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*Research Article*

# Optimal Bounds for Neuman Means in Terms of Harmonic and Contraharmonic Means

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For $a, b > 0$ with $a \neq b$, the Schwab-Borchardt mean $\mathrm{SB}(a, b)$ is defined as $\mathrm{SB}(a, b) = \{\sqrt{b^2 - a^2}/\cos^{-1}(a/b)$ if $a < b$, $\sqrt{a^2 - b^2}/\cosh^{-1}(a/b)$ if a $> b$. In this paper, we find the greatest values of $\alpha_1$ and $\alpha_2$ and the least values of $\beta_1$ and $\beta_2$ in $[0, 1/2]$ such that $H(\alpha_1 a+(1-\alpha_1)b, \alpha_1 b+(1-\alpha_1)a) < S_{AH}(a, b) < H(\beta_1 a+(1-\beta_1)b, \beta_1 b+(1-\beta_1)a)$ and $H(\alpha_2 a+(1-\alpha_2)b, \alpha_2 b+(1-\alpha_2)a) < S_{HA}(a, b) < H(\beta_2 a+(1-\beta_2)b, \beta_2 b+(1-\beta_2)a)$. Similarly, we also find the greatest values of $\alpha_3$ and $\alpha_4$ and the least values of $\beta_3$ and $\beta_4$ in $[1/2, 1]$ such that $C(\alpha_3 a+(1-\alpha_3)b, \alpha_3 b+(1-\alpha_3)a) < S_{CA}(a, b) < C(\beta_3 a+(1-\beta_3)b, \beta_3 b+(1-\beta_3)a)$ and $C(\alpha_4 a+(1-\alpha_4)b, \alpha_4 b+(1-\alpha_4)a) < S_{AC}(a, b) < C(\beta_4 a+(1-\beta_4)b, \beta_4 b+(1-\beta_4)a)$. Here, $H(a, b) = 2ab/(a+b)$, $A(a, b) = (a+b)/2$, and $C(a, b) = (a^2+b^2)/(a+b)$ are the harmonic, arithmetic, and contraharmonic means, respectively, and $S_{HA}(a, b) = \mathrm{SB}(H, A)$, $S_{AH}(a, b) = \mathrm{SB}(A, H)$, $S_{CA}(a, b) = \mathrm{SB}(C, A)$, and $S_{AC}(a, b) = \mathrm{SB}(A, C)$ are four Neuman means derived from the Schwab-Borchardt mean.

## 1. Introduction

For $a, b > 0$ with $a \neq b$, the Schwab-Borchardt mean $\mathrm{SB}(a, b)$ is defined as

$$\mathrm{SB}(a, b) = \begin{cases} \dfrac{\sqrt{b^2 - a^2}}{\cos^{-1}(a/b)} & \text{if } a < b, \\ \dfrac{\sqrt{a^2 - b^2}}{\cosh^{-1}(a/b)} & \text{if } a > b. \end{cases} \quad (1)$$

It is well known that the mean $\mathrm{SB}(a, b)$ is strictly increasing in both $a$ and $b$, nonsymmetric, and homogeneous of degree 1 in its variables. Several symmetric bivariate means are special cases of the Schwab-Borchardt mean; for example,

$$\begin{aligned} P(a, b) &= \frac{a - b}{2\sin^{-1}[(a - b)/(a + b)]} \\ &= \mathrm{SB}(G, A) \quad \text{is the first Seiffert mean,} \\ T(a, b) &= \frac{a - b}{2\tan^{-1}[(a - b)/(a + b)]} \\ &= \mathrm{SB}(A, Q) \quad \text{is the second Seiffert mean,} \\ M(a, b) &= \frac{a - b}{2\sinh^{-1}[(a - b)/(a + b)]} \\ &= \mathrm{SB}(Q, A) \quad \text{is the Neuman-Sándor mean,} \\ L(a, b) &= \frac{a - b}{2\tanh^{-1}[(a - b)/(a + b)]} \\ &= \mathrm{SB}(A, G) \quad \text{is the logarithmic mean,} \end{aligned} \quad (2)$$

where $G(a, b) = \sqrt{ab}$, $A(a, b) = (a + b)/2$ and $Q(a, b) = \sqrt{(a^2 + b^2)/2}$ denote the classical geometric mean, arithmetic mean, and quadratic mean, respectively.

The Schwab-Borchardt mean $\mathrm{SB}(a, b)$ was firstly investigated in [1–4]. In [3], the authors pointed out that the logarithmic mean, two Seiffert means, and the Neuman-Sándor mean are particular cases of the Schwab-Borchardt mean.

