Preliminaries

We first recall some definitions, notations, and conclusions which will be needed in proving our main results.

Let E be a Banach space. A mapping $T : E \to E$ is said to be demiclosed at origin, if for any sequence $\{x_n\} \subset E$ with $x_n \rightharpoonup x^*$ and $\|(I-T)x_n\| \to 0$, then $x^* = Tx^*$, where $x_n \rightharpoonup x^*$ denotes that $\{x_n\}$ converges weakly to x^*.

A Banach space E is said to satisfy Opial's condition, if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x^*$ implies that

$$\liminf_{n\to\infty} \|x_n - x^*\| < \liminf_{n\to\infty} \|x_n - y\|, \quad \forall y \in E \text{ with } y \neq x^*. \tag{3}$$

It is well known that every Hilbert space satisfies Opial's condition.

Definition 1. Let H be a real Hilbert space, T be a mapping from H into itself and the fixed point set $F(T)$ of T be nonempty.

(1) T is called a $(\gamma, \{k_n\})$-*asymptotically strict pseudocontraction* if there exists a constant $\gamma \in [0,1)$ and a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ such that

$$\|T^n x - T^n y\|^2 \le k_n\|x-y\|^2 + \gamma\|(I-T^n)x - (I-T^n)y\|^2, \quad \forall x, y \in H. \tag{4}$$

Especially, if $k_n = 1$ for each $n \geq 1$ in (4) and there exists a $\gamma \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \gamma\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in H, \tag{5}$$

then $T : H \to H$ is called a *γ-strict pseudocontraction.*

(2) T is said to be *uniformly L-Lipschitzian* if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\|^2 \leq L\|x - y\|, \quad \forall x, y \in H, n \geq 1. \tag{6}$$

(3) T is said to be *semicompact* if for any bounded sequence $\{x_n\} \subset H$ with $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges strongly to a point $x^* \in H$.

Remark 2. (1) If we put $\gamma = 0$ in (4), then the mapping $T : H \to H$ is asymptotically nonexpansive.

(2) If we put $\gamma = 0$ in (5), then the mapping $T : H \to H$ is nonexpansive.

(3) Each $(\gamma, \{k_n\})$-asymptotically strict pseudocontraction and each γ-strict pseudocontraction both are demiclosed at origin [9].

In 2011, Moudafi [10] proposed the following iterative algorithm for solving split common fixed problem of quasi-nonexpansive mappings: for arbitrarily chosen $x_1 \in H_1$,

$$u_n = x_n + \gamma\beta A^*(T - I)Ax_n,$$
$$x_{n+1} = (1 - \alpha_n)u_n + \alpha_n U u_n, \quad n \in \mathbb{N}, \tag{7}$$

and proved that $\{x_n\}$ converges weakly to a split common fixed point $x^* \in \Gamma$, where $U : H_1 \to H_1$ and $T : H_2 \to H_2$ are two quasinonexpansive mappings, $A : H_1 \to H_2$ is a bounded linear operator, and A^* denotes the adjoint of A.

Motivated and inspired by the studies of Moudafi [10, 11] and Chang et al. [12], in this paper, we introduce an algorithm for solving the split feasibility problems for countable families of asymptotically strict pseudocontractions and prove some strong and weak convergence theorems for such mappings in Hilbert spaces. The results extend those of the authors [12] whose related research studies are restricted to the situation of at most finite families of such mappings.

By using the well-known inequality $\langle x, y\rangle = (1/2)\|x\|^2 + (1/2)\|y\|^2 - (1/2)\|x - y\|^2$ in Hilbert spaces, we can easily show the following proposition, whose proof is omitted.

Proposition 3 (see [12]). *Let $T : H \to H$ be a $(\gamma, \{k_n\})$-asymptotically strict pseudocontraction. If $\Gamma \neq \emptyset$, then for each $p \in F(T)$ and $x \in H$, the following inequalities hold and they are equivalent:*

$$\|T^n x - p\|^2 \leq k_n\|x - p\|^2 + \gamma\|x - T^n x\|^2; \tag{8}$$

$$\langle x - T^n x, x - p\rangle \geq \frac{1 - \gamma}{2}\|x - T^n x\|^2 - \frac{k_n - 1}{2}\|x - p\|^2; \tag{9}$$

$$\langle x - T^n x, p - T^n x\rangle \leq \frac{1 + \gamma}{2}\|x - T^n x\|^2 + \frac{k_n - 1}{2}\|x - p\|^2. \tag{10}$$

Lemma 4 (see [13]). *Let $\{a_n\}$, $\{b_n\}$, and $\{\delta_n\}$ be the sequences of nonnegative real numbers satisfying*

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n. \tag{11}$$

If $\sum_{n=1}^{\infty}\delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then the $\lim_{n\to\infty} a_n$ exists.

