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Research Article

Certain Sequence Spaces over the Non-Newtonian Complex Field

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It is known from functional analysis that in classical calculus, the sets ω, ℓ_∞, c, c_0 and ℓ_p of all bounded, convergent, null and p-absolutely summable sequences are Banach spaces with their natural norms and they are complete according to the metric reduced from their norm, where $0 < p < \infty$. In this study, our main goal is to construct the spaces ω^*, ℓ_∞^*, c^*, c_0^* and ℓ_p^* over the non-Newtonian complex field $\mathbb{C}^*$ and to obtain the corresponding results for these spaces, where $\ddot{0} \ddot{<} p \ddot{<} \infty$.

1. Preliminaries, Background, and Notations

A *complete ordered field* is a system consisting of a set X, four binary operations $\dot{+}$, $\dot{-}$, $\dot{\times}$, $\dot{/}$ for X, and an ordering relation $\dot{<}$ for X, all of which behave with respect to the set X exactly as $+, -, \times, /, <$ behave with respect to the set $\mathbb{R}$ of real numbers. We call X the *realm* of the complete ordered field, [1, page 32]. A complete ordered field is called *arithmetic* if its realm is a subset of $\mathbb{R}$. A bijective function with domain $\mathbb{R}$ and range a subset of $\mathbb{R}$ is called a *generator*. For example, the identity function I, exponential function, and the function x^3 are generators.

Bashirov et al. [2] have recently emphasized on the multiplicative calculus and gave the results with applications corresponding to the well-known properties of derivative and integral in the classical calculus. Quite recently, Uzer [3] has extended the multiplicative calculus to the complex valued functions and gave the statements of some fundamental theorems and concepts of multiplicative complex calculus, and demonstrated some analogies between the multiplicative complex calculus and classical calculus by theoretical and numerical examples. Bashirov and Rıza [4] have studied the multiplicative differentiation for complex-valued functions and established the multiplicative Cauchy-Riemann conditions. Bashirov et al. [5] have investigated various problems from different fields which can be modelled more efficiently using multiplicative calculus, in place of Newtonian calculus.

Let α be a generator with range A. An arithmetic with range A, and operations and ordering relation defined as follows, is called *α-arithmetic*. Let $y, z \in A$. Then, we define the operations α-addition ($\dot{+}$), α-subtraction ($\dot{-}$), α-multiplication ($\dot{\times}$), α-division ($\dot{/}$), and α-ordering ($\dot{\leq}$) as follows:

$$\begin{aligned}
&\alpha\text{-addition} : y \dot{+} z = \alpha\left\{\alpha^{-1}(y) + \alpha^{-1}(z)\right\} \\
&\alpha\text{-subtraction} : y \dot{-} z = \alpha\left\{\alpha^{-1}(y) - \alpha^{-1}(z)\right\} \\
&\alpha\text{-multiplication} : y \dot{\times} z = \alpha\left\{\alpha^{-1}(y) \times \alpha^{-1}(z)\right\} \\
&\alpha\text{-division}\,(z \neq \dot{0}) : y \dot{/} z = \alpha\left\{\frac{\alpha^{-1}(y)}{\alpha^{-1}(z)}\right\} \\
&\alpha\text{-ordering} : y \dot{\leq} z \Longleftrightarrow \alpha^{-1}(y) \leq \alpha^{-1}(z).
\end{aligned} \tag{1}$$

With the above new operations, $(A, \dot{+}, \dot{-}, \dot{\times}, \dot{/}, \dot{\leq})$ is an α-arithmetic. In other words, one can easily show that $(A, \dot{+}, \dot{-}, \dot{\times}, \dot{/}, \dot{\leq})$ is a complete ordered field. As was seen, α-generator generates α-arithmetic. For example, the identity function generates classical arithmetic, and exponential function generates geometric arithmetic. Each generator generates exactly one arithmetic and each arithmetic is generated by exactly one generator. We denote *α-zero* by $\dot{0}$ and *α-one* by $\dot{1}$ which are obtained from $\alpha(0)$ and $\alpha(1)$, respectively. $\dot{0}$ and numbers obtained by successive α-addition of $\dot{1}$ to $\dot{0}$ with numbers obtained by successive α-subtraction of $\dot{1}$ from $\dot{0}$ are called *α-integers*. Thus, α-integers are given as follows:

$$\ldots, \alpha(-2), \alpha(-1), \alpha(0), \alpha(1), \alpha(2), \ldots \quad (2)$$

Thus, we have for all $n \in \mathbb{Z}$ that $\dot{n} = \alpha(n)$. Let $x \in A$. If $x \dot{>} \dot{0}$ then x is called *α-positive* and if $x \dot{<} \dot{0}$ then x is called *α-negative*. The *α-absolute value* $\dot{|}x\dot{|}$ of $x \in A$ is defined by

$$\dot{|}x\dot{|} = \begin{cases} x, & x \dot{>} \dot{0}, \\ \dot{0}, & x = \dot{0}, \\ \dot{0} \dot{-} x, & x \dot{<} \dot{0}. \end{cases} \quad (3)$$

For any elements r and s in A with $r \dot{<} s$, the set of all elements x in A such that $r \dot{\leq} x \dot{\leq} s$ is called an *α-interval*, is denoted by $\dot{[}r, s\dot{]}$, has *α-extent* of $s \dot{-} r$, and has the *α-interior* consisting of all elements x in A such that $r \dot{<} x \dot{<} s$. Let (u_n) be an infinite sequence of the elements in A. Then there is at most one element u in A such that every α-interval with u in its α-interior contains all but finitely many terms of (u_n). If there is such a number u, then (u_n) is said to be *α-convergent* to u, which is called the *α-limit* of (u_n). In other words,

$$\begin{aligned} &\lim_{n \to \infty} u_n = u \, (\alpha\text{-convergent}) \\ &\Longleftrightarrow \forall \varepsilon \dot{>} \dot{0}, \exists n_0 \in \mathbb{N} \ni \dot{|}u_n \dot{-} u\dot{|} \dot{<} \varepsilon \\ &\qquad \forall n \geq n_0 \text{ and some } u \in A. \end{aligned} \quad (4)$$

Let α and β be two arbitrarily selected generators and let $*$-("*star*") also be the ordered pair of arithmetics (α-arithmetic,β-arithmetic). $(B, \ddot{+}, \ddot{-}, \ddot{\times}, \ddot{/}, \ddot{\leq})$ is a β-arithmetic. Definitions given for α-arithmetic are also valid for β-arithmetic. For example, β-convergence is defined by means of β-intervals and their β-interiors.

In the $*$-calculus, α-arithmetic is used for arguments and β-arithmetic is used for values; in particular, changes in arguments and values are measured by α-differences and β-differences, respectively. The operators of the $*$-calculus are applied only to functions with arguments in A and values in B. The *$*$-limit* of a function f at an element a in A is, if it exists, the unique number b in B such that for every sequence (a_n) of arguments of f distinct from a, if (a_n) is α-convergent to a, then $\{f(a_n)\}$ β-converges to b and is denoted by $\lim^*_{x \to a} f(x) = b$. That is,

$$\begin{aligned} &\lim^*_{x \to a} f(x) = b \Longleftrightarrow \forall \epsilon \ddot{>} \ddot{0}, \exists \delta \dot{>} \dot{0} \ni \ddot{|}f(x) \ddot{-} b\ddot{|} \ddot{<} \epsilon \\ &\qquad \forall x \in A \text{ with } \dot{|}x \dot{-} a\dot{|} \dot{<} \delta. \end{aligned} \quad (5)$$

A function f is *$*$-continuous* at a point a in A if and only if a is an argument of f and $\lim^*_{x \to a} f(x) = f(a)$. When α and β are the identity function I, the concepts of $*$-limit and $*$-continuity are identical with those of classical limit and classical continuity.

The isomorphism from α-arithmetic to β-arithmetic is the unique function ι(*iota*) that possesses the following three properties:

(i) ι is one to one.

(ii) ι is from A onto B.

(iii) For any numbers u and v in A,

$$\begin{aligned} \iota(u \dot{+} v) &= \iota(u) \ddot{+} \iota(v) \\ \iota(u \dot{-} v) &= \iota(u) \ddot{-} \iota(v) \\ \iota(u \dot{\times} v) &= \iota(u) \ddot{\times} \iota(v) \\ \iota(u \dot{/} v) &= \iota(u) \ddot{/} \iota(v), \quad v \neq \dot{0} \\ u \dot{\leq} v &\Longleftrightarrow \iota(u) \ddot{\leq} \iota(v). \end{aligned} \quad (6)$$

It turns out that $\iota(x) = \beta\{\alpha^{-1}(x)\}$ for every x in A, and that $\iota(\dot{n}) = \ddot{n}$ for every integer n. Since, for example, $u \dot{+} v = \iota^{-1}\{\iota(u) \ddot{+} \iota(v)\}$, it should be clear that any statement in α-arithmetic can readily be transformed into a statement in β-arithmetic.