Later, SB and its special cases have been the subject of intensive research. In particular, many inequalities for them can be found in the literature [3–13].

Let $H(a,b) = 2ab/(a+b)$, $C(a,b) = (a^2+b^2)/(a+b)$ be the harmonic and contraharmonic means of two positive numbers $a$ and $b$, respectively. Then, it is well known that

$$H < G < L < P < A < M < T < Q < C. \quad (3)$$

for $a, b > 0$ with $a \neq b$.

Recently, the second author of this paper reviewed two elegant papers [14, 15] by Neuman and found that the bivariate means $S_{AH}$, $S_{HA}$, $S_{CA}$, and $S_{AC}$, derived from the Schwab-Borchardt mean are very interesting. They are defined as follows:

$$S_{AH} = \mathrm{SB}(A,H), \qquad S_{HA} = \mathrm{SB}(H,A),$$
$$S_{CA} = \mathrm{SB}(C,A), \qquad S_{AC} = \mathrm{SB}(A,C). \quad (4)$$

We call the means $S_{AH}$, $S_{HA}$, $S_{CA}$, and $S_{AC}$, defined in (4) the Neuman means. Moreover, if we let $v = (a-b)/(a+b) \in (-1,1)$, then explicit formulas for $S_{AH}$, $S_{HA}$, $S_{AC}$, and $S_{CA}$ are in the following:

$$S_{AH} = A\frac{\tanh(p)}{p}, \qquad S_{HA} = A\frac{\sin(q)}{q}, \quad (5)$$

$$S_{CA} = A\frac{\sinh(r)}{r}, \qquad S_{AC} = A\frac{\tan(s)}{s}, \quad (6)$$

where $p$, $q$, $r$, and $s$ are defined implicitly as $\operatorname{sech}(p) = 1 - v^2$, $\cos(q) = 1 - v^2$, $\cosh(r) = 1 + v^2$ and $\sec(s) = 1 + v^2$, respectively. Clearly, $p \in (0,\infty)$, $q \in (0,\pi/2)$, $r \in (0, \log(2+\sqrt{3}))$, and $s \in (0,\pi/3)$.

Neuman [14, 15] presented several optimal bounds for $S_{HA}$, $S_{AH}$, $S_{CA}$, and $S_{AC}$. The bounding quantities are arithmetic convex, geometric convex, and harmonic convex combinations of their generating means. Besides, he also proved that

$$H < S_{AH} < L < S_{HA} < P,$$
$$T < S_{CA} < Q < S_{AC} < C, \quad (7)$$

for $a, b > 0$ with $a \neq b$.

For fixed $a, b > 0$ with $a \neq b$, $x \in [0,1/2]$ and $y \in [1/2,1]$. Let

$$f(x) = H(xa + (1-x)b, xb + (1-x)a),$$
$$g(y) = C(ya + (1-y)b, yb + (1-y)a). \quad (8)$$

Then, it is not difficult to verify that $f(x)$ and $g(y)$ are continuous and strictly increasing on $[0,1/2]$ and $[1/2,1]$, respectively. Note that $f(0) = H < S_{AH} < S_{HA} < A = f(1/2)$, $g(1/2) = A < S_{CA} < S_{AC} < C = g(1)$. Therefore, it is natural to ask what are the greatest values of $\alpha_1$ and $\alpha_2$ and the least values of $\beta_1$ and $\beta_2$ in $[0,1/2]$ such that $H(\alpha_1 a + (1-\alpha_1)b, \alpha_1 b + (1-\alpha_1)a) < S_{AH}(a,b) < H(\beta_1 a + (1-\beta_1)b, \beta_1 b + (1-\beta_1)a)$ and $H(\alpha_2 a + (1-\alpha_2)b, \alpha_2 b + (1-\alpha_2)a) < S_{HA}(a,b) < H(\beta_2 a + (1-\beta_2)b, \beta_2 b + (1-\beta_2)a)$? And what are the greatest values of $\alpha_3$ and $\alpha_4$ and the least values of $\beta_3$ and $\beta_4$ in $[1/2,1]$ such that $C(\alpha_3 a + (1-\alpha_3)b, \alpha_3 b + (1-\alpha_3)a) < S_{CA}(a,b) < C(\beta_3 a + (1-\beta_3)b, \beta_3 b + (1-\beta_3)a)$ and $C(\alpha_4 a + (1-\alpha_4)b, \alpha_4 b + (1-\alpha_4)a) < S_{AC}(a,b) < C(\beta_4 a + (1-\beta_4)b, \beta_4 b + (1-\beta_4)a)$? The main purpose of this paper is to answer these questions. Our main results are in Theorems 1 and 2.

