

Research Article

Exact Solutions and Conservation Laws of a (2+1)-Dimensional Nonlinear KP-BBM Equation

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In this paper, we study the two-dimensional nonlinear Kadomtsov-Petviashvilli-Benjamin-Bona-Mahony (KP-BBM) equation. This equation is the Benjamin-Bona-Mahony equation formulated in the KP sense. We first obtain exact solutions of this equation using the Lie group analysis and the simplest equation method. The solutions obtained are solitary waves. In addition, the conservation laws for the KP-BBM equation are constructed by using the multiplier method.

1. Introduction

It is well known that the nonlinear evolution equations (NLEEs) are extensively used as models to describe physical phenomena in various fields of sciences, especially in fluid mechanics, solid state physics, plasma physics, plasma waves, and biology. One of the basic physical problems for such models is to obtain their exact solutions. In this respect, during the past few decades, various methods have been developed by researchers to find explicit solutions for the NLEEs. See, for example, Wang et al. [1] and references therein.

The purpose of this paper is to study one such NLEE, namely, the (2+1)-dimensional nonlinear Kadomtsov-Petviashvilli-Benjamin-Bona-Mahony (KP-BBM) equation that is, given by

$$(u_t + u_x - \alpha(u^2)_x - \beta u_{xxt})_x + \gamma u_{yy} = 0. \quad (1)$$

Here, in (1), α , β , and γ are real valued constants.

The solutions of (1) have been studied in various aspects. See, for example, the recent papers [2–5]. Wazwaz [2, 3] used the sine-cosine method, the tanh method and the extended tanh method for finding solitary solutions of this equation. Abdou [4] used the extended mapping method with symbolic computation to obtain some periodic solutions, solitary wave solution, and triangular wave solution.

In this paper, Lie group analysis [6–9] in conjunction with the simplest equation method [10] is employed to obtain some exact solutions of (1). In addition to this conservation laws will be derived for (1) using the multiplier method [11].

2. Symmetry Analysis

In this section, we first calculate the Lie point symmetries of (1) and later use them to construct exact solutions.

2.1. Lie Point Symmetries. A Lie point symmetry of a partial differential equation (PDE) is an invertible transformation of the dependent and independent variables that leaves the equation unchanged. In general, determining all the symmetries of a partial differential equation is a formidable task. However, Sophus Lie observed that if we restrict ourselves to symmetries that depend continuously on a small parameter and that form a group (continuous one-parameter group of transformations), one can linearize the symmetry conditions and end up with an algorithm for calculating continuous symmetries.

The symmetry group of KP-BBM equation (1) will be generated by the vector field of the form

$$V = \xi^1 \frac{\partial}{\partial x} + \xi^2 \frac{\partial}{\partial y} + \xi^3 \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial u}, \quad (2)$$

where ξ^i , $i = 1, 2, 3$ and η depend on x, y, t , and u . Applying the fourth prolongation $\text{pr}^{(4)}V$ to (1), we obtain an overdetermined system of linear partial differential equations. Solving resultant system of linear overdetermined partial differential equations one obtains the following four Lie point symmetries:

$$\begin{aligned} V_1 &= \frac{\partial}{\partial x}, \\ V_2 &= \frac{\partial}{\partial y}, \\ V_3 &= \frac{\partial}{\partial t}, \\ V_4 &= -\alpha y \frac{\partial}{\partial y} - 2\alpha t \frac{\partial}{\partial t} + (2\alpha u - 1) \frac{\partial}{\partial u}. \end{aligned} \tag{3}$$

2.2. Exact Solutions. One of the main reasons for finding symmetries of a differential equation is to use them for finding exact solutions. In this subsection, we will utilize the symmetries calculated in the previous subsection to deduce exact solutions of (1).

One way to obtain exact solutions of (1) is by reducing it to ordinary differential equations. This can be achieved with the use of Lie point symmetries admitted by (1). It is well known that the reduction of a partial differential equation with respect to r -dimensional (solvable) subalgebra of its Lie symmetry algebra leads to reducing the number of independent variables by r .