2. Non-Newtonian Complex Field and Some Inequalities

Let $\dot{a} \in (A, \dot{+}, \dot{-}, \dot{\times}, \dot{/}, \dot{\leq})$ and $\ddot{b} \in (B, \ddot{+}, \ddot{-}, \ddot{\times}, \ddot{/}, \ddot{\leq})$ be arbitrarily chosen elements from corresponding arithmetics. Then the ordered pair $(\dot{a}, \ddot{b})$ is called as a *$*$-point*. The set of all $*$-points is called the *set of non-Newtonian complex numbers* and is denoted by $\mathbb{C}^*$; that is,

$$\begin{aligned} \mathbb{C}^* &:= \left\{z^* = \dot{a} \oplus \left(i^* \odot \dot{b}\right) : a, b \in \mathbb{R}, i^* = (\dot{0}, \ddot{1})\right\} \\ &= \left\{z^* = \left(\dot{a}, \ddot{b}\right) : \dot{a} \in A \subseteq \mathbb{R}, \ddot{b} \in B \subseteq \mathbb{R}\right\}. \end{aligned} \quad (7)$$

The binary operations addition ($\oplus$) and multiplication ($\odot$) of non-Newtonian complex numbers $z_1^* = (\dot{a}_1, \ddot{b}_1)$ and $z_2^* = (\dot{a}_2, \ddot{b}_2)$ are defined, as follows:

$$\begin{aligned} &\oplus : \mathbb{C}^* \times \mathbb{C}^* \longrightarrow \mathbb{C}^* \\ &\qquad (z_1^*, z_2^*) \longmapsto z_1^* \oplus z_2^* = \left(\dot{a}_1 \dot{+} \dot{a}_2, \ddot{b}_1 \ddot{+} \ddot{b}_2\right), \\ &\odot : \mathbb{C}^* \times \mathbb{C}^* \longrightarrow \mathbb{C}^* \\ &\qquad (z_1^*, z_2^*) \longmapsto z_1^* \odot z_2^* = \left(\alpha\left(\bar{\dot{a}}_1 \bar{\dot{a}}_2 - \bar{\ddot{b}}_1 \bar{\ddot{b}}_2\right),\right. \\ &\qquad\qquad\qquad \left.\beta\left(\bar{\dot{a}}_1 \bar{\ddot{b}}_2 + \bar{\ddot{b}}_1 \bar{\dot{a}}_2\right)\right), \end{aligned} \quad (8)$$

where $\dot{a}_1, \dot{a}_2 \in A$ and $\ddot{b}_1, \ddot{b}_2 \in B$ with

$$\begin{aligned} \bar{\dot{a}}_1 &= \alpha^{-1}(\dot{a}_1) = \alpha^{-1}(\alpha(a_1)) = a_1 \in \mathbb{R}, \\ \bar{\ddot{b}}_1 &= \beta^{-1}(\ddot{b}_1) = \beta^{-1}(\beta(b_1)) = b_1 \in \mathbb{R}. \end{aligned} \quad (9)$$

Then we have the following.

Theorem 1. *$(\mathbb{C}^*, \oplus, \odot)$ is a field.*

Proof. A straightforward calculation leads to the following statements:

(i) $(\mathbb{C}^*, \oplus)$ is an Abelian group;

(ii) $(\mathbb{C}^* \setminus \{\theta^*\}, \odot)$ is an Abelian group;

(iii) the operation $\odot$ is distributive over the operation $\oplus$

which conclude that $(\mathbb{C}^*, \oplus, \odot)$ is a field. □

Let $\ddot{b} \in B \subseteq \mathbb{R}$. Then the number $\ddot{b} \ddot{\times} \ddot{b}$ is called the *β-square* of $\ddot{b}$ and is denoted by $b^{\ddot{2}}$. Let $\ddot{b}$ be a non-negative number in B. Then, $\beta[\sqrt{\beta^{-1}(\ddot{b})}]$ is called the *β-square root* of $\ddot{b}$ and is denoted by $\ddot{\sqrt{\ddot{b}}}$. The *$*$-distance* d^* between two arbitrarily elements $z_1^* = (\dot{a}_1, \ddot{b}_1)$ and $z_2^* = (\dot{a}_2, \ddot{b}_2)$ of the set $\mathbb{C}^*$ is defined by

$$d^* : \mathbb{C}^* \times \mathbb{C}^* \longrightarrow \ddot{[}\ddot{0}, \infty\ddot{)} = B' \subset B$$

$$\begin{aligned}(z_1^*, z_2^*) \longrightarrow d^*(z_1^*, z_2^*) &= \ddot{\sqrt{[\iota(\dot{a}_1 \dot{-} \dot{a}_2)]^{\ddot{2}} \ddot{+} \left(\ddot{b}_1 \ddot{-} \ddot{b}_2\right)^{\ddot{2}}}} \\ &= \beta\left[\sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2}\right]. \end{aligned} \tag{10}$$

Up to now, we know that $\mathbb{C}^*$ is a field and the distance between two points in $\mathbb{C}^*$ is computed by the function d^*, defined by (10). Now, we define the $*$-norm and next derive some required inequalities in the sense of non-Newtonian complex calculus.

Let $z^* \in \mathbb{C}^*$ be an arbitrary element. $d^*(z^*, \theta^*)$ is called *$*$-norm* of z^* and is denoted by $\ddot{\|} \cdot \ddot{\|}$. In other words,

$$\begin{aligned}\ddot{\|} z^* \ddot{\|} &= d^*(z^*, \theta^*) \\ &= \ddot{\sqrt{[\iota(\dot{a} \dot{-} \dot{0})]^{\ddot{2}} \ddot{+} \left(\ddot{b} \ddot{-} \ddot{0}\right)^{\ddot{2}}}} \\ &= \beta\left(\sqrt{a^2 + b^2}\right), \end{aligned} \tag{11}$$

where $z^* = (\dot{a}, \ddot{b})$ and $\theta^* = (\dot{0}, \ddot{0})$. Moreover, since for all $z_1^*, z_2^* \in \mathbb{C}^*$ we have $d^*(z_1^*, z_2^*) = \ddot{\|} z_1^* \ominus z_2^* \ddot{\|}$, d^* is the induced metric from the norm $\ddot{\|} \cdot \ddot{\|}$.

Lemma 2 ($*$-Triangle inequality). *Let $z_1^*, z_2^* \in \mathbb{C}^*$. Then,*

$$\ddot{\|} z_1^* \oplus z_2^* \ddot{\|} \ddot{\leq} \ddot{\|} z_1^* \ddot{\|} \ddot{+} \ddot{\|} z_2^* \ddot{\|}. \tag{12}$$

Proof. Let $z_1^*, z_2^* \in \mathbb{C}^*$. Then, a straightforward calculation gives that

$$\begin{aligned}&\ddot{\|} z_1^* \oplus z_2^* \ddot{\|} \\ &\quad = \ddot{\sqrt{[\iota(\dot{a}_1 \dot{+} \dot{a}_2)]^{\ddot{2}} \ddot{+} \left(\ddot{b}_1 \ddot{+} \ddot{b}_2\right)^{\ddot{2}}}} \\ &\quad = \beta\left[\sqrt{(a_1 + a_2)^2 + (b_1 + b_2)^2}\right] \\ &\quad \ddot{\leq} \beta\left(\sqrt{a_1^2 + b_1^2} + \sqrt{a_2^2 + b_2^2}\right) \\ &\quad = \beta\left\{\beta^{-1}\left[\beta\left(\sqrt{a_1^2 + b_1^2}\right)\right] + \beta^{-1}\left[\beta\left(\sqrt{a_2^2 + b_2^2}\right)\right]\right\} \\ &\quad = \beta\left[\beta^{-1}\left(\ddot{\|} z_1^* \ddot{\|}\right) + \beta^{-1}\left(\ddot{\|} z_2^* \ddot{\|}\right)\right] \\ &\quad = \ddot{\|} z_1^* \ddot{\|} \ddot{+} \ddot{\|} z_2^* \ddot{\|}. \end{aligned} \tag{13}$$

Hence, the inequality (12) holds. □

Lemma 3. $\ddot{\|} z_1^* \odot z_2^* \ddot{\|} = \ddot{\|} z_1^* \ddot{\|} \ddot{\times} \ddot{\|} z_2^* \ddot{\|}$ *for all $z_1^*, z_2^* \in \mathbb{C}^*$.*

Proof. Let $z_1^*, z_2^* \in \mathbb{C}^*$. In this case, one can observe by a routine verification that

$$\begin{aligned}&\ddot{\|} z_1^* \odot z_2^* \ddot{\|} \\ &\quad = \ddot{\sqrt{\left\{\iota\left[\alpha\left(\bar{\bar{a}}_1 \bar{\bar{a}}_2 - \bar{\bar{b}}_1 \bar{\bar{b}}_2\right)\right]\right\}^{\ddot{2}} \ddot{+} \left[\beta\left(\bar{\bar{a}}_1 \bar{\bar{b}}_2 + \bar{\bar{b}}_1 \bar{\bar{a}}_2\right)\right]^{\ddot{2}}}} \\ &\quad = \beta\left[\sqrt{(a_1 a_2 - b_1 b_2)^2 + (a_1 b_2 + b_1 a_2)^2}\right] \\ &\quad = \beta\left(\sqrt{a_1^2 + b_1^2}\sqrt{a_2^2 + b_2^2}\right) \\ &\quad = \beta\left\{\beta^{-1}\left[\beta\left(\sqrt{a_1^2 + b_1^2}\right)\right] \times \beta^{-1}\left[\beta\left(\sqrt{a_2^2 + b_2^2}\right)\right]\right\} \\ &\quad = \beta\left[\beta^{-1}\left(\ddot{\|} z_1^* \ddot{\|}\right) \times \beta^{-1}\left(\ddot{\|} z_2^* \ddot{\|}\right)\right] \\ &\quad = \ddot{\|} z_1^* \ddot{\|} \ddot{\times} \ddot{\|} z_2^* \ddot{\|}, \end{aligned} \tag{14}$$

as required. □

Lemma 4. *Let $z_1^*, z_2^* \in \mathbb{C}^*$. Then, the following inequality holds:*

$$\begin{aligned}&\ddot{\|} z_1^* \oplus z_2^* \ddot{\|} \ddot{/} \left(\ddot{1} \ddot{+} \ddot{\|} z_1^* \oplus z_2^* \ddot{\|}\right) \\ &\quad \ddot{\leq} \ddot{\|} z_1^* \ddot{\|} \ddot{/} \left(\ddot{1} \ddot{+} \ddot{\|} z_1^* \ddot{\|}\right) \ddot{+} \ddot{\|} z_2^* \ddot{\|} \ddot{/} \left(\ddot{1} \ddot{+} \ddot{\|} z_2^* \ddot{\|}\right). \end{aligned} \tag{15}$$