**Theorem 1.** *Let $\alpha_1, \alpha_2, \beta_1, \beta_2 \in [0,1/2]$. Then, the double inequality*

$$H(\alpha_1 a + (1-\alpha_1)b, \alpha_1 b + (1-\alpha_1)a) < S_{AH} < H(\beta_1 a + (1-\beta_1)b, \beta_1 b + (1-\beta_1)a) \quad (9)$$

*holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_1 = 0$ and $\beta_1 \geq [3-\sqrt{6}]/6$. Also the double inequality*

$$H(\alpha_2 a + (1-\alpha_2)b, \alpha_2 b + (1-\alpha_2)a) < S_{HA} < H(\beta_2 a + (1-\beta_2)b, \beta_2 b + (1-\beta_2)a) \quad (10)$$

*holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_2 \leq [1-\sqrt{1-2/\pi}]/2$ and $\beta_2 \geq [3-\sqrt{3}]/6$.*

**Theorem 2.** *Let $\alpha_3, \alpha_4, \beta_3, \beta_4 \in [1/2,1]$. Then, the double inequality*

$$C(\alpha_3 a + (1-\alpha_3)b, \alpha_3 b + (1-\alpha_3)a) < S_{CA} < C(\beta_3 a + (1-\beta_3)b, \beta_3 b + (1-\beta_3)a) \quad (11)$$

*holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_3 \leq [1+\sqrt{\sqrt{3}/\log(2+\sqrt{3})-1}]/2$ and $\beta_3 \geq (3+\sqrt{3})/6$. Also the double inequality*

$$C(\alpha_4 a + (1-\alpha_4)b, \alpha_4 b + (1-\alpha_4)a) < S_{AC} < C(\beta_4 a + (1-\beta_4)b, \beta_4 b + (1-\beta_4)a) \quad (12)$$

*holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_4 \leq [1+\sqrt{3\sqrt{3}/\pi-1}]/2$ and $\beta_4 \geq (3+\sqrt{6})/6$.*

## 2. Two Lemmas

In order to prove the desired theorems, we first give two lemmas.

**Lemma 1** (see [16, Theorem 1.25]). *For $-\infty < a < b < \infty$, let $f, g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$, and be differentiable on $(a,b)$, let $g'(x) \neq 0$ on $(a,b)$. If $f'(x)/g'(x)$ is increasing (decreasing) on $(a,b)$, then so are*

$$\frac{f(x)-f(a)}{g(x)-g(a)}, \qquad \frac{f(x)-f(b)}{g(x)-g(b)}. \quad (13)$$

*If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.*

**Lemma 2.** *(1) The function $\varphi(x) = (x\cosh(x) - \sinh(x))/[x(\cosh(x)-1)]$ is strictly increasing from $(0,\infty)$ onto $(2/3,1)$.*

*(2) The function* $\phi(x) = (x - \sin(x))/[x(1-\cos(x))]$ *is strictly increasing from* $(0, \pi/2)$ *onto* $(1/3, (\pi-2)/\pi)$.

*(3) The function* $\xi(x) = (\sinh(x) - x)/[x(\cosh(x)-1)]$ *is strictly decreasing from* $(0, \log(2+\sqrt{3}))$ *onto* $([\sqrt{3} - \log(2+\sqrt{3})]/\log(2+\sqrt{3}), 1/3)$.

*(4) The function* $\eta(x) = (\sin(x) - x\cos(x))/[x(1-\cos(x))]$ *is strictly decreasing from* $(0, \pi/3)$ *onto* $((3\sqrt{3}-\pi)/\pi, 2/3)$.

*Proof.* From part (1), let $\varphi_1(x) = x\cosh(x) - \sinh(x)$ and $\varphi_2(x) = x(\cosh(x)-1)$. Then, $\varphi(x) = \varphi_1(x)/\varphi_2(x)$, $\varphi_1(0) = \varphi_2(0) = 0$, and

$$\frac{\varphi_1'(x)}{\varphi_2'(x)} = \frac{x\sinh(x)}{\cosh(x) - 1 + x\sinh(x)} = \frac{1}{1 + (\cosh(x)-1)/(x\sinh(x))} = \frac{1}{1 + (1/2)\tanh(x/2)/(x/2)}. \tag{14}$$

It is well known that $x \to \tanh(x)/x$ is strictly decreasing on $(0, \infty)$. Then, Lemma 1 and (14) lead to the conclusion that $\varphi(x)$ is strictly increasing on $(0, \infty)$. Moreover, by l'Hôptial's rule we have $\varphi(0^+) = 2/3$ and $\lim_{x\to+\infty}\varphi(x) = 1$.