First of all, we utilize the symmetry $V = V_1 + V_2 + V_3$ and reduce the KP-BBM equation (1) to a PDE in two independent variables. It can be seen that the symmetry V yields the following three invariants:

$$f = t - y, \quad g = x - y, \quad \theta = u. \tag{4}$$

Now treating θ as the new dependent variable and f and g as new independent variables, the KP-BBM equation (1) transforms to

$$\begin{aligned} \theta_{fg} + \theta_{gg} - 2\alpha\theta_g^2 - 2\alpha\theta\theta_{gg} - \beta\theta_{fgg} \\ + \gamma\theta_{ff} + 2\gamma\theta_{fg} + \gamma\theta_{gg} = 0, \end{aligned} \tag{5}$$

which is a nonlinear PDE in two independent variables. We now use the Lie point symmetries of (5) and transform it to an ordinary differential equation (ODE). Equation (5) has the following two translational symmetries:

$$\begin{aligned} \Gamma_1 &= \frac{\partial}{\partial g}, \\ \Gamma_2 &= \frac{\partial}{\partial f}. \end{aligned} \tag{6}$$

The combination $\rho\Gamma_1 + \Gamma_2$, where $\rho \neq 0$ is a constant, of the two symmetries Γ_1 and Γ_2 yields the following two invariants:

$$r = \frac{\rho f - g}{\rho}, \quad \psi = \theta. \tag{7}$$

This gives rise to a group invariant solution $\psi = \psi(r)$ and consequently using these invariants, (5) is transformed into the fourth-order nonlinear ODE

$$\begin{aligned} -2\alpha\rho\psi'^2 - 2\alpha\rho\psi\psi'' \\ + (\rho + \gamma\rho - \rho^2 - 2\gamma\rho^2 + \gamma\rho^3)\psi'' + \beta\psi'''' = 0. \end{aligned} \tag{8}$$

Integrating the above equation twice and taking the constants of integration to be zero, we obtain a second-order ODE

$$\rho(-1 - \gamma + \gamma\rho)(1 - \rho)\psi - \alpha\rho\psi^2 + \beta\psi'' = 0. \tag{9}$$

Multiplying (9) by ψ' , integrating once and taking the constant of integration to be zero, we obtain the first-order ODE

$$\frac{1}{2}\rho(-1 - \gamma + \gamma\rho)(1 - \rho)\psi^2 - \frac{1}{3}\alpha\rho\psi^3 + \frac{1}{2}\beta\psi'^2 = 0. \tag{10}$$

One can integrate the above equation by separating the variables. After integrating and reverting back to the original variables, we obtain the following group-invariant solutions of the KP-BBM equation (1):

$$u(x, y, t) = A_1 \text{sech}^2 \left[\frac{1}{2} (\pm A_2 r - A_3 C) \right], \tag{11}$$

where

$$\begin{aligned} A_1 &= \frac{1}{2\alpha^3} (-1 + \rho)(-1 - \gamma + \gamma\rho), \\ A_2 &= \frac{\sqrt{\rho(1 - \rho)(-1 - \gamma + \gamma\rho)}}{\sqrt{\beta}}, \\ A_3 &= \sqrt{3(-1 + \rho)(-1 - \gamma + \gamma\rho)}, \\ r &= \frac{t\rho - x + (1 - \rho)y}{\rho}. \end{aligned} \tag{12}$$

To find another group-invariant solution of the KP-BBM equation (1), we now make use of the symmetry $\Gamma = a_1V_1 + a_2V_2 + a_3V_3 + a_4V_4$ ($a_i, i = 1, \dots, 4$ are constants). The symmetry Γ yields the following three invariants:

$$\begin{aligned} f &= \frac{2a_3\alpha t - a_4}{2a_3\alpha(a_2 - a_3\alpha y)^2}, \quad g = x + \frac{a_1 \ln(a_2 - a_3\alpha y)}{a_3\alpha}, \\ \theta &= a_2^2 u - 2a_2 a_3 \alpha \gamma u + a_3^2 \alpha^2 \gamma^2 u + a_2 a_3 \gamma - \frac{1}{2} a_3^2 \alpha \gamma^2. \end{aligned} \tag{13}$$