Proof. Let $z_1^*, z_2^* \in \mathbb{C}^*$. Then, one can see that

$$\begin{aligned}&\ddot{\|} z_1^* \oplus z_2^* \ddot{\|} \ddot{/} \left(\ddot{1} \ddot{+} \ddot{\|} z_1^* \oplus z_2^* \ddot{\|}\right) \\ &\quad = \beta\left[\sqrt{(a_1 + a_2)^2 + (b_1 + b_2)^2}\right] \\ &\qquad \ddot{/} \left\{\ddot{1} \ddot{+} \beta\left[\sqrt{(a_1 + a_2)^2 + (b_1 + b_2)^2}\right]\right\} \\ &\quad = \beta\left[\frac{\sqrt{(a_1 + a_2)^2 + (b_1 + b_2)^2}}{1 + \sqrt{(a_1 + a_2)^2 + (b_1 + b_2)^2}}\right] \\ &\quad \ddot{\leq} \beta\left[\left(\frac{\sqrt{a_1^2 + b_1^2}}{1 + \sqrt{a_1^2 + b_1^2}}\right) + \left(\frac{\sqrt{a_2^2 + b_2^2}}{1 + \sqrt{a_2^2 + b_2^2}}\right)\right] \end{aligned}$$

$$= \beta \left\{ \frac{\beta^{-1}\left[\beta\left(\sqrt{a_1^2+b_1^2}\right)\right]}{\beta^{-1}\left\{\beta\left[\beta^{-1}(\ddot{1})+\beta^{-1}\left[\beta\left(\sqrt{a_1^2+b_1^2}\right)\right]\right]\right\}} + \frac{\beta^{-1}\left[\beta\left(\sqrt{a_2^2+b_2^2}\right)\right]}{\beta^{-1}\left\{\beta\left\{\beta^{-1}(\ddot{1})+\beta^{-1}\left[\beta\left(\sqrt{a_2^2+b_2^2}\right)\right]\right\}\right\}} \right\}$$
$$= \beta\left\{\left[\frac{\beta^{-1}\left(\ddot{\|}z_1^*\ddot{\|}\right)}{\beta^{-1}\left(\ddot{1}\ddot{+}\ddot{\|}z_1^*\ddot{\|}\right)}\right]+\left[\frac{\beta^{-1}\left(\ddot{\|}z_2^*\ddot{\|}\right)}{\beta^{-1}\left(\ddot{1}\ddot{+}\ddot{\|}z_2^*\ddot{\|}\right)}\right]\right\}$$
$$= \beta\left\{\beta^{-1}\left\{\beta\left[\frac{\beta^{-1}\left(\ddot{\|}z_1^*\ddot{\|}\right)}{\beta^{-1}\left(\ddot{1}\ddot{+}\ddot{\|}z_1^*\ddot{\|}\right)}\right]\right\} + \beta^{-1}\left\{\beta\left[\frac{\beta^{-1}\left(\ddot{\|}z_2^*\ddot{\|}\right)}{\beta^{-1}\left(\ddot{1}\ddot{+}\ddot{\|}z_2^*\ddot{\|}\right)}\right]\right\}\right\}$$
$$= \beta\left\{\beta^{-1}\left[\ddot{\|}z_1^*\ddot{\|}\,\ddot{/}\left(\ddot{1}\ddot{+}\ddot{\|}z_1^*\ddot{\|}\right)\right]+\beta^{-1}\left[\ddot{\|}z_2^*\ddot{\|}\,\ddot{/}\left(\ddot{1}\ddot{+}\ddot{\|}z_2^*\ddot{\|}\right)\right]\right\}$$
$$= \ddot{\|}z_1^*\ddot{\|}\,\ddot{/}\left(\ddot{1}\ddot{+}\ddot{\|}z_1^*\ddot{\|}\right)\ddot{+}\ddot{\|}z_2^*\ddot{\|}\,\ddot{/}\left(\ddot{1}\ddot{+}\ddot{\|}z_2^*\ddot{\|}\right). \tag{16}$$

This means that the inequality (15) holds. □

Lemma 5 ($*$-Minkowski inequality). *Let* $p \ddot{\geq} \ddot{1}$ *and* $z_k^*, t_k^* \in \mathbb{C}^*$ *for all* $k \in \{1,2,3,\dots,n\}$. *Then,*

$$\left(\ddot{\sum_{k=0}^{n}}\ddot{\|}z_k^*\oplus t_k^*\ddot{\|}^p\right)^{\ddot{1}\ddot{/}p} \ddot{\leq} \left(\ddot{\sum_{k=0}^{n}}\ddot{\|}z_k^*\ddot{\|}^p\right)^{\ddot{1}\ddot{/}p} \ddot{+} \left(\ddot{\sum_{k=0}^{n}}\ddot{\|}t_k^*\ddot{\|}^p\right)^{\ddot{1}\ddot{/}p}. \tag{17}$$

Proof. Let $p \ddot{\geq} \ddot{1}$ and $z_k^*, t_k^* \in \mathbb{C}^*$ for all $k \in \{1,2,3,\dots,n\}$. Then,

$$\ddot{\sum_{k=0}^{n}}\ddot{\|}z_k^*\oplus t_k^*\ddot{\|}^p = \ddot{\|}z_0^*\oplus t_0^*\ddot{\|}^p \ddot{+} \ddot{\|}z_1^*\oplus t_1^*\ddot{\|}^p \ddot{+}\cdots\ddot{+} \ddot{\|}z_n^*\oplus t_n^*\ddot{\|}^p$$
$$= \beta\left[\beta^{-1}\left(\ddot{\|}z_0^*\oplus t_0^*\ddot{\|}^p\right)+\beta^{-1}\left(\ddot{\|}z_1^*\oplus t_1^*\ddot{\|}^p\right) + \cdots + \beta^{-1}\left(\ddot{\|}z_n^*\oplus t_n^*\ddot{\|}^p\right)\right]. \tag{18}$$

Let us take $z_k^* = (\dot{x}_k, \ddot{y}_k)$, $z_k = (x_k, y_k)$, $t_k^* = (\dot{u}_k, \ddot{v}_k)$, $t_k = (u_k, v_k)$. Then, since the equality

$$\ddot{\|}z_k^*\oplus t_k^*\ddot{\|} = \ddot{\sqrt{\left[\imath\left(\dot{x}_k\dot{+}\dot{u}_k\right)\right]^{\ddot{2}}\ddot{+}\left(\ddot{y}_k\ddot{+}\ddot{v}_k\right)^{\ddot{2}}}} = \beta\left[\sqrt{\left(x_k+u_k\right)^2+\left(y_k+v_k\right)^2}\right] = \beta\left(\left|z_k+t_k\right|\right) \tag{19}$$

holds for every fixed k, we obtain

$$\ddot{\|}z_k^*\oplus t_k^*\ddot{\|}^p = \underbrace{\ddot{\|}z_k^*\oplus t_k^*\ddot{\|}\,\ddot{\times}\cdots\ddot{\times}\,\ddot{\|}z_k^*\oplus t_k^*\ddot{\|}}_{p\text{-times}}$$
$$= \beta\left[\beta^{-1}\left(\ddot{\|}z_k^*\oplus t_k^*\ddot{\|}\right)\times\cdots\times\beta^{-1}\left(\ddot{\|}z_k^*\oplus t_k^*\ddot{\|}\right)\right]$$
$$= \beta\left\{\beta^{-1}\left[\beta\left(\left|z_k+t_k\right|\right)\right]\times\cdots\times\beta^{-1}\left[\beta\left(\left|z_k+t_k\right|\right)\right]\right\}$$
$$= \beta\left(\left|z_k+t_k\right|^{\beta^{-1}(p)}\right) \tag{20}$$

by (18) and Minkowski inequality in the complex field leads us to

$$\ddot{\sum_{k=0}^{n}}\ddot{\|}z_k^*\oplus t_k^*\ddot{\|}^p = \beta\left\{\beta^{-1}\left[\beta\left(\left|z_0\oplus t_0\right|^{\beta^{-1}(p)}\right)\right]+\beta^{-1}\left[\beta\left(\left|z_1\oplus t_1\right|^{\beta^{-1}(p)}\right)\right] + \cdots + \beta^{-1}\left[\beta\left(\left|z_n\oplus t_n\right|^{\beta^{-1}(p)}\right)\right]\right\}$$
$$= \beta\left(\sum_{k=0}^{n}\left|z_k+t_k\right|^{\beta^{-1}(p)}\right)$$
$$\ddot{\leq} \beta\left\{\left[\left(\sum_{k=0}^{n}\left|z_k\right|^{\beta^{-1}(p)}\right)^{1/\beta^{-1}(p)} + \left(\sum_{k=0}^{n}\left|t_k\right|^{\beta^{-1}(p)}\right)^{1/\beta^{-1}(p)}\right]^{\beta^{-1}(p)}\right\}, \tag{21}$$