Lemma 5 (see [14]). *Let K be a nonempty closed convex subset of a real Hilbert space H and T a nonexpansive mapping from K into itself. If T has a fixed point, then $I - T$ is demiclosed at zero, where I is the identity mapping of H.*

Lemma 6 (see [15]). *The unique solutions to the positive integer equation*

$$n = i + \frac{(m - 1)m}{2}, \quad m \geq i, \; n = 1, 2, 3, \ldots \tag{12}$$

are

$$i = n - \frac{(m - 1)m}{2}, \qquad m = -\left[\frac{1}{2} - \sqrt{2n + \frac{1}{4}}\right], \quad n = 1, 2, 3, \ldots, \tag{13}$$

where $[x]$ denotes the maximal integer that is not larger than x.

3. Main Results

In the sequel, we assume that the following conditions are satisfied:

(a) H_1 and H_2 are two real Hilbert spaces, $A : H_1 \to H_2$ is a bounded linear operator, and A^* denotes the adjoint of A;

(b) $\{S_i\} : H_1 \to H_1$ is a sequence of uniformly L_1-Lipschitzian and $(\beta_i, \{k_{1,n}^{(i)}\})$-asymptotically strict pseudocontractions and $\{T_i\} : H_2 \to H_2$ is a sequence of uniformly L_2-Lipschitzian and $(\mu_i, \{k_{2,n}^{(i)}\})$-asymptotically strict pseudocontractions satisfying the following conditions:

(1) $C := \cap_{i=1}^{\infty} F(S_i) \neq \emptyset$, $Q := \cap_{i=1}^{\infty} F(T_i) \neq \emptyset$;

(2) $\beta := \sup_{i\geq 1}\{\beta_i\} < 1$ and $\mu := \sup_{i\geq 1}\{\mu_i\} < 1$;

(3) for each $i \geq 1$, $k_n^{(i)} := \max\{k_{1,n}^{(i)}, k_{2,n}^{(i)}\}$, and $\sum_{i=1}^{\infty}\sum_{n=1}^{\infty}(k_n^{(i)} - 1) < \infty$.

The multiple-set split feasibility problem for infinite families of nonlinear mappings $\{S_i\}$ and $\{T_i\}$ is to find a point

$$q \in C \text{ such that } Aq \in Q, \tag{14}$$

whose set of solutions is denoted by Γ.

Lemma 7. *Let $H_1, H_2, A, \{S_i\}, \{T_i\}, C, Q, \beta, \mu, L_1, L_2$ and $\{k_n^{(i)}\}$ be the same as those mentioned above. Let $\{x_n\}$ be the following sequence generated by an arbitrarily chosen $x_1 \in H_1$*

$$\begin{gathered} u_n = x_n + \gamma A^* \left((T_n^*)^{m_n} - I \right) A x_n, \\ x_{n+1} = (1-\alpha_n) u_n + \alpha_n (S_n^*)^{m_n} u_n, \quad n \in \mathbb{N}, \end{gathered} \tag{15}$$

where $T_n^* = T_{i_n}$, $S_n^* = S_{i_n}$ with i_n and m_n being the solutions to the positive integer equation: $n = i + (m-1)m/2$ $(m \geq i, n = 1, 2, \ldots)$; that is, for each $n \geq 1$, there exist unique i_n and m_n such that

$$\begin{gathered} i_1 = 1,\ i_2 = 1,\ i_3 = 2,\ i_4 = 1, \\ i_5 = 2,\ i_6 = 3,\ i_7 = 1,\ i_8 = 2, \ldots; \\ m_1 = 1,\ m_2 = 2,\ m_3 = 2,\ m_4 = 3,\ m_5 = 3, \\ m_6 = 3,\ m_7 = 4,\ m_8 = 4, \ldots; \end{gathered} \tag{16}$$

$\{\alpha_n\}$ is a sequence in $[0, 1]$, and γ is a constant satisfying the following condition:

(4) : $\alpha_n \in (\delta, 1-\beta)$ *for all $n \geq 1$ and $\gamma \in (0, (1-\mu)/\|A\|^2)$, where $\delta \in (0, 1-\beta)$ is a positive constant. If $\Gamma \neq \emptyset$, for any $q \in \Gamma$, then*

(I) $\lim_{n\to\infty}\|x_n - q\|$ *and* $\lim_{n\to\infty}\|u_n - q\|$ *exist and have the same values;*

(II) *for each $i \geq 1$, there exists a corresponding subsequence $\{x_n\}_{n\in\mathbb{N}_i}$ of $\{x_n\}$ such that*

$$\lim_{\mathbb{N}_i \ni n \to \infty} \|u_n - S_i u_n\| = \|A x_n - T_i A x_n\| = 0, \tag{17}$$

where $\mathbb{N}_i := \{n \in \mathbb{N} : n = i + (m-1)m/2, m \geq i, m \in \mathbb{N}\}$.