Treating θ as the new dependent variable and f and g as new independent variables, the KP-BBM equation (1) transforms to a nonlinear fourth-order PDE in two independent variables, namely,

$$\begin{aligned} a_2^2\theta_{gg} - 3a_2^2 a_3^2 \alpha \gamma + 14a_3^2 \alpha^2 \gamma f \theta_f + a_1^2 \gamma \theta_{gg} \\ - 4a_1 a_3 \alpha \gamma f \theta_{fg} - 2\alpha\theta_{gg} \theta_{fg} \\ - 2\alpha\theta_g^2 + 6a_3^2 \alpha^2 \gamma \theta + 4a_3^2 \alpha^2 \gamma f^2 \theta_{ff} \\ - 5a_1 a_3 \alpha \gamma \theta_g - \beta\theta_{fgg} = 0. \end{aligned} \tag{14}$$

Equation (14) has a single Lie point symmetry, namely,

$$\Gamma_1 = \frac{\partial}{\partial g}, \tag{15}$$

and this symmetry yields the two invariants

$$r = f, \quad \psi = \theta, \tag{16}$$

which gives a group-invariant solution $\psi = \psi(r)$ and consequently (14) is then transformed to

$$4\alpha r^2 \psi'' + 14\alpha r \psi' + 6\alpha \psi - 3a_2^2 = 0, \tag{17}$$

which is a second-order linear ODE. Now, solving this equation and reverting back to the original variables, we obtain the following solutions of the KP-BBM equation (1):

$$u(x, y, t) = A_1 (A_2 + C_1 r^{-3/2} + C_2 r^{-1} + A_3), \tag{18}$$

where C_1 and C_2 are constants of integration and

$$\begin{aligned} A_1 &= \frac{1}{a_2^2 - 2a_2 a_3 \alpha y + a_3^2 \alpha^2 y^2}, \\ A_2 &= \frac{1}{2} a_3^2 \alpha y^2 - a_2 a_3 y, \\ A_3 &= \frac{a_2^2}{2\alpha}, \\ r &= \frac{2a_3 \alpha t - a_4}{2a_3 \alpha (a_2 - a_3 \alpha y)^2}. \end{aligned} \tag{19}$$

3. Simplest Equation Method

We now use the simplest equation method, which was introduced by Kudryashov [10] and modified by Vitanov [12], to solve the fourth-order ODE (8). The simplest equations that will be used are the Bernoulli and Riccati equations. Their solutions can be written in elementary functions [13].

Let us consider the solution of (8) in the form

$$\psi(r) = \sum_{i=0}^M \mathcal{A}_i (H(r))^i, \tag{20}$$

where $H(r)$ satisfies the Bernoulli and Riccati equations, M is a positive integer that can be determined by balancing procedure as in [12], and A_0, \dots, A_M are parameters to be determined.

3.1. Solutions of (8) Using the Equation of Bernoulli as the Simplest Equation. The balancing procedure [12] yields $M = 2$ so the solutions of (20) are of the form

$$\psi(r) = A_0 + A_1 H + A_2 H^2. \tag{21}$$

Substituting (21) into (8) and making use of the Bernoulli equation [13] and then equating all coefficients of the functions H^i to zero, we obtain an algebraic system of equations in terms of A_0, A_1 , and A_2 .

Solving the system of algebraic equations, we obtain

$$\begin{aligned} A_0 &= \frac{a^2 \beta + \rho + \gamma \rho - \rho^2 - 2\gamma \rho^2 + \gamma \rho^3}{2\alpha \rho}, \\ A_1 &= \frac{6ab\beta}{\alpha}, \\ A_2 &= \frac{6b^2\beta}{\alpha \rho}. \end{aligned} \tag{22}$$

Therefore, the solution of (1) is given by

$$\begin{aligned} u(x, y, t) &= A_0 + A_1 \left(\frac{\cosh[a(r+C)] + \sinh[a(r+C)]}{1 - b \cosh[a(r+C)] - b \sinh[a(r+C)]} \right) \\ &\quad + A_2 \left(\frac{\cosh[a(r+C)] + \sinh[a(r+C)]}{1 - b \cosh[a(r+C)] - b \sinh[a(r+C)]} \right)^2, \end{aligned} \tag{23}$$

where $r = t\rho - x + (1 - \rho)y/\rho$, and C is a constant of integration.