$$\beta^{-1}\left\{\left[\left(\ddot{\sum_{k=0}^{n}}\ddot{\|}z_k^*\ddot{\|}^p\right)^{\ddot{1}\ddot{/}p}\ddot{+}\left(\ddot{\sum_{k=0}^{n}}\ddot{\|}t_k^*\ddot{\|}^p\right)^{\ddot{1}\ddot{/}p}\right]^p\right\}$$
$$= \beta^{-1}\left\{\left[\left(\ddot{\sum_{k=0}^{n}}\ddot{\|}z_k^*\ddot{\|}^p\right)^{\ddot{1}\ddot{/}p}\ddot{+}\left(\ddot{\sum_{k=0}^{n}}\ddot{\|}t_k^*\ddot{\|}^p\right)^{\ddot{1}\ddot{/}p}\right] \ddot{\times}\cdots\ddot{\times} \left[\left(\ddot{\sum_{k=0}^{n}}\ddot{\|}z_k^*\ddot{\|}^p\right)^{\ddot{1}\ddot{/}p}\ddot{+}\left(\ddot{\sum_{k=0}^{n}}\ddot{\|}t_k^*\ddot{\|}^p\right)^{\ddot{1}\ddot{/}p}\right]\right\}$$
$$= \beta^{-1}\left[\left(\ddot{\sum_{k=0}^{n}}\ddot{\|}z_k^*\ddot{\|}^p\right)^{\ddot{1}\ddot{/}p}\ddot{+}\left(\ddot{\sum_{k=0}^{n}}\ddot{\|}t_k^*\ddot{\|}^p\right)^{\ddot{1}\ddot{/}p}\right] \times\cdots\times \beta^{-1}\left[\left(\ddot{\sum_{k=0}^{n}}\ddot{\|}z_k^*\ddot{\|}^p\right)^{\ddot{1}\ddot{/}p}\ddot{+}\left(\ddot{\sum_{k=0}^{n}}\ddot{\|}t_k^*\ddot{\|}^p\right)^{\ddot{1}\ddot{/}p}\right]$$

$$
\begin{aligned}
&= \left\{ \beta^{-1}\left[\left(\ddot{\sum_{k=0}^{n}}\ddot{\|}z_k^*\ddot{\|}^p\right)^{1\ddot{/}p}\right] + \beta^{-1}\left[\left(\ddot{\sum_{k=0}^{n}}\ddot{\|}t_k^*\ddot{\|}^p\right)^{1\ddot{/}p}\right]\right\} \\
&\quad \times\cdots\times \left\{ \beta^{-1}\left[\left(\ddot{\sum_{k=0}^{n}}\ddot{\|}z_k^*\ddot{\|}^p\right)^{1\ddot{/}p}\right] \right. \\
&\qquad \left. + \beta^{-1}\left[\left(\ddot{\sum_{k=0}^{n}}\ddot{\|}t_k^*\ddot{\|}^p\right)^{1\ddot{/}p}\right]\right\} \\
&= \left\{ \beta^{-1}\left[\left(\ddot{\sum_{k=0}^{n}}\ddot{\|}z_k^*\ddot{\|}^p\right)^{1\ddot{/}p}\right] + \beta^{-1}\left[\left(\ddot{\sum_{k=0}^{n}}\ddot{\|}t_k^*\ddot{\|}^p\right)^{1\ddot{/}p}\right]\right\}^{\beta^{-1}(p)}.
\end{aligned}
\tag{22}
$$

On the other hand, let us prove

$$
\beta\left[\left(\sum_{k=0}^{n}|z_k|^{\beta^{-1}(p)}\right)^{1/\beta^{-1}(p)}\right] = \left(\ddot{\sum_{k=0}^{n}}\ddot{\|}z_k^*\ddot{\|}^p\right)^{1\ddot{/}p}. \tag{23}
$$

Indeed,

$$
\begin{aligned}
&\left\{\beta\left[\left(\sum_{k=0}^{n}|z_k|^{\beta^{-1}(p)}\right)^{1/\beta^{-1}(p)}\right]\right\}^{p} \\
&= \underbrace{\beta\left[\left(\sum_{k=0}^{n}|z_k|^{\beta^{-1}(p)}\right)^{1/\beta^{-1}(p)}\right]\ddot{\times}\cdots\ddot{\times}\,\beta\left[\left(\sum_{k=0}^{n}|z_k|^{\beta^{-1}(p)}\right)^{1/\beta^{-1}(p)}\right]}_{p\text{-times}} \\
&= \beta\left[\left(\sum_{k=0}^{n}|z_k|^{\beta^{-1}(p)}\right)^{1/\beta^{-1}(p)} \right. \\
&\qquad \left. \times\cdots\times\left(\sum_{k=0}^{n}|z_k|^{\beta^{-1}(p)}\right)^{1/\beta^{-1}(p)}\right] \\
&= \beta\left\{\left[\left(\sum_{k=0}^{n}|z_k|^{\beta^{-1}(p)}\right)^{1/\beta^{-1}(p)}\right]^{\beta^{-1}(p)}\right\} \\
&= \beta\left(\sum_{k=0}^{n}|z_k|^{\beta^{-1}(p)}\right) \\
&= \ddot{\sum_{k=0}^{n}}\ddot{\|}z_k^*\ddot{\|}^p.
\end{aligned}
\tag{24}
$$

Substituting the relation (24) in (22) we obtain,

$$
\begin{aligned}
&\beta^{-1}\left\{\left[\left(\ddot{\sum_{k=0}^{n}}\ddot{\|}z_k^*\ddot{\|}^p\right)^{1\ddot{/}p} \ddot{+} \left(\ddot{\sum_{k=0}^{n}}\ddot{\|}t_k^*\ddot{\|}^p\right)^{1\ddot{/}p}\right]^{p}\right\} \\
&= \left[\left(\sum_{k=0}^{n}|z_k|^{\beta^{-1}(p)}\right)^{1/\beta^{-1}(p)} + \left(\sum_{k=0}^{n}|t_k|^{\beta^{-1}(p)}\right)^{1/\beta^{-1}(p)}\right]^{\beta^{-1}(p)}.
\end{aligned}
\tag{25}
$$

By using this equality in (21), we get

$$
\begin{aligned}
&\left(\ddot{\sum_{k=0}^{n}}\ddot{\|}z_k^*\oplus t_k^*\ddot{\|}^p\right)^{1\ddot{/}p} \\
&\quad \ddot{\leq} \left(\ddot{\sum_{k=0}^{n}}\ddot{\|}z_k^*\ddot{\|}^p\right)^{1\ddot{/}p} \ddot{+} \left(\ddot{\sum_{k=0}^{n}}\ddot{\|}t_k^*\ddot{\|}^p\right)^{1\ddot{/}p},
\end{aligned}
\tag{26}
$$

as desired. □

Theorem 6. *$(\mathbb{C}^*, d^*)$ is a complete metric space, where d^* is defined by* (10).

Proof. First, we show that d^*, defined by (10), is a metric on $\mathbb{C}^*$.

It is immediate for $z_1^*, z_2^* \in \mathbb{C}^*$ that

$$
\begin{aligned}
d^*(z_1^*, z_2^*) &= \ddot{\sqrt{[\iota(\dot{a}_1 \dot{-} \dot{a}_2)]^{\ddot{2}} \ddot{+} (\ddot{b}_1 \ddot{-} \ddot{b}_2)^{\ddot{2}}}} \\
&= \beta\left[\sqrt{(a_1-a_2)^2+(b_1-b_2)^2}\right] \ddot{\geq} \ddot{0}.
\end{aligned}
\tag{27}
$$

(i) Now we show that $d^*(z_1^*, z_2^*) = \ddot{0}$ if and only if $z_1^* = z_2^*$ for $z_1^*, z_2^* \in \mathbb{C}^*$. Indeed,

$$
\begin{aligned}
d^*(z_1^*, z_2^*) = \ddot{0} &\Longleftrightarrow \ddot{\sqrt{[\iota(\dot{a}_1 \dot{-} \dot{a}_2)]^{\ddot{2}} \ddot{+} (\ddot{b}_1 \ddot{-} \ddot{b}_2)^{\ddot{2}}}} = \ddot{0} \\
&\Longleftrightarrow \beta\left[\sqrt{(a_1-a_2)^2+(b_1-b_2)^2}\right] = \beta(0) \\
&\Longleftrightarrow (a_1-a_2)^2+(b_1-b_2)^2 = 0 \\
&\Longleftrightarrow a_1-a_2=0, \qquad b_1-b_2=0 \\
&\Longleftrightarrow a_1=a_2, \qquad b_1=b_2 \\
&\Longleftrightarrow \dot{a}_1=\dot{a}_2, \qquad \ddot{b}_1=\ddot{b}_2 \\
&\Longleftrightarrow z_1^* = (\dot{a}_1, \ddot{b}_1) = (\dot{a}_2, \ddot{b}_2) = z_2^*.
\end{aligned}
\tag{28}
$$

(ii) One can easily establish for all $z_1^*, z_2^* \in \mathbb{C}^*$ that

$$\begin{aligned} d^*(z_1^*, z_2^*) &= \sqrt[\ddot{}]{[\iota(\dot{a}_1 \dot{-} \dot{a}_2)]^{\ddot{2}} \ddot{+} (\ddot{b}_1 \ddot{-} \ddot{b}_2)^{\ddot{2}}} \\ &= \beta\left[\sqrt{(a_1-a_2)^2 + (b_1-b_2)^2}\right] \\ &= \beta\left[\sqrt{(a_2-a_1)^2 + (b_2-b_1)^2}\right] \\ &= d^*(z_2^*, z_1^*). \end{aligned} \tag{29}$$

(iii) We show that the inequality $d^*(z_1^*, z_2^*) \ddot{+} d^*(z_2^*, z_3^*) \ddot{\geq} d^*(z_1^*, z_3^*)$ holds for all $z_1^*, z_2^*, z_3^* \in \mathbb{C}^*$. In fact,

$$\begin{aligned} &d^*(z_1^*, z_2^*) \ddot{+} d^*(z_2^*, z_3^*) \\ &= \sqrt[\ddot{}]{[\iota(\dot{a}_1 \dot{-} \dot{a}_2)]^{\ddot{2}} \ddot{+} (\ddot{b}_1 \ddot{-} \ddot{b}_2)^{\ddot{2}}} \ddot{+} \sqrt[\ddot{}]{[\iota(\dot{a}_2 \dot{-} \dot{a}_3)]^{\ddot{2}} \ddot{+} (\ddot{b}_2 \ddot{-} \ddot{b}_3)^{\ddot{2}}} \\ &= \beta\left[\sqrt{(a_1-a_2)^2 + (b_1-b_2)^2}\right] \\ &\quad \ddot{+} \beta\left[\sqrt{(a_2-a_3)^2 + (b_2-b_3)^2}\right] \\ &= \beta\left[\sqrt{(a_1-a_2)^2 + (b_1-b_2)^2} + \sqrt{(a_2-a_3)^2 + (b_2-b_3)^2}\right] \\ &\ddot{\geq} \beta\left[\sqrt{(a_1-a_3)^2 + (b_1-b_3)^2}\right] \\ &= \sqrt[\ddot{}]{[\iota(\dot{a}_1 \dot{-} \dot{a}_3)]^{\ddot{2}} \ddot{+} (\ddot{b}_1 \ddot{-} \ddot{b}_3)^{\ddot{2}}} \\ &= d^*(z_1^*, z_3^*). \end{aligned} \tag{30}$$

Therefore, d^* is a metric over $\mathbb{C}^*$.