Proof. (I) Taking $q \in \Gamma$, that is, $q \in C$ and $Aq \in Q$, and using (15) and (9), we have

$$\begin{aligned} \|x_{n+1} - q\|^2 &\leq \left\| u_n - q - \alpha_n \left(u_n - (S_n^*)^{m_n} u_n \right) \right\|^2 \\ &= \|u_n - q\|^2 - 2\alpha_n \left\langle u_n - q, u_n - (S_n^*)^{m_n} u_n \right\rangle \\ &\quad + \alpha_n^2 \left\| u_n - (S_n^*)^{m_n} u_n \right\|^2 \leq \|u_n - q\|^2 \\ &\quad - \alpha_n \left[(1-\beta) \left\| u_n - (S_n^*)^{m_n} u_n \right\|^2 \right. \\ &\quad \left. - \left(k_{m_n}^{(i_n)} - 1 \right) \|u_n - q\|^2 \right] \\ &\quad + \alpha_n^2 \left\| u_n - (S_n^*)^{m_n} u_n \right\|^2 \\ &= \left(1 + \alpha_n \left(k_{m_n}^{(i_n)} - 1 \right) \right) \|u_n - q\|^2 \\ &\quad - \alpha_n (1 - \beta - \alpha_n) \left\| u_n - (S_n^*)^{m_n} u_n \right\|^2, \end{aligned} \tag{18}$$

$$\begin{aligned} \|u_n - q\|^2 &= \left\| x_n + \gamma A^* \left((T_n^*)^{m_n} - I \right) A x_n - q \right\|^2 \\ &= \|x_n - q\|^2 + \gamma^2 \left\| A^* \left((T_n^*)^{m_n} - I \right) A x_n \right\|^2 \\ &\quad + 2\gamma \left\langle x_n - q, A^* \left((T_n^*)^{m_n} - I \right) A x_n \right\rangle \\ &= \|x_n - q\|^2 \\ &\quad + \gamma^2 \left\langle \left((T_n^*)^{m_n} - I \right) A x_n, A A^* \left((T_n^*)^{m_n} - I \right) A x_n \right\rangle \\ &\quad + 2\gamma \left\langle x_n - q, A^* \left((T_n^*)^{m_n} - I \right) A x_n \right\rangle, \end{aligned} \tag{19}$$

where

$$\begin{aligned} &\gamma^2 \left\langle \left((T_n^*)^{m_n} - I \right) A x_n, A A^* \left((T_n^*)^{m_n} - I \right) A x_n \right\rangle \\ &\quad \leq \|A\|^2 \gamma^2 \left\langle \left((T_n^*)^{m_n} - I \right) A x_n, \left((T_n^*)^{m_n} - I \right) A x_n \right\rangle \\ &\quad \leq \|A\|^2 \gamma^2 \left\| \left((T_n^*)^{m_n} - I \right) A x_n \right\|^2, \end{aligned} \tag{20}$$

$$\begin{aligned} &2\gamma \left\langle x_n - q, A^* \left((T_n^*)^{m_n} - I \right) A x_n \right\rangle \\ &\quad = 2\gamma \left\langle A (x_n - q), \left((T_n^*)^{m_n} - I \right) A x_n \right\rangle \\ &\quad = 2\gamma \left\langle A (x_n - q) + \left((T_n^*)^{m_n} - I \right) A x_n \right. \\ &\qquad \left. - \left((T_n^*)^{m_n} - I \right) A x_n, \left((T_n^*)^{m_n} - I \right) A x_n \right\rangle \\ &\quad = 2\gamma \left(\left\langle (T_n^*)^{m_n} A x_n - Aq, \left((T_n^*)^{m_n} - I \right) A x_n \right\rangle \right. \\ &\qquad \left. - \left\| \left((T_n^*)^{m_n} - I \right) A x_n \right\|^2 \right). \end{aligned} \tag{21}$$

Further, letting $x = Ax_n$, $T^n = (T_n^*)^{m_n}$, $p = Aq$, $\gamma = \mu$ in (10) and noting $Aq \in F(T_n^*)$, we have

$$\begin{aligned} &\left\langle (T_n^*)^{m_n} A x_n - Aq, \left((T_n^*)^{m_n} - I \right) A x_n \right\rangle \\ &\quad \leq \frac{1+\mu}{2} \left\| \left((T_n^*)^{m_n} - I \right) A x_n \right\|^2 + \frac{k_{m_n}^{(i_n)} - 1}{2} \|A x_n - Aq\|^2 \\ &\quad \leq \frac{1+\mu}{2} \left\| \left((T_n^*)^{m_n} - I \right) A x_n \right\|^2 + \frac{\left(k_{m_n}^{(i_n)} - 1 \right) \|A\|^2}{2} \|x_n - q\|^2. \end{aligned} \tag{22}$$

Substituting (22) into (21) and simplifying it, we have

$$\begin{aligned} &2\gamma \left\langle x_n - q, A^* \left((T_n^*)^{m_n} - I \right) A x_n \right\rangle \\ &\quad \leq \gamma (\mu - 1) \left\| \left((T_n^*)^{m_n} - I \right) A x_n \right\|^2 \\ &\qquad + \left(k_{m_n}^{(i_n)} - 1 \right) \gamma \|A\|^2 \|x_n - q\|^2. \end{aligned} \tag{23}$$