3.2. Solutions of (8) Using Riccati Equation as the Simplest Equation. The balancing procedure yields $M = 2$, so the solutions of (20) are of the form

$$\psi(r) = A_0 + A_1 H + A_2 H^2. \tag{24}$$

Substituting (24) into (8) and making use of the Riccati equation [13], we obtain algebraic equations in terms of A_0, A_1 , and A_2 by equating all coefficients of the functions H^i to zero.

Solving the algebraic equations, one obtains

$$\begin{aligned} A_0 &= \frac{b^2 \beta + \rho + \gamma \rho - \rho^2 - 2\gamma \rho^2 + \gamma \rho^3}{2\alpha \rho}, \\ A_1 &= 0, \\ A_2 &= \frac{2(2a^3 \beta + 11a^2 b \beta + 2ab^2 \beta)}{\alpha \rho (2a + 3b)}, \end{aligned} \tag{25}$$

and hence the solutions of (1) are

$$\begin{aligned} u(x, y, t) &= A_0 + A_2 \left(-\frac{b}{2a} - \frac{\theta}{2a} \tanh \left[\frac{1}{2} \theta (r+C) \right] \right)^2, \\ u(x, y, t) &= A_0 + A_2 \left(-\frac{b}{2a} - \frac{\theta}{2a} \tanh \left(\frac{1}{2} \theta r \right) \right. \\ &\quad \left. + \frac{\operatorname{sech}(\theta r/2)}{C \cosh(\theta r/2) - (2a/\theta) \sinh(\theta r/2)} \right)^2, \end{aligned} \tag{26}$$

where $r = t\rho - x + (1 - \rho)y/\rho$, and C is a constant of integration.

4. Conservation Laws

In this section, we construct conservation laws for (1). The multiplier method will be used [11]. See also [14].

Consider a k th-order system of PDEs of n independent variables $x = (x^1, x^2, \dots, x^n)$ and m dependent variables $u = (u^1, u^2, \dots, u^m)$, namely,

$$E_\alpha(x, u, u_{(1)}, \dots, u_{(k)}) = 0, \quad \alpha = 1, \dots, m, \quad (27)$$

where $u_{(1)}, u_{(2)}, \dots, u_{(k)}$ denote the collections of all first, second, ..., k th-order partial derivatives, that is, $u_i^\alpha = D_i(u^\alpha)$, $u_{ij}^\alpha = D_j D_i(u^\alpha), \dots$, respectively, with the *total derivative operator* with respect to x^i is given by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \dots, \quad i = 1, \dots, n, \quad (28)$$

where the summation convention is used whenever appropriate [9].

The following are known (see, e.g., [9] and the references therein).

The *Euler-Lagrange operator*, for each α , is given by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} (-1)^s D_{i_1} \dots D_{i_s} \frac{\partial}{\partial u_{i_1 i_2 \dots i_s}^\alpha}, \quad \alpha = 1, \dots, m. \quad (29)$$

The n -tuple vector $T = (T^1, T^2, \dots, T^n)$, $T^j \in \mathcal{A}$, $j = 1, \dots, n$, is a *conserved vector* of (27) if T^i satisfies

$$D_i T^i|_{(27)} = 0. \quad (30)$$

Equation (30) defines a *local conservation law* of system (27).

A multiplier $\Lambda_\alpha(x, u, u_{(1)}, \dots)$ has the property that

$$\Lambda_\alpha E_\alpha = D_i T^i \quad (31)$$

holds identically. Here, we will consider multipliers of the second order, that is, $\Lambda_\alpha = \Lambda_\alpha(t, x, u, v, w, u_x, v_x, w_x, u_{xx}, v_{xx}, w_{xx})$. The right hand side of (31) is a divergence expression. The determining equation for the multiplier Λ_α is

$$\frac{\delta(\Lambda_\alpha E_\alpha)}{\delta u^\alpha} = 0. \quad (32)$$

Once the multipliers are obtained, the conserved vectors are calculated via a homotopy formula [11].