Now, we can show that the metric space $(\mathbb{C}^*, d^*)$ is complete. Let $(z_n^*)_{n\in\mathbb{N}}$ be an arbitrary Cauchy sequence in $\mathbb{C}^*$. In this case, for all $\varepsilon \ddot{>} \ddot{0}$ there exists an $n_0 \in \mathbb{N}$ such that $d^*(z_m^*, z_n^*) \ddot{<} \varepsilon$ for all $m, n \geq n_0$. Let $z_m^* = (\dot{a}_m, \ddot{b}_m) \in \mathbb{C}^*$ and $m \in \mathbb{N}$. Then,

$$\begin{aligned} d^*(z_m^*, z_n^*) &= \sqrt[\ddot{}]{[\iota(\dot{a}_m \dot{-} \dot{a}_n)]^{\ddot{2}} \ddot{+} (\ddot{b}_m \ddot{-} \ddot{b}_n)^{\ddot{2}}} \\ &= \beta\left[\sqrt{(a_m-a_n)^2 + (b_m-b_n)^2}\right] \\ &\ddot{<} \varepsilon = \beta(\varepsilon'). \end{aligned} \tag{31}$$

Thus we obtain that

$$\sqrt{(a_m-a_n)^2 + (b_m-b_n)^2} < \varepsilon'. \tag{32}$$

On the other hand, since the following inequalities:

$$\begin{aligned} |a_m - a_n| &= \sqrt{(a_m-a_n)^2} \leq \sqrt{(a_m-a_n)^2 + (b_m-b_n)^2}, \\ |b_m - b_n| &= \sqrt{(b_m-b_n)^2} \leq \sqrt{(a_m-a_n)^2 + (b_m-b_n)^2} \end{aligned} \tag{33}$$

hold we therefore have by (32) that $|a_m - a_n| < \varepsilon'$ and $|b_m - b_n| < \varepsilon'$. This means that (a_n) and (b_n) are Cauchy sequences with real terms. Since $\mathbb{R}$ is complete, it is clear that for every $\varepsilon' > 0$ there exists an $n_1 \in \mathbb{N}$ such that $|a_n - a| < \varepsilon'/2$ for all $n \geq n_1$ and for every $\varepsilon' > 0$ there exists an $n_2 \in \mathbb{N}$ such that $|b_n - b| < \varepsilon'/2$ for all $n \geq n_2$.

Define, $z^* \in \mathbb{C}^*$ by $z^* = (\dot{a}, \ddot{b})$. Then, we have

$$\begin{aligned} d^*(z_n^*, z^*) &= \sqrt[\ddot{}]{[\iota(\dot{a}_n \dot{-} \dot{a})]^{\ddot{2}} \ddot{+} [\ddot{b}_n \ddot{-} \ddot{b}]^{\ddot{2}}} \\ &= \beta\left[\sqrt{(a_n-a)^2 + (b_n-b)^2}\right] \\ &\ddot{\leq} \beta\left[\sqrt{(a_n-a)^2} + \sqrt{(b_n-b)^2}\right] \\ &= \beta(|a_n - a| + |b_n - b|) \\ &\ddot{\leq} \beta\left(\frac{\varepsilon'}{2} + \frac{\varepsilon'}{2}\right) \\ &= \beta(\varepsilon') = \varepsilon. \end{aligned} \tag{34}$$

Hence, $(\mathbb{C}^*, d^*)$ is a complete metric space. □

Since $\mathbb{C}^*$ is a complete metric space with the metric d^* defined by (10) induced by the norm $\ddot{\|} \cdot \ddot{\|}$, as a direct consequence of Theorem 6, we have the following.

Corollary 7. *$\mathbb{C}^*$ is a Banach space with the norm $\ddot{\|} \cdot \ddot{\|}$ defined by*

$$\ddot{\|} z^* \ddot{\|} = \sqrt[\ddot{}]{[\iota(\dot{a})]^{\ddot{2}} \ddot{+} (\ddot{b})^{\ddot{2}}}; \quad z^* = (\dot{a}, \ddot{b}) \in \mathbb{C}^*. \tag{35}$$

3. Sequence Spaces over Non-Newtonian Complex Field

In this section, we define the sets ω^*, ℓ_∞^*, c^*, c_0^*, and ℓ_p^* of all, bounded, convergent, null, and absolutely p-summable sequences over the non-Newtonian complex field $\mathbb{C}^*$ which correspond to the sets ω, ℓ_∞, c, c_0, and ℓ_p over the complex field $\mathbb{C}$, respectively. That is to say that

$$\begin{aligned} \omega^* &:= \{z^* = (z_k^*) : z_k^* \in \mathbb{C}^* \ \forall k \in \mathbb{N}\}, \\ \ell_\infty^* &:= \left\{z^* = (z_k^*) \in \omega^* : \sup_{k\in\mathbb{N}} \ddot{\|} z_k^* \ddot{\|} \ddot{<} \infty\right\}, \\ c^* &:= \left\{z^* = (z_k^*) \in \omega^* : \exists l^* \in \mathbb{C}^* \ni {\lim_{k\to\infty}}^* z_k^* = l^*\right\}, \\ c_0^* &:= \left\{z^* = (z_k^*) \in \omega^* : {\lim_{k\to\infty}}^* z_k^* = \theta^*\right\}, \\ \ell_p^* &:= \left\{z^* = (z_k^*) \in \omega^* : \ddot{\sum_k} \ddot{\|} z_k^* \ddot{\|}^p \ddot{<} \infty\right\}. \end{aligned} \tag{36}$$

For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. One can easily see that the set ω^* forms a vector space over $\mathbb{C}^*$ with respect to the algebraic operations addition (+) and scalar multiplication (×) defined on ω^*, as follows:

$$\begin{aligned}
&+ : \omega^* \times \omega^* \longrightarrow \omega^* \\
&\quad (z^*, t^*) \longmapsto z^* + t^* = (z_k^* \oplus t_k^*); \\
&\qquad z^* = (z_k^*),\ t^* = (t_k^*) \in \omega^*, \\
&\times : \mathbb{C}^* \times \omega^* \longrightarrow \omega^* \\
&\quad (\alpha, z^*) \longmapsto \alpha \times z^* = (\alpha \odot z_k^*); \\
&\qquad z^* = (z_k^*) \in \omega^*,\ \alpha \in \mathbb{C}^*.
\end{aligned} \tag{37}$$

Theorem 8. *Define the function d_{ω^*} by*

$$\begin{aligned}
&d_{\omega^*} : \omega^* \times \omega^* \longrightarrow B' \subseteq B \\
&(z^*, t^*) \longmapsto d_{\omega^*}(z^*, t^*) \\
&= \ddot{\sum_k} \mu_k \ddot{\times} \left\{ d^*(z_k^*, t_k^*) \ddot{/} [\ddot{1} \ddot{+} d^*(z_k^*, t_k^*)] \right\},
\end{aligned} \tag{38}$$

where $(\mu_k) \in B' \subseteq B$ such that $\ddot{\sum}_k \mu_k$ is convergent with $\mu_k \ddot{>} \ddot{0}$ for all $k \in \mathbb{N}$. Then, (ω^*, d_{ω^*}) is a metric space.

Proof. We show that d_{ω^*} satisfies the metric axioms on the space ω^* of all non-Newtonian complex valued sequences.

(i) First we show that $d_{\omega^*}(z^*, t^*) \ddot{\geq} \ddot{0}$ for all $z^*, t^* \in \omega^*$.