Substituting (20) and (23) into (19) and simplifying it, we have

$$\begin{aligned}\|u_n-q\|^2 &\le \|x_n-q\|^2+\gamma^2\|A\|^2\left\|\left((T_n^*)^{m_n}-I\right)Ax_n\right\|^2\\&\quad+\gamma(\mu-1)\left\|\left((T_n^*)^{m_n}-I\right)Ax_n\right\|^2\\&\quad+\left(k_{m_n}^{(i_n)}-1\right)\gamma\|A\|^2\|x_n-q\|^2\\&=\|x_n-q\|^2-\gamma\left(1-\mu-\gamma\|A\|^2\right)\\&\quad\times\left\|\left((T_n^*)^{m_n}-I\right)Ax_n\right\|^2\\&\quad+\left(k_{m_n}^{(i_n)}-1\right)\gamma\|A\|^2\|x_n-q\|^2.\end{aligned}\tag{24}$$

Again, substituting (24) into (18) and simplifying it, we have

$$\begin{aligned}\|x_{n+1}-q\|^2 &\le \left(1+\alpha_n\left(k_{m_n}^{(i_n)}-1\right)\right)\\&\quad\times\Big\{\|x_n-q\|^2\\&\quad-\gamma\left(1-\mu-\gamma\|A\|^2\right)\\&\quad\times\left\|\left((T_n^*)^{m_n}-I\right)Ax_n\right\|^2\\&\quad+\left(k_{m_n}^{(i_n)}-1\right)\gamma\|A\|^2\|x_n-q\|^2\Big\}\\&\quad-\alpha_n(1-\beta-\alpha_n)\left\|u_n-(S_n^*)^{m_n}u_n\right\|^2\\&\le\left(1+\alpha_n\left(k_{m_n}^{(i_n)}-1\right)\right)\|x_n-q\|^2\\&\quad-\gamma\left(1-\mu-\gamma\|A\|^2\right)\left\|\left((T_n^*)^{m_n}-I\right)Ax_n\right\|^2\\&\quad+\left(1+\alpha_n\left(k_{m_n}^{(i_n)}-1\right)\right)\left(k_{m_n}^{(i_n)}-1\right)\gamma\|A\|^2\|x_n-q\|^2\\&\quad-\alpha_n(1-\beta-\alpha_n)\left\|u_n-(S_n^*)^{m_n}u_n\right\|^2.\end{aligned}\tag{25}$$

By condition (4), we have

$$\begin{aligned}\|x_{n+1}-q\|^2 &\le \left(1+\alpha_n\left(k_{m_n}^{(i_n)}-1\right)\right)\|x_n-q\|^2\\&\quad+\left(1+\alpha_n\left(k_{m_n}^{(i_n)}-1\right)\right)\left(k_{m_n}^{(i_n)}-1\right)\\&\quad\times\gamma\|A\|^2\|x_n-q\|^2\\&\le\left(1+K\left(k_{m_n}^{(i_n)}-1\right)\right)\|x_n-q\|^2,\end{aligned}\tag{26}$$

where

$$K=\sup_{n\ge1}\left\{\alpha_n+\left(1+\alpha_n\left(k_{m_n}^{(i_n)}-1\right)\right)\gamma\|A\|^2\right\}<\infty.\tag{27}$$

Note that $\sum_{n=1}^{\infty}(k_{m_n}^{(i_n)}-1)=\sum_{i=1}^{\infty}\sum_{n=i}^{\infty}(k_n^{(i)}-1)\le\sum_{i=1}^{\infty}\sum_{n=1}^{\infty}(k_n^{(i)}-1)<\infty$. Hence, from Lemma 4, we know that the following limit exists:

$$\lim_{n\to\infty}\|x_n-q\|.\tag{28}$$

We now prove that for each $q\in\Gamma$, the limit

$$\lim_{n\to\infty}\|u_n-q\|\tag{29}$$

exists. In fact, from (25) and (28), it follows that

$$\begin{aligned}&\gamma\left(1-\mu-\gamma\|A\|^2\right)\left\|\left((T_n^*)^{m_n}-I\right)Ax_n\right\|^2\\&\quad+\alpha_n(1-\beta-\alpha_n)\left\|u_n-(S_n^*)^{m_n}u_n\right\|^2\\&\le\|x_n-q\|^2-\|x_{n+1}-q\|^2\\&\quad+K\left(k_{m_n}^{(i_n)}-1\right)\|x_n-q\|^2\longrightarrow0\quad(n\longrightarrow\infty).\end{aligned}\tag{30}$$

This, combined with condition (4), implies that

$$\lim_{n\to\infty}\left\|u_n-(S_n^*)^{m_n}u_n\right\|=0,\tag{31}$$

$$\lim_{n\to\infty}\left\|\left((T_n^*)^{m_n}-I\right)Ax_n\right\|=0.\tag{32}$$

Therefore, it follows from (19), (28), and (32) that $\lim_{n\to\infty}\|u_n-q\|$ exists.