From part (2), similarly let $\phi_1(x) = x - \sin(x)$ and $\phi_2(x) = x(1-\cos(x))$. Then $\phi(x) = \phi_1(x)/\phi_2(x)$, $\phi_1(0) = \phi_2(0) = 0$ and

$$\frac{\phi_1'(x)}{\phi_2'(x)} = \frac{1-\cos(x)}{1-\cos(x) + x\sin(x)} = \frac{1}{1 + x\sin(x)/(1-\cos(x))} = \frac{1}{1 + 2(x/2)/\tan(x/2)}. \tag{15}$$

It is well known that $x \to \tan(x)/x$ is strictly increasing on $(0, \pi/2)$. Then, by Lemma 1 and (15) we know that $\phi(x)$ is strictly increasing on $(0, \pi/2)$. Clearly, $\phi(\pi/2) = (\pi-2)/\pi$, while by l'Hôptial's rule we have $\phi(0^+) = 1/3$.

Parts (3) and (4) have been proven in [14, Theorem 3]. □

## 3. Proofs of Theorems 1 and 2

*Proof of Theorem 1.* Without loss of generality, we assume that $a > b$. Let $v = (a-b)/(a+b) \in (0,1)$ and $\lambda \in [0, 1/2]$; then,

$$\begin{aligned} &H(\lambda a + (1-\lambda)b, \lambda b + (1-\lambda)a) - S_{AH} \\ &= A\left[1 - (1-2\lambda)^2 v^2\right] - A\frac{\tanh(p)}{p} \\ &= A\left[1 - (1-2\lambda)^2(1-\operatorname{sech}(p)) - \frac{\tanh(p)}{p}\right] \\ &= A(1-\operatorname{sech}(p))\left[\frac{p\cosh(p) - \sinh(p)}{p(\cosh(p)-1)} - (1-2\lambda)^2\right] \end{aligned} \tag{16}$$

provided that $\operatorname{sech}(p) = 1 - v^2 (p > 0)$. Thus, inequality (9) follows from (16) and Lemma 2(1). Similarly,

$$\begin{aligned} &H(\lambda a + (1-\lambda)b, \lambda b + (1-\lambda)a) - S_{HA} \\ &= A\left[1 - (1-2\lambda)^2 v^2\right] - A\frac{\sin(q)}{q} \\ &= A\left[1 - (1-2\lambda)^2(1-\cos(q)) - \frac{\sin(q)}{q}\right] \\ &= A(1-\cos(q))\left[\frac{q - \sin(q)}{q(1-\cos(q))} - (1-2\lambda)^2\right] \end{aligned} \tag{17}$$

provided that $\cos(q) = 1 - v^2$ $(q \in (0, \pi/2))$. Thus, inequality (10) follows from (17) and Lemma 2(2). □

*Proof of Theorem 2.* Without loss of generality, we assume that $a > b$. Let $v = (a-b)/(a+b) \in (0,1)$ and $\mu \in [1/2, 1]$, then

$$\begin{aligned} &C(\mu a + (1-\mu)b, \mu b + (1-\mu)a) - S_{CA} \\ &= A\left[1 + (1-2\mu)^2 v^2\right] - A\frac{\sinh(r)}{r} \\ &= A\left[1 + (1-2\mu)^2(\cosh(r)-1) - \frac{\sinh(r)}{r}\right] \\ &= A(\cosh(r)-1)\left[(1-2\mu)^2 - \frac{\sinh(r) - r}{r(\cosh(r)-1)}\right] \end{aligned} \tag{18}$$

provided that $\cosh(r) = 1 + v^2$ $(r \in (0, \cosh^{-1}(2)))$. Thus, inequality (11) follows from (18) and Lemma 2(3). Similarly,

$$\begin{aligned} &C(\mu a + (1-\mu)b, \mu b + (1-\mu)a) - S_{AC} \\ &= A\left[1 + (1-2\mu)^2 v^2\right] - A\frac{\tan(s)}{s} \\ &= A\left[1 + (1-2\mu)^2(\sec(s)-1) - \frac{\tan(s)}{s}\right] \\ &= A(\sec(s)-1)\left[(1-2\mu)^2 - \frac{\sin(s) - s\cos(s)}{s(1-\cos(s))}\right] \end{aligned} \tag{19}$$

provided that $\sec(s) = 1 + v^2$ $(s \in (0, \pi/3))$. Thus, inequality (12) follows from (19) and Lemma 2(4). □

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