Because $(\mathbb{C}^*, d^*)$ is a metric space, we have $d^*(z_k^*, t_k^*) \ddot{\geq} \ddot{0}$; $z_k^*, t_k^* \in \mathbb{C}^*$ and $\ddot{1} \ddot{+} d^*(z_k^*, t_k^*) \ddot{\geq} \ddot{0}$. Hence, we obtain that $d^*(z_k^*, t_k^*) \ddot{/} [\ddot{1} \ddot{+} d^*(z_k^*, t_k^*)] \ddot{\geq} \ddot{0}$. Moreover, since $\mu_k \ddot{>} \ddot{0}$ for all $k \in \mathbb{N}$, we have

$$\mu_k \ddot{\times} \left\{ d^*(z_k^*, t_k^*) \ddot{/} [\ddot{1} \ddot{+} d^*(z_k^*, t_k^*)] \right\} \ddot{\geq} \ddot{0}. \tag{39}$$

This means that

$$d_{\omega^*}(z^*, t^*) = \ddot{\sum_k} \mu_k \ddot{\times} \left\{ d^*(z_k^*, t_k^*) \ddot{/} [\ddot{1} \ddot{+} d^*(z_k^*, t_k^*)] \right\} \ddot{\geq} \ddot{0}. \tag{40}$$

(ii) We show that $d_{\omega^*}(z^*, t^*) = \ddot{0}$ iff $z^* = t^*$. In this situation, one can see that

$$\begin{aligned}
&d_{\omega^*}(z^*, t^*) \\
&= \ddot{\sum_k} \mu_k \ddot{\times} \left\{ d^*(z_k^*, t_k^*) \ddot{/} [\ddot{1} \ddot{+} d^*(z_k^*, t_k^*)] \right\} = \ddot{0} \\
&\Longleftrightarrow \mu_k \ddot{\times} \left\{ d^*(z_k^*, t_k^*) \ddot{/} [\ddot{1} \ddot{+} d^*(z_k^*, t_k^*)] \right\} = \ddot{0}, \\
&\qquad \forall k \in \mathbb{N} \\
&\Longleftrightarrow d^*(z_k^*, t_k^*) \ddot{/} [\ddot{1} \ddot{+} d^*(z_k^*, t_k^*)] = \ddot{0}, \\
&\qquad \mu_k \ddot{>} \ddot{0},\ \forall k \in \mathbb{N} \\
&\Longleftrightarrow d^*(z_k^*, t_k^*) = \ddot{0}; \quad d^*(z_k^*, t_k^*) \ddot{\geq} \ddot{0},\ \forall k \in \mathbb{N} \\
&\Longleftrightarrow z_k^* = t_k^*; \quad d^* \text{metric},\ \forall k \in \mathbb{N} \\
&\Longleftrightarrow z^* = (z_k^*) = (t_k^*) = t^*.
\end{aligned} \tag{41}$$

(iii) We show that $d_{\omega^*}(z^*, t^*) = d_{\omega^*}(t^*, z^*)$ for $z^* = (z_k^*)$, $t^* = (t_k^*) \in \omega^*$. First, we know that d^* is a metric over $\mathbb{C}^*$. Thus,

$$\begin{aligned}
d_{\omega^*}(z^*, t^*) &= \ddot{\sum_k} \mu_k \ddot{\times} \left\{ d^*(z_k^*, t_k^*) \ddot{/} [\ddot{1} \ddot{+} d^*(z_k^*, t_k^*)] \right\} \\
&= \ddot{\sum_k} \mu_k \ddot{\times} \left\{ d^*(t_k^*, z_k^*) \ddot{/} [\ddot{1} \ddot{+} d^*(t_k^*, z_k^*)] \right\} \\
&= d_{\omega^*}(t^*, z^*).
\end{aligned} \tag{42}$$

(iv) We show that $d_{\omega^*}(z^*, t^*) \ddot{+} d_{\omega^*}(t^*, u^*) \ddot{\geq} d_{\omega^*}(z^*, u^*)$ holds for $z^* = (z_k^*)$, $t^* = (t_k^*)$, $u^* = (u_k^*) \in \omega^*$. Again, using the fact that $(\mathbb{C}^*, d^*)$ is a metric space, it is easy to see by Lemma 4 that

$$\begin{aligned}
&d_{\omega^*}(z^*, u^*) \\
&= \ddot{\sum_k} \mu_k \ddot{\times} \left\{ d^*(z_k^*, u_k^*) \ddot{/} [\ddot{1} \ddot{+} d^*(z_k^*, u_k^*)] \right\} \\
&\ddot{\leq} \ddot{\sum_k} \mu_k \ddot{\times} \left\{ [d^*(z_k^*, t_k^*) \ddot{+} d^*(t_k^*, u_k^*)] \right. \\
&\qquad \left. \ddot{/} \{\ddot{1} \ddot{+} [d^*(z_k^*, t_k^*) \ddot{+} d^*(t_k^*, u_k^*)]\} \right\} \\
&\ddot{\leq} \ddot{\sum_k} \mu_k \ddot{\times} \left\{ \left\{ d^*(z_k^*, t_k^*) \ddot{/} [\ddot{1} \ddot{+} d^*(z_k^*, t_k^*)] \right\} \right. \\
&\qquad \left. \ddot{+} \left\{ d^*(t_k^*, u_k^*) \ddot{/} [\ddot{1} \ddot{+} d^*(t_k^*, u_k^*)] \right\} \right\} \\
&= \ddot{\sum_k} \mu_k \ddot{\times} \left\{ d^*(z_k^*, t_k^*) \ddot{/} [\ddot{1} \ddot{+} d^*(z_k^*, t_k^*)] \right\} \\
&\quad \ddot{+} \ddot{\sum_k} \mu_k \ddot{\times} \left\{ d^*(t_k^*, u_k^*) \ddot{/} [\ddot{1} \ddot{+} d^*(t_k^*, u_k^*)] \right\} \\
&= d_{\omega^*}(z^*, t^*) \ddot{+} d_{\omega^*}(t^*, u^*),
\end{aligned} \tag{43}$$

as required. □

Theorem 9. *The set ℓ_∞^* is a sequence space.*

Proof. It is trivial that the inclusion $\ell_\infty^* \subset \omega^*$ holds.

(i) We show that $z^* + t^* \in \ell_\infty^*$ for $z^* = (z_k^*)$, $t^* = (t_k^*) \in \ell_\infty^*$. Indeed, combining the hypothesis

$$\sup_{k \in \mathbb{N}} \ddot{\|} z_k^* \ddot{\|} \ddot{<} \infty, \qquad \sup_{k \in \mathbb{N}} \ddot{\|} t_k^* \ddot{\|} \ddot{<} \infty \tag{44}$$

with the fact $\ddot{\|} z_k^* \oplus t_k^* \ddot{\|} \ddot{\leq} \ddot{\|} z_k^* \ddot{\|} \dot{+} \ddot{\|} t_k^* \ddot{\|}$ obtained from Lemma 2, we can easily derive that

$$\sup_{k\in\mathbb{N}} \ddot{\|} z_k^* \oplus t_k^* \ddot{\|} \ddot{\leq} \sup_{k\in\mathbb{N}} \ddot{\|} z_k^* \ddot{\|} \dot{+} \sup_{k\in\mathbb{N}} \ddot{\|} t_k^* \ddot{\|} \ddot{<} \infty. \quad (45)$$

Hence, $z^* + t^* \in \ell_\infty^*$.

(ii) We show that $\alpha \times z^* \in \ell_\infty^*$ for any $\alpha \in \mathbb{C}^*$ and for $z^* = (z_k^*) \in \ell_\infty^*$.

Since $\ddot{\|} \alpha \odot z_k^* \ddot{\|} = \ddot{\|} \alpha \ddot{\|} \ddot{\times} \ddot{\|} z_k^* \ddot{\|}$ by Lemma 3 and $\sup_{k\in\mathbb{N}} \ddot{\|} z_k^* \ddot{\|} \ddot{<} \infty$, it is immediate that

$$\sup_{k\in\mathbb{N}} \ddot{\|} \alpha \odot z_k^* \ddot{\|} = \ddot{\|} \alpha \ddot{\|} \ddot{\times} \sup_{k\in\mathbb{N}} \ddot{\|} z_k^* \ddot{\|} \ddot{<} \infty \quad (46)$$

which means that $\alpha \times z^* \in \ell_\infty^*$.

Therefore, we have proved that ℓ_∞^* is a subspace of the space ω^*. □

Theorem 10. *Define the relation d_∞^* by*

$$\begin{aligned} d_\infty^* : \ell_\infty^* \times \ell_\infty^* &\longrightarrow B' \subseteq B \\ (z^*, t^*) &\longmapsto d_\infty^*(z^*, t^*) = \sup_{k\in\mathbb{N}} \ddot{\|} z_k^* \ominus t_k^* \ddot{\|}. \end{aligned} \quad (47)$$

Then, $(\ell_\infty^*, d_\infty^*)$ is a complete metric space.

Proof. One can easily show by a routine verification that d_∞^* satisfies the metric axioms on the space ℓ_∞^*. So, we omit the details.

Now, we prove the second part of the theorem. Let (z_m^*) be a Cauchy sequence in ℓ_∞^*, where $z_m^* = (z_k^{*m})_{k\in\mathbb{N}}$. Then, there exists a positive integer k_0 such that $d_\infty^*(z_m^*, z_r^*) = \sup_{k\in\mathbb{N}} \ddot{\|} z_k^{*m} \ominus z_k^{*r} \ddot{\|} \ddot{<} \varepsilon$ for all $m, r \in \mathbb{N}$ with $m, r > k_0$. For any fixed k, if $m, r > k_0$ then

$$\ddot{\|} z_k^{*m} \ominus z_k^{*r} \ddot{\|} \ddot{<} \varepsilon. \quad (48)$$

In this case for any fixed k, $(z_k^{*0}, z_k^{*1}, \ldots)$ is a Cauchy sequence of non-Newtonian complex numbers and since $\mathbb{C}^*$ is complete, it converges to a $z_k^* \in \mathbb{C}^*$. Define $z^* = (z_0^*, z_1^*, \ldots)$ with infinitely many limits $z_0^*, z_1^*, \ldots$. Let us show $z^* \in \ell_\infty^*$ and $z_m^* \to z^*$, as $m \to \infty$. Indeed, by (48), by letting $r \to \infty$, for $m > k_0$ we obtain that

$$\ddot{\|} z_k^{*m} \ominus z_k^* \ddot{\|} \ddot{\leq} \varepsilon. \quad (49)$$