(II) We firstly prove that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$ and $\lim_{n\to\infty}\|u_{n+1}-u_n\|=0$. As a matter of fact, it follows from (15) that

$$\begin{aligned}\|x_{n+1}-x_n\|&=\left\|(1-\alpha_n)u_n+\alpha_n(S_n^*)^{m_n}u_n-x_n\right\|\\&=\Big\|(1-\alpha_n)\left(x_n+\gamma A^*\left((T_n^*)^{m_n}-I\right)Ax_n\right)\\&\quad+\alpha_n\left((S_n^*)^{m_n}u_n-x_n\right)\Big\|\\&=\Big\|(1-\alpha_n)\gamma A^*\left((T_n^*)^{m_n}-I\right)Ax_n\\&\quad+\alpha_n\left((S_n^*)^{m_n}u_n-x_n\right)\Big\|\\&=\Big\|(1-\alpha_n)\gamma A^*\left((T_n^*)^{m_n}-I\right)Ax_n\\&\quad+\alpha_n\left((S_n^*)^{m_n}u_n-u_n\right)\\&\quad+\alpha_n(u_n-x_n)\Big\|\\&=\Big\|(1-\alpha_n)\gamma A^*\left((T_n^*)^{m_n}-I\right)Ax_n\\&\quad+\alpha_n\left((S_n^*)^{m_n}u_n-u_n\right)\\&\quad+\alpha_n\gamma A^*\left((T_n^*)^{m_n}-I\right)Ax_n\Big\|\\&=\Big\|\gamma A^*\left((T_n^*)^{m_n}-I\right)Ax_n\\&\quad+\alpha_n\left((S_n^*)^{m_n}u_n-u_n\right)\Big\|.\end{aligned}\tag{33}$$

In view of (31) and (32), we have that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0.\tag{34}$$

Similarly, it follows from (15), (32), and (34) that

$$\begin{aligned}\|u_{n+1}-u_n\| &= \Big\| x_{n+1} + \gamma A^*\left(\left(T_{n+1}^*\right)^{m_{n+1}} - I\right)Ax_{n+1} \\ &\quad - \left(x_n + \gamma A^*\left(\left(T_n^*\right)^{m_n} - I\right)Ax_n\right)\Big\| \\ &= \|x_{n+1}-x_n\| + \gamma\left\|A^*\left(\left(T_{n+1}^*\right)^{m_{n+1}} - I\right)Ax_{n+1}\right\| \\ &\quad + \gamma\left\|A^*\left(\left(T_n^*\right)^{m_n} - I\right)Ax_n\right\| \longrightarrow 0 \quad (n \longrightarrow \infty).\end{aligned} \tag{35}$$

Next, for each $i \in \mathbb{N}$, we consider the corresponding subsequence $\{u_n\}_{n\in\mathbb{N}_i}$ of $\{u_n\}$. For example, by Lemma 6 and the definition of $\mathbb{N}_1$, we have $\mathbb{N}_1 = \{1,2,4,7,11,16,\ldots\}$ and $i_1 = i_2 = i_4 = i_7 = i_{11} = i_{16} = \cdots = 1$. Note that $\{m_n\}_{n\in\mathbb{N}_i} = \{i, i+1, i+2,\ldots\}$, that is, $m_n - 1 = m_{n-1}$, and $S_n^* = S_{i_n} = S_i$ whenever $n \in \mathbb{N}_i$. Set $\eta_n := \|u_n - S_i^{m_n}u_n\|$. Since $\{S_i\}$ are uniformly L_1-Lipschitzian and $m_n \geq 1$ for $n \geq 1$, we have, for each $n \in \mathbb{N}_i$ and $n \geq 2$,

$$\begin{aligned}\|u_n - S_iu_n\| &\leq \|u_n - S_i^{m_n}u_n\| + \|S_i^{m_n}u_n - S_iu_n\| \\ &\leq \eta_n + L_1\left\|S_i^{m_n-1}u_n - u_n\right\| \\ &\leq \eta_n + L_1\left(\left\|S_i^{m_n-1}u_n - S_i^{m_n-1}u_{n-1}\right\| + \left\|S_i^{m_n-1}u_{n-1} - u_n\right\|\right) \\ &\leq \eta_n + L_1\left\|S_i^{m_n-1}u_n - S_i^{m_n-1}u_{n-1}\right\| \\ &\quad + L_1\left(\left\|S_i^{m_n-1}u_{n-1} - u_{n-1}\right\| + \|u_n - u_{n-1}\|\right) \\ &\leq \eta_n + L_1^2\|u_n - u_{n-1}\| \\ &\quad + L_1\left(\left\|S_i^{m_n-1}u_{n-1} - u_{n-1}\right\| + \|u_{n-1} - u_n\|\right) \\ &\leq \eta_n + L_1(1+L_1)\|u_n - u_{n-1}\| + L_1\eta_{n-1}.\end{aligned} \tag{36}$$

Thus, it follows from (31) and (35) that, for each $i \geq 1$,

$$\lim_{\mathbb{N}_i \ni n\to\infty}\|u_n - S_iu_n\| = 0. \tag{37}$$

Similarly, we have, for each $i \geq 1$,

$$\lim_{\mathbb{N}_i \ni n\to\infty}\|Ax_n - T_iAx_n\| = 0. \tag{38}$$

This completes the proof. □

Theorem 8. *Let $H_1, H_2, A, \{S_i\}, \{T_i\}, C, Q, \beta, \mu, L_1, L_2$ and $\{k_n^{(i)}\}$ be the same as those in Lemma 7. Suppose that $\{x_n\}$ is a sequence defined by* (15)*. If $\Gamma \neq \emptyset$ and there exist mappings $S_{i_0} \in \{S_i\}_{i=1}^\infty$ and $T_{i_0} \in \{T_i\}_{i=1}^\infty$ and nondecreasing functions $f, h : [0,\infty) \to [0,\infty)$ with $f(0) = h(0) = 0$ and $f(r), h(r) > 0$ for all $r \in (0,\infty)$ such that $f(d(x_n,\Gamma)) \leq \|u_n - S_{i_0}u_n\|$ and $h(d(Ax_n,Q)) \leq \|Ax_n - T_{i_0}Ax_n\|$ for all $n \geq 1$, then $\{x_n\}$ converges strongly to some member of Γ.*

