Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2013, Article ID 714715, 8 pages
http://dx.doi.org/10.1155/2013/714715

Research Article

Product Selection and Components Replenishment Model of ATO Manufacturer under Heterogeneous Demand

Yuyu Li[1] and Bo Huang[2]

[1] *College of Computer and Information Science, Chongqing Normal University, Chongqing 400047, China*
[2] *College of Economics and Business Administration, Chongqing University, Chongqing 400044, China*

Correspondence should be addressed to Bo Huang; huangbo@cqu.edu.cn

Received 16 July 2013; Revised 29 September 2013; Accepted 2 October 2013

Academic Editor: Chong Lin

How to satisfy customers' heterogeneous demands by limited models of product is the key problem in an ATO (Assemble-to-Order) manufacturer's operations management. Based on the condition that customers' demands depend on their types which follow uniform distribution, we develop an inventory model to study how an ATO manufacturer selects the products from the candidate products in a same product family and determines components replenishment quantities, in order to maximize its profit. We get the optimal set of products which an ATO manufacturer should assemble, as well as the components replenishment quantities through theoretical, numerical, and case analyses. Furthermore, we propose an algorithm to obtain the optimal solution on the set of products and the components replenishment quantities. We found that an ATO manufacturer should make the performance and qualities of products over a certain level, and determine the optimal selling duration of products according to their selling prices and qualities, as well as the purchase prices and holding costs of their components. This paper offers consultation for ATO manufacturers to make policies on product selection and components replenishment.

1. Introduction

Due to the heterogeneity of customers and their demands, which means different customers have different income, price sensitivity, and preference on product usage; manufacturers in reality have to assemble a series of products with different usages and qualities (i.e., product family) to meet heterogeneous demands [1–3]. Take PC manufacturers as an example, they provide PC of low configurations (e.g., low speed processor and integrated graphics video card) at a low price to price sensitive customers while they provide PC of high configurations (e.g., high speed processor and discrete graphics video card) at a high price to customers with demand for high performance. Meanwhile, in order to reduce the costs due to the diversity of products, many manufacturers have adopted ATO (Assemble-to-Order), that is, only after receiving customers' order, do they assemble products according to it [4, 5].

Different with make-to-stock [6], there is no product inventory under ATO model, whose core is components replenishment accordingly [7]. Many researches on components replenishment of multiproduct ATO system have been made [8, 9]. Lu and Song studied a multiproduct ATO system in which customers may order different but possibly overlapping subsets of items, and determined the right base-stock level [10]. Betts and Johnston used a deterministic batch sizing model assuming that ATO inventory is finite to analyze the JIT replenishment or component substitution decisions [11], and further treated the stochastic version of the same problem and developed a tractable solution method for ATO decision problem [12]. DeCroix et al. identified several ways in which returns complicate the behavior of ATO system, and demonstrated how to handle these additional complexities when calculating or approximating key order-based performance metrics [13]. Zhao derived exact expressions for key performance metrics of a multiproduct and multicomponent ATO system where demand follows compound Poisson processes, and develop an efficient sampling method to estimate these metrics [14]. However, these researches did not consider the heterogeneity of customers and their demands, and did not study the product selection problem.

While the study of product selection forms a growing part of the literature in operation research [15–17], most of them are either about how to select and design product line according to product selection in order to enhance its performance, reduce costs, and raise profits [18], or about the product selection during the competition between traditional retailer or cross-channel competition [19]. To the best of our knowledge, there is little research on how ATO manufacturers select the optimal products and determine the components replenishment quantities to meet the heterogeneous customer demands.

The remainder of the paper is organized as follows. Section 2 is dedicated to the model development. The model and its optimal solution are analyzed in Section 3. We propose an algorithm for the optimal solution in Section 4. A numerical example is used to illuminate the model in Section 5. Section 6 is a case study. Finally, conclusions are drawn in Section 7.

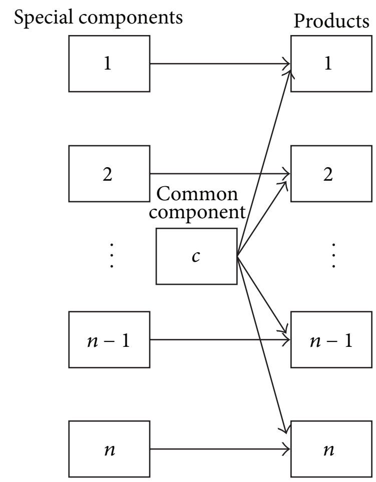

Figure 1: The assemble-to-order system.

2. Model Development

2.1. Problem Description and Assumption. A product family ATO manufacture is able to use common components and special components to assemble n models of seasonal products belonging to the same family. These products with different performances or qualities and selling price are candidate ones for the ATO manufacturer's final optimal set of products, which maximize its profit. In reality, a product consists of multiple common components and special components. However, the parameters of these components, such as purchase price and holding cost per unit per time, are fixed constants. Therefore, for simplifying the model and analysis, we can use a common component c to represent all common components of product family, and special component i for all special components of product i, $i = 1, 2, \ldots, n$. So, a product i consists of one unit each of component c and i [20, 21]. In additional, we number all products in order of their values or qualities, that is, the values or qualities of product i follows $V_1 < V_2 < \cdots < V_n$. The structure of the assemble-to-order system is shown in Figure 1.

The potential demand of whole product family is D and the market segmentation or actual demand of product i is m_i, $i = 1, 2, \ldots, n$. The ATO manufacture only selects the products with positive market segmentation to assemble from the whole family according to the heterogeneous preference and demand of customers, as well as the quality V_i and selling price S_i of product i. In practice, the price of many ATO products, PC, for example, is decided by market because of the substitution among different types of product and a number of competitors. Therefore, we consider the selling price, S_i, is exogenous and follows $S_1 < S_2 < \cdots < S_n$.

Customers are heterogeneous