On the other hand, since $z_m^* = (z_k^{*m})_{k\in\mathbb{N}} \in \ell_\infty^*$, there exists $t_m \in B \subseteq \mathbb{R}$ such that $\ddot{\|} z_k^{*m} \ddot{\|} \ddot{\leq} t_m$ for all $k \in \mathbb{N}$. Hence, by triangle inequality (12), the inequality

$$\ddot{\|} z_k^* \ddot{\|} \ddot{\leq} \ddot{\|} z_k^* \ominus z_k^{*m} \ddot{\|} \dot{+} \ddot{\|} z_k^{*m} \ddot{\|} \ddot{\leq} \varepsilon \dot{+} t_m \quad (50)$$

holds for all $k \in \mathbb{N}$ which is independent of k. Hence, $z^* = (z_k^*)_{k\in\mathbb{N}} \in \ell_\infty^*$. By (49), since $m > k_0$, we obtain $d_\infty^*(z_m^*, z^*) = \sup_{k\in\mathbb{N}} \ddot{\|} z_k^{*m} \ominus z_k^* \ddot{\|} \ddot{\leq} \varepsilon$. Therefore, the sequence (z_m^*) converges to z^* which means that ℓ_∞^* is complete. □

Since it is known by Theorem 10 that ℓ_∞ is a complete metric space with the metric d_∞^* induced by the norm $\ddot{\|} \cdot \ddot{\|}_\infty$, defined by

$$\ddot{\|} z^* \ddot{\|}_\infty = \sup_{k\in\mathbb{N}} \ddot{\|} z_k^* \ddot{\|}; \quad z^* = (z_k^*) \in \ell_\infty^*, \quad (51)$$

we have the following.

Corollary 11. *ℓ_∞^* is a Banach space with the norm $\ddot{\|} \cdot \ddot{\|}_\infty$ defined by* (51).

Now, we give the following lemma required in proving the fact that ℓ_p^* is a sequence space in the case $\ddot{0} \ddot{<} p \ddot{<} \ddot{1}$.

Lemma 12. *Let $\ddot{0} \ddot{<} p \ddot{<} \ddot{1}$. Then, the inequality $(\ddot{a} \dot{+} \ddot{b})^p \ddot{<} \ddot{a}^p \dot{+} \ddot{b}^p$ holds for all $\ddot{a}, \ddot{b} \ddot{>} \ddot{0}$.*

Proof. Let $\ddot{0} \ddot{<} p \ddot{<} \ddot{1}$ and $\ddot{a}, \ddot{b} \ddot{>} \ddot{0}$. Then, one can easily see that

$$\begin{aligned}
\left(\ddot{a} \dot{+} \ddot{b}\right)^p &= \underbrace{\left(\ddot{a} \dot{+} \ddot{b}\right) \ddot{\times} \cdots \ddot{\times} \left(\ddot{a} \dot{+} \ddot{b}\right)}_{p\text{-times}} \\
&= \beta\{a+b\} \ddot{\times} \cdots \ddot{\times} \beta\{a+b\} \\
&= \beta\left\{\underbrace{\beta^{-1}[\beta(a+b)] \times \cdots \times \beta^{-1}[\beta(a+b)]}_{\beta^{-1}(p)\text{-times}}\right\} \\
&= \beta\left\{(a+b)^{\beta^{-1}(p)}\right\} \\
&\ddot{<} \beta\left\{a^{\beta^{-1}(p)} + b^{\beta^{-1}(p)}\right\} \\
&= \beta\left\{\beta^{-1}\left(\ddot{a}^p\right) + \beta^{-1}\left(\ddot{b}^p\right)\right\} \\
&= \ddot{a}^p \dot{+} \ddot{b}^p,
\end{aligned} \quad (52)$$

as desired. □

Theorem 13. *The sets c^*, c_0^*, and ℓ_p^* are sequence spaces, where $\ddot{0} \ddot{<} p \ddot{<} \infty$.*

Proof. It is not hard to establish by the similar way that c_0^* and ℓ_p^* are the sequence spaces. So, to avoid the repetition of the similar statements, we consider only the set c^*.

It is obvious that the inclusion $c^* \subset \omega^*$ strictly holds.

(i) Let $z^* = (z_k^*)$, $t^* = (t_k^*) \in c^*$. Then, there exist $l_1^*, l_2^* \in \mathbb{C}^*$ such that $\lim^*_{k\to\infty} z_k^* = l_1^*$ and $\lim^*_{k\to\infty} t_k^* = l_2^*$. Thus, there exist $k_1, k_2 \in \mathbb{N}$ such that

$$\begin{aligned} \forall \varepsilon \ddot{>} \ddot{0}, \quad \ddot{\|} z_k^* \ominus l_1^* \ddot{\|} \ddot{\leq} \varepsilon \ddot{/} \ddot{2} \quad \forall k \geq k_1, \\ \forall \varepsilon \ddot{>} \ddot{0}, \quad \ddot{\|} t_k^* \ominus l_2^* \ddot{\|} \ddot{\leq} \varepsilon \ddot{/} \ddot{2} \quad \forall k \geq k_2. \end{aligned} \quad (53)$$

Thus if we set $k_0 = \max\{k_1, k_2\}$, by (53) we obtain for all $k \geq k_0$ that

$$\begin{aligned}
&\|(z_k^* \oplus t_k^*) \ominus (l_1^* \oplus l_2^*)\| \\
&\quad = \|(z_k^* \ominus l_1^*) \oplus (t_k^* \ominus l_2^*)\| \\
&\quad \ddot{\leq} \|z_k^* \ominus l_1^*\| \ddot{+} \|t_k^* \ominus l_2^*\| \\
&\quad \ddot{\leq} \varepsilon \ddot{/} \ddot{2} \ddot{+} \varepsilon \ddot{/} \ddot{2} \\
&\quad = \varepsilon
\end{aligned} \tag{54}$$

which means that

$$\lim_{k\to\infty}{}^* (z_k^* \oplus t_k^*) = l_1^* \oplus l_2^* = \lim_{k\to\infty}{}^* z_k^* \oplus \lim_{k\to\infty}{}^* t_k^*. \tag{55}$$

Therefore, $z^* + t^* \in c^*$.

(ii) Let $z^* = (z_k^*) \in c^*$ and $\alpha \in \mathbb{C}^* \setminus \{\theta^*\}$. Since $z^* \in c^*$ there exists an $l^* \in \mathbb{C}^*$ such that $\lim^*_{k\to\infty} z_k^* = l^*$, we have

$$\forall \varepsilon \ddot{>} \ddot{0}, \exists k_0 \in \mathbb{N} \text{ such that } \|z_k^* \ominus l^*\| \ddot{\leq} \varepsilon \ddot{/} \|\alpha\| \quad \forall k \geq k_0. \tag{56}$$

Thus, for $k \geq k_0$, we have

$$\begin{aligned}
&\|(\alpha \odot z_k^*) \ominus (\alpha \odot \ell^*)\| \\
&\quad = \|\alpha \odot (z_k^* \ominus \ell^*)\| \\
&\quad = \|\alpha\| \ddot{\times} \|z_k^* \ominus \ell^*\| \\
&\quad \ddot{\leq} \|\alpha\| \ddot{\times} \varepsilon \ddot{/} \|\alpha\| \\
&\quad = \varepsilon
\end{aligned} \tag{57}$$

which implies that $\lim^*_{k\to\infty}(\alpha \odot z_k^*) = \alpha \odot l^* = \alpha \odot \lim^*_{k\to\infty} z_k^*$ Hence, $\alpha \times z^* \in c^*$.

That is to say that c^* is a subspace of ω^*. □

Theorem 14. *(c^*, d_∞^*), (c_0^*, d_∞^*), and (ℓ_p^*, d_p^*) are complete metric spaces, where d_p^* is defined as follows:*

$$d_p^*(z^*, t^*) = \begin{cases} \ddot{\sum_k} \|z_k^* \ominus t_k^*\|^p, & \ddot{0} \ddot{<} p \ddot{<} \ddot{1} \\ \left(\ddot{\sum_k} \|z_k^* \ominus t_k^*\|^p\right)^{\ddot{1}\ddot{/}p}, & p \ddot{\geq} \ddot{1}; \end{cases} \tag{58}$$

$$z^* = (z_k^*), \quad t^* = (t_k^*) \in \ell_p^*.$$

Proof. We consider only the space ℓ_p^* with $p \ddot{\geq} \ddot{1}$.

(i) For $z^* = (z_k^*)$, $t^* = (t_k^*) \in \ell_p^*$, we establish that the two sided implication $d_p^*(z^*, t^*) = \ddot{0} \Leftrightarrow z^* = t^*$ holds. In fact,

$$\begin{aligned}
&d_p^*(z^*, t^*) = \left(\ddot{\sum_k} \|z_k^* \ominus t_k^*\|^p\right)^{\ddot{1}\ddot{/}p} = \ddot{0} \\
&\quad \Longleftrightarrow \ddot{\sum_k} \|z_k^* \ominus t_k^*\|^p = \ddot{0} \\
&\quad \Longleftrightarrow \forall k \in \mathbb{N}, \quad \|z_k^* \ominus t_k^*\|^p = \ddot{0} \\
&\quad \Longleftrightarrow \forall k \in \mathbb{N}, \quad \|z_k^* \ominus t_k^*\| = \ddot{0} \\
&\quad \Longleftrightarrow \forall k \in \mathbb{N}, \quad z_k^* = t_k^* \\
&\quad \Longleftrightarrow z^* = (z_k^*) = (t_k^*) = t^*.
\end{aligned} \tag{59}$$

(ii) For $z^* = (z_k^*)$, $t^* = (t_k^*) \in \ell_p^*$, we show that $d_p^*(z^*, t^*) = d_p^*(t^*, z^*)$. In this situation, since $\|z_k^* \ominus t_k^*\| = \|t_k^* \ominus z_k^*\|$ holds for every fixed $k \in \mathbb{N}$ it is immediate that

$$\begin{aligned}
d_p^*(z^*, t^*) &= \left(\ddot{\sum_k} \|z_k^* \ominus t_k^*\|^p\right)^{\ddot{1}\ddot{/}p} \\
&= \left(\ddot{\sum_k} \|t_k^* \ominus z_k^*\|^p\right)^{\ddot{1}\ddot{/}p} \\
&= d_p^*(t^*, z^*).
\end{aligned} \tag{60}$$

(iii) By Minkowski inequality in Lemma 5, we have for $z^* = (z_k^*)$, $t^* = (t_k^*)$, $u^* = (u_k^*) \in \ell_p^*$ that

$$\begin{aligned}
d_p^*(z^*, t^*) &= \left(\ddot{\sum_k} \|z_k^* \ominus t_k^*\|^p\right)^{\ddot{1}\ddot{/}p} \\
&= \left[\ddot{\sum_k} \|(z_k^* \ominus u_k^*) \oplus (u_k^* \ominus t_k^*)\|^p\right]^{\ddot{1}\ddot{/}p} \\
&\ddot{\leq} \left(\ddot{\sum_k} \|z_k^* \ominus u_k^*\|^p\right)^{\ddot{1}\ddot{/}p} \ddot{+} \left(\ddot{\sum_k} \|u_k^* \ominus t_k^*\|^p\right)^{\ddot{1}\ddot{/}p} \\
&= d_p^*(z^*, u^*) \ddot{+} d_p^*(u^*, t^*).
\end{aligned} \tag{61}$$

Hence, triangle inequality is satisfied by d_p^* on the space ℓ_p^*. Therefore, the function d_p^* is a metric over the space ℓ_p^*.

Now we show that the metric space (ℓ_p^*, d_p^*) is complete. Let $(z_m^*)_{m\in\mathbb{N}}$ be an arbitrary Cauchy sequence in the space ℓ_p^*, where $z_m^* = (z_1^{*m}, z_2^{*m}, \ldots)$. Then, for any $\varepsilon \ddot{>} \ddot{0}$ there exists an $n_0 \in \mathbb{N}$ such that

$$d_p^*(z_m^*, z_n^*) = \left(\ddot{\sum_k} \|z_k^{*m} \ominus z_k^{*n}\|^p\right)^{\ddot{1}\ddot{/}p} \ddot{<} \varepsilon \tag{62}$$

for all $m, n \geq n_0$. Hence, for $m, n \geq n_0$ and every fixed $k \in \mathbb{N}$, we obtain

$$\|z_k^{*m} \ominus z_k^{*n}\| \ddot{<} \varepsilon. \tag{63}$$

If we set k fixed then, it follows by (63) that $(z_k^{*0}, z_k^{*1}, \ldots)$ is a Cauchy sequence. Since $\mathbb{C}^*$ is complete, this sequence converges to a point say z_k^*. Let us define the sequence $z^* = (z_0^*, z_1^*, \ldots)$ with these limits and show $z^* \in \ell_p^*$ and $z_m^* \to z^*$, as $m \to \infty$. Indeed, by (62), we obtain the inequality for all $m, n \in \mathbb{N}$ with $m, n \geq n_0$ that

$$\ddot{\sum_{k=0}^{j}} \ddot{\|} z_k^{*m} \ominus z_k^{*n} \ddot{\|}^p \ddot{<} \varepsilon^p \quad \forall j \in \mathbb{N} \tag{64}$$

and thus we have by letting $n \to \infty$ and $m > n_0$ that

$$\ddot{\sum_{k=0}^{j}} \ddot{\|} z_k^{*m} \ominus z_k^{*} \ddot{\|}^p \ddot{<} \varepsilon^p \quad \forall j \in \mathbb{N} \tag{65}$$

which gives as $j \to \infty$ and for all $m > n_0$ that

$$\left[d_p^*\left(z_m^*, z^*\right)\right]^p = \ddot{\sum_{k}} \ddot{\|} z_k^{*m} \ominus z_k^{*} \ddot{\|}^p \ddot{<} \varepsilon^p. \tag{66}$$

Setting $z_k^* = z_k^{*m} \oplus (z_k^* \ominus z_k^{*m})$ and applying Lemma 5 we obtain by (66) and the fact $(z_k^{*m}) \in \ell_p^*$ that

$$\begin{aligned}\left(\ddot{\sum_{k}} \ddot{\|} z_k^* \ddot{\|}^p\right)^{\ddot{1}\ddot{/}p} &= \left(\ddot{\sum_{k}} \ddot{\|} z_k^{*m} \oplus \left(z_k^* \ominus z_k^{*m}\right) \ddot{\|}^p\right)^{\ddot{1}\ddot{/}p} \\ &\ddot{\leq} \left(\ddot{\sum_{k}} \ddot{\|} z_k^{*m} \ddot{\|}^p\right)^{\ddot{1}\ddot{/}p} \\ &\ddot{+} \left(\ddot{\sum_{k}} \ddot{\|} z_k^{*} \ominus z_k^{*m} \ddot{\|}^p\right)^{\ddot{1}\ddot{/}p} \ddot{<} \infty\end{aligned} \tag{67}$$

which means that $z^* = (z_k^*) \in \ell_p^*$. Therefore, we see from (66) that $z_m^* \to z^*$. Since the arbitrary Cauchy sequence $(z_m^*) = (z_k^{*m})_{k, m \in \mathbb{N}} \in \ell_p^*$ is convergent, the space ℓ_p^* is complete. □

Corollary 15. *c^* and c_0^* are the Banach spaces equipped with the norm $\ddot{\|} \cdot \ddot{\|}_\infty$ defined in* (51).

Since it is known by Theorem 14 that ℓ_p^* is a complete metric space with the metric d_p^* induced in the case $p \ddot{\geq} \ddot{1}$ by the norm $\ddot{\|} \cdot \ddot{\|}_p$ and in the case $\ddot{0} \ddot{<} p \ddot{<} \ddot{1}$ by the p-norm $\ddot{\|} \cdot \widehat{\ddot{\|}}_p$, defined for $z^* = (z_k^*) \in \ell_p^*$ by

$$\ddot{\|} z^* \ddot{\|}_p = \left(\ddot{\sum_{k}} \ddot{\|} z_k^* \ddot{\|}^p\right)^{\ddot{1}\ddot{/}p} \qquad \ddot{\|} z^* \widehat{\ddot{\|}}_p = \ddot{\sum_{k}} \ddot{\|} z_k^* \ddot{\|}^p, \tag{68}$$

we have the following.

Corollary 16. *The space ℓ_p^* is a Banach space with the norm $\ddot{\|} \cdot \ddot{\|}_p$ and p-norm $\ddot{\|} \cdot \widehat{\ddot{\|}}_p$ defined by* (68).

4. Conclusion

We present some important inequalities such as triangle, Minkowski, and some other inequalities in the sense of non-Newtonian complex calculus which are frequently used. We state the classical sequence spaces over the non-Newtonian complex field $\mathbb{C}^*$ and try to understand their structure of being non-Newtonian complex vector space. There are lots of techniques that have been developed in the sense of non-Newtonian complex calculus. If the non-Newtonian complex calculus is employed instead of the classical calculus in the formulations, then many of the complicated phenomena in physics or engineering may be analyzed more easily.

As an alternative to the classical (additive) calculus, Grossman and Katz [1] introduced some new kind of calculus named as non-Newtonian calculus, geometric calculus, anageometric calculus, and bigeometric calculus. Türkmen and Başar [6, 7] have recentlystudied the classical sequence spaces and related topics, in the sense of geometric calculus. Quite recently, Çakmak and Başar [8] have also worked on the same subject by using non-Newtonian calculus. In the present paper, we use the non-Newtonian complex calculus instead of non-Newtonian real calculus and geometric calculus. It is trivial that in the special cases of the generators α and β, the non-Newtonian complex calculus gives the special kind of the following calculus:

(i) if $\alpha = \beta = I$, the identity function, then the non-Newtonian complex calculus is reduced to the classical calculus;

(ii) if $\alpha = I$ and $\beta = \exp$, then the non-Newtonian complex calculus is reduced to the geometric calculus;

(iii) if $\alpha = \exp$ and $\beta = I$, then the non-Newtonian complex calculus is reduced to the anageometric calculus;

(iv) if $\alpha = \beta = \exp$, then the non-Newtonian complex calculus is reduced to the bigeometric calculus.

Since our results are obtained by using the non-Newtonian complex calculus, they are much more general and comprehensive than those of Türkmen and Başar [6, 7], Çakmak and Başar [8].

Quite recently, Talo and Başar have studied the certain sets of sequences of fuzzy numbers and introduced the classical sets $\ell_\infty(F)$, $c(F)$, $c_0(F)$, and $\ell_p(F)$ consisting of the bounded, convergent, null, and absolutely p-summable sequences of fuzzy numbers in [9]. Nextly, they have defined the alpha-, beta-, and gamma-duals of a set of sequences of fuzzy numbers and gave the duals of the classical sets of sequences of fuzzy numbers together with the characterization of the classes of infinite matrices of fuzzy numbers transforming one of the classical set into another one. Following Bashirov et al. [2] and Uzer [3], we give the corresponding results for multiplicative calculus to the results derived for the sets of fuzzy valued sequences in Talo and Başar [9], as a beginning. As a natural continuation of this paper, we should record from now on that it is meaningful to define the alpha-, beta-, and gamma-duals of a set of sequences over the non-Newtonian complex field $\mathbb{C}^*$ and to determine the duals of classical spaces ℓ_∞^*, c^*, c_0^*, and ℓ_p^*. Further, one can obtain the similar results by using another type of calculus instead of non-Newtonian complex calculus.

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