Proof. By Lemma 7, there exists a subsequence $\{u_n\}_{n\in\mathbb{N}_{i_0}}$ of $\{u_n\}$ such that

$$\lim_{\mathbb{N}_{i_0} \ni n\to\infty}\left\|u_n - S_{i_0}u_n\right\| = 0. \tag{39}$$

Since for all $n \in \mathbb{N}_{i_0}$,

$$f(d(x_n,\Gamma)) \leq \left\|u_n - S_{i_0}u_n\right\|, \tag{40}$$

by taking lim sup as $\mathbb{N}_{i_0} \ni n \to \infty$ on both sides in the inequality above, we have

$$\lim_{\mathbb{N}_{i_0} \ni n\to\infty} f(d(x_n,\Gamma)) = 0, \tag{41}$$

which implies $\lim_{\mathbb{N}_{i_0} \ni n\to\infty} d(x_n,\Gamma) = 0$ by the definition of the function f.

Now we will show that $\{x_n\}_{n\in\mathbb{N}_{i_0}}$ is a Cauchy sequence. By Lemma 7, there exists a constant $M > 0$ such that $\|x_n - q\|^2 \leq M\|x_m - q\|^2$ for any $q \in \Gamma$ and all $n > m$. And for any $\epsilon > 0$, there exists a positive integer N such that $d^2(x_n,\Gamma) < \epsilon/4M$ for all $n \geq N$ and $n \in \mathbb{N}_{i_0}$. Then, for any $q \in \Gamma$ and $n, m \geq N$ and $n, m \in \mathbb{N}_{i_0}$, we have

$$\begin{aligned}\|x_n - x_m\|^2 &\leq 2\left(\|x_n - q\|^2 + \|x_m - q\|^2\right) \\ &\leq 4M\|x_N - q\|^2.\end{aligned} \tag{42}$$

Taking the infimum in the above inequalities for all $q \in \Gamma$ yields that

$$\|x_n - x_m\|^2 \leq 4Md^2(x_N,\Gamma) < \epsilon, \tag{43}$$

which implies that $\{x_n\}_{n\in\mathbb{N}_{i_0}}$ is a Cauchy sequence. Therefore, there exists a $p \in H_1$ such that $x_n \to p$ as $\mathbb{N}_{i_0} \ni n \to \infty$ since H_1 is complete. Firstly, we show that $p \in C$. $\lim_{\mathbb{N}_{i_0} \ni n\to\infty} d(x_n,\Gamma) = 0$ shows that $d(p,\Gamma) = 0$, which implies that $p \in C$ since $\Gamma \subset C$. Secondly, we show that $Ap \in Q$. Since $\{x_n\}_{n\in\mathbb{N}_{i_0}}$ converges to p and $h(d(Ax_n,Q)) \leq \|Ax_n - T_{i_0}Ax_n\|$ for all $n \in \mathbb{N}_{i_0}$, then $d(Ap,Q) = 0$. This implies that $Ap \in Q$ because of the closedness of Q, and so $p \in \Gamma$. It finally follows from the existence of $\lim_{n\to\infty}\|x_n - p\|$ that $x_n \to p$ as $n \to \infty$. This completes the proof. □

Example 9. Let $H_1 = H_2 = \mathbb{R}^1$ with the standard norm $\|\cdot\| = |\cdot|$ and $C = [-1,1]$. Let $\{S_i\} = \{T_i\} : C \to C$ be two sequences of mappings defined by

$$S_ix = \begin{cases} \dfrac{x}{i+1}, & x \in (0,1], \\ x, & x \in [-1,0]. \end{cases} \tag{44}$$

It is easily shown that $\{S_i\}$ is uniformly L-Lipschitzian and a sequence of $(0,\{k_n = 1\})$-asymptotically strict pseudocontractions. We now prove that the sequence $\{x_n\}$ defined by (15) converges strongly to some member of Γ. Let

$Ax = A^*x = x/2$ for all $x \in C$ with $\|A\| = 1/2$ and $\gamma = 3 \in (0, 1/\|A\|^2)$. If $\{x_n\} \subset (0, 1]$, we then have

$$\begin{aligned} |u_n - S_1 u_n| &= \frac{1}{2}|u_n| \\ &= \frac{1}{2}\left|x_n + \frac{3}{2}\left[\left(\frac{1}{i_n+1}\right)^{m_n}\frac{x_n}{2} - \frac{x_n}{2}\right]\right| \\ &= \kappa_n |x_n|, \end{aligned} \quad (45)$$

where $\kappa_n := (1/2)|1 + (3/4)[(1/(i_n+1))^{m_n} - 1]|$ with $\kappa := \inf_{n\geq 1}\kappa_n > 0$. Define a nondecreasing function $f : [0,\infty) \to [0,\infty)$ by $f(x) = \kappa x$. Since $\Gamma = [-1, 0]$, we then have

$$f(d(x_n, \Gamma)) = f(|x_n|) = \kappa|x_n| \leq |u_n - S_1 u_n|. \quad (46)$$

Similarly, we also can define a nondecreasing function $h : [0,\infty) \to [0,\infty)$ with $h(0) = 0$ such that

$$h(d(Ax_n, Q)) \leq |Ax_n - T_{i_0}Ax_n| \quad (47)$$

for some $i_0 \geq 1$, which implies that, by Lemma 7 and Theorem 8, $x_n \to x^* \in \Gamma$ as $n \to \infty$.

Theorem 10. *Let H_1, H_2, A, C and Q be the same as those in Lemma 7. Let $\{S_i\} : H_1 \to H_1$ and $\{T_i\} : H_2 \to H_2$ be two sequences of nonexpansive mappings. Let $\{x_n\}$ be the following sequence generated by an arbitrarily chosen $x_1 \in H_1$*

$$\begin{aligned} u_n &= x_n + \gamma A^*(T_n^* - I)Ax_n, \\ x_{n+1} &= (1-\alpha_n)u_n + \alpha_n S_n^* u_n, \quad n \geq 1, \end{aligned} \quad (48)$$

where $\{\alpha_n\}$ is a sequence in $[\alpha, 1-\alpha]$ for some $\alpha \in (0,1)$; $\gamma \in (0, 1/\|A\|^2)$; $T_n^* = T_{i_n}$, $S_n^* = S_{i_n}$ with i_n satisfying the positive integer equation: $n = i + (m-1)m/2$ $(m \geq i, n = 1, 2, \ldots)$. Then $\{x_n\}$ converges weakly to an $x^* \in \Gamma$.

Proof. It is clear that both $\{S_i\}$ and $\{T_i\}$ are asymptotically strict pseudocontractions. Then, by the proof of Lemma 7, we have

$$\lim_{n\to\infty}\|u_n - S_n^* u_n\| = 0, \quad (49)$$

$$\lim_{n\to\infty}\|(T_n^* - I)Ax_n\| = 0. \quad (50)$$

In addition, we also have

$$\lim_{n\to\infty}\|u_{n+1} - u_n\| = \|x_{n+1} - x_n\| = 0, \quad (51)$$

which implies that, by induction, for any nonnegative integer k,

$$\lim_{n\to\infty}\|u_{n+k} - u_n\| = \lim_{n\to\infty}\|x_{n+k} - x_n\| = 0. \quad (52)$$

For each $k \geq 1$, since

$$\begin{aligned} \|u_n - S_{n+k}^* u_n\| &\leq \|u_n - u_{n+k}\| + \|u_{n+k} - S_{n+k}^* u_n\| \\ &\leq \|u_n - u_{n+k}\| + \|u_{n+k} - S_{n+k}^* u_{n+k}\| \\ &\quad + \|S_{n+k}^* u_{n+k} - S_{n+k}^* u_n\| \\ &\leq 2\|u_n - u_{n+k}\| + \|u_{n+k} - S_{n+k}^* u_{n+k}\|, \end{aligned} \quad (53)$$

it follows from (49) and (52) that

$$\lim_{n\to\infty}\|u_n - S_{n+k}^* u_n\| = 0. \quad (54)$$

Now, for each $i \geq 1$, we claim that

$$\lim_{n\to\infty}\|u_n - S_i u_n\| = 0. \quad (55)$$

As a matter of fact, setting

$$n = N_m + i, \quad (56)$$

where $N_m = (m-1)m/2, m \geq i$, we obtain that

$$\begin{aligned} \|u_n - S_i u_n\| &\leq \|u_n - u_{N_m}\| + \|u_{N_m} - S_i u_n\| \\ &\leq \|u_n - u_{N_m}\| + \|u_{N_m} - S_{N_m+i}^* u_{N_m}\| \\ &\quad + \|S_{N_m+i}^* u_{N_m} - S_i u_n\| \\ &= \|u_n - u_{N_m}\| + \|u_{N_m} - S_{N_m+i}^* u_{N_m}\| \\ &\quad + \|S_i u_{N_m} - S_i u_n\| \\ &\leq 2\|u_n - u_{N_m}\| + \|u_{N_m} - S_{N_m+i}^* u_{N_m}\| \\ &= 2\|u_n - u_{n-i}\| + \|u_{N_m} - S_{N_m+i}^* u_{N_m}\|. \end{aligned} \quad (57)$$

Then, since $N_m \to \infty$ as $n \to \infty$, it follows from (52) and (54) that (55) holds obviously. Similarly, we have, for each $i \geq 1$,

$$\lim_{n\to\infty}\|Ax_n - T_i Ax_n\| = 0. \quad (58)$$

Next, since $\{u_n\}$ is bounded, there exists a subsequence $\{u_{n_i}\} \subset \{u_n\}$ such that $u_{n_i} \rightharpoonup x^*$ (some point in H_1). From (55) we have $\lim_{i\to\infty}\|u_{n_i} - S_j u_{n_i}\| = 0$ for each $j \geq 1$. By Lemma 5, each S_j is demiclosed at zero, so we know that $x^* \in \cap_{j=1}^{\infty} F(S_j)$. Moreover, it follows from (48) and (50) that

$$x_{n_i} = u_{n_i} - \gamma A^*(T_{n_i}^* - I)Ax_{n_i} \rightharpoonup x^* \quad (i \longrightarrow \infty). \quad (59)$$

Since A is a linear bounded operator, it yields that $Ax_{n_i} \rightharpoonup Ax^*$. In view of (58) we have

$$\lim_{i\to\infty}\|Ax_{n_i} - T_j Ax_{n_i}\| = 0, \quad \forall j \geq 1. \quad (60)$$

Again since each T_j is demiclosed at zero, we know that $Ax^* \in \cap_{j=1}^{\infty} F(T_j)$. This implies that $x^* \in \Gamma$.

Note that each Hilbert space possesses Opial property, which guarantees that the weakly subsequential limit of $\{u_n\}$ is unique. Consequently, $\{u_n\}$ converges weakly to the point $x^* \in \Gamma$. Since $x_n = u_n - \gamma A^*(T_n^* - I)Ax_n$, we know that $\{x_n\}$ converges weakly to $x^* \in \Gamma$. The proof is completed. □

Remark 11. Note that, from Remark 2(3), the class of $(\gamma, \{k_n\})$-asymptotically strict pseudocontractions is demiclosed at zero. Then, together with nonexpansiveness replaced by Lipschitz continuity, the two sequences of nonexpansive mappings $\{S_i\}$ and $\{T_i\}$ in Theorem 10 can be extended to $(\gamma, \{k_n\})$-asymptotically strict pseudocontractions as in Lemma 7.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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References

[1] Y. Censor and T. Elfving, "A multiprojection algorithm using Bregman projections in a product space," *Numerical Algorithms*, vol. 8, no. 2–4, pp. 221–239, 1994.

[2] C. Byrne, "Iterative oblique projection onto convex sets and the split feasibility problem," *Inverse Problems*, vol. 18, no. 2, pp. 441–453, 2002.

[3] Y. Censor, T. Bortfeld, B. Martin, and A. Trofimov, "A unified approach for inversion problems in intensity-modulated radiation therapy," *Physics in Medicine and Biology*, vol. 51, no. 10, pp. 2353–2365, 2006.

[4] Y. Censor, T. Elfving, N. Kopf, and T. Bortfeld, "The multiple-sets split feasibility problem and its applications for inverse problems," *Inverse Problems*, vol. 21, no. 6, pp. 2071–2084, 2005.

[5] Y. Censor, A. Motova, and A. Segal, "Perturbed projections and subgradient projections for the multiple-sets split feasibility problem," *Journal of Mathematical Analysis and Applications*, vol. 327, no. 2, pp. 1244–1256, 2007.

[6] H. K. Xu, "A variable Krasnosel'skii–Mann algorithm and the multiple-set split feasibility problem," *Inverse Problems*, vol. 22, no. 6, pp. 2021–2034, 2006.

[7] Q. Yang, "The relaxed CQ algorithm solving the split feasibility problem," *Inverse Problems*, vol. 20, no. 4, pp. 1261–1266, 2004.

[8] J. Zhao and Q. Yang, "Several solution methods for the split feasibility problem," *Inverse Problems*, vol. 21, no. 5, pp. 1791–1799, 2005.

[9] T. H. Kim and H. K. Xu, "Convergence of the modified Mann's iteration method for asymptotically strict pseudo-contractions," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 68, no. 9, pp. 2828–2836, 2008.

[10] A. Moudafi, "A note on the split common fixed-point problem for quasi-nonexpansive operators," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 74, no. 12, pp. 4083–4087, 2011.

[11] A. Moudafi, "The split common fixed-point problem for demicontractive mappings," *Inverse Problems*, vol. 26, no. 5, Article ID 055007, 6 pages, 2010.

[12] S. S. Chang, Y. J. Cho, J. K. Kim, W. B. Zhang, and L. Yang, "Multiple-set split feasibility problems for asymptotically strict pseudocontractions," *Abstract and Applied Analysis*, vol. 2012, Article ID 491760, 12 pages, 2012.

[13] K. Aoyama, Y. Kimura, W. Takahashi, and M. Toyoda, "Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 67, no. 8, pp. 2350–2360, 2007.

[14] K. Goebel and W. A. Kirk, *Topics in Metric Fixed Point Theory*, vol. 28 of *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge, UK, 1990.

[15] W. Q. Deng and P. Bai, "An implicit iteration process for common fixed points of two infinite families of asymptotically nonexpansive mappings in Banach spaces," *Journal of Applied Mathematics*, vol. 2013, Article ID 602582, 6 pages, 2013.