4.1. Construction of Conservation Laws for KP-BBM Equation. We now construct conservation laws for the (2+1)-dimensional nonlinear KP-BBM equation (1). The zeroth-order multiplier for (1) is, $\Lambda(t, x, y, u)$, that is, given by

$$\Lambda = C_1 x + C_1 t x - \frac{C_1 y^2}{2\gamma} + y F_2(t) + F_3(t), \quad (33)$$

where C_1 is a constant, and $F_2(t)$ and $F_3(t)$ are arbitrary functions of t . Corresponding to the above multiplier, we have the following conserved vectors of (33):

$$\begin{aligned} T_1^t &= \frac{1}{8\gamma} \left\{ -4\gamma t u - 4\gamma u + 2\beta\gamma t u_{xx} - 2\beta\gamma t x u_{xxx} + 4\gamma t x u_x \right. \\ &\quad \left. + 2\beta\gamma u_{xx} - 2\beta\gamma x u_{xxx} \right. \\ &\quad \left. + 4\gamma x u_x + \beta y^2 u_{xxx} - 2y^2 u_x \right\}, \\ T_1^x &= \frac{1}{8\gamma} \left\{ 8\alpha y^2 u_x u - 16\alpha\gamma t x u_x u - 16\alpha\gamma x u_x u \right. \\ &\quad \left. + 8\alpha\gamma t u^2 + 8\alpha\gamma u^2 - 8\gamma t u - 4\gamma x u - 8\gamma u \right. \\ &\quad \left. + 4\beta\gamma t u_{tx} + 4\beta\gamma u_{tx} - 6\beta\gamma t x u_{txx} \right. \\ &\quad \left. - 6\beta\gamma x u_{txx} + 8\gamma t x u_x + 4\gamma t x u_t + 4\gamma x u_t \right. \\ &\quad \left. + 3\beta y^2 u_{txx} - 2y^2 u_t - 4\beta\gamma u_x \right. \\ &\quad \left. + 2\beta\gamma x u_{xx} + 8\gamma x u_x - 4y^2 u_x \right\}, \\ T_1^y &= \frac{1}{2} \left\{ 2\gamma u + 2\gamma t x u_y + 2\gamma x u_y - y^2 u_y \right\}; \\ T_2^t &= \frac{1}{2} \left\{ 2y F_2(t) u_x - \beta y F_2(t) u_{xxx} \right\}, \\ T_2^x &= \frac{1}{4} \left\{ -8\alpha y F_2(t) u_x u - 2\gamma F_2'(t) u - 3\beta y F_2(t) u_{txx} \right. \\ &\quad \left. + 4y F_2(t) u_x + 2y F_2(t) u_t + \beta y F_2'(t) u_{xx} \right\}, \\ T_2^y &= \gamma y F_2(t) u_y - \gamma F_2(t) u; \\ T_3^t &= \frac{1}{4} \left\{ 2F_3(t) u_x - \beta F_3(t) u_{xxx} \right\}, \\ T_3^x &= \frac{1}{4} \left\{ -8\alpha F_3(t) u_x u - 2F_3'(t) u - 3\beta F_3(t) u_{txx} + 4F_3(t) u_x \right. \\ &\quad \left. + 2F_3(t) u_t + \beta F_3'(t) u_{xx} \right\}, \\ T_3^y &= \gamma F_3(t) u_y. \end{aligned} \quad (34)$$

Remark 1. Due to the presence of the arbitrary function in the multiplier, one can obtain an infinitely many conservation laws for the (2+1)-dimensional nonlinear KP-BBM equation.

5. Concluding Remarks

In this paper, we obtained the solutions of the two-dimensional nonlinear Kadomtsov-Petviashvilli-Benjamin-Bona-Mahony (KP-BBM) equation by employing the Lie group analysis and the simplest equation method. The solutions obtained are solitary waves and nontopological solutions. The conservation laws for the underlying equation were also derived by using the multiplier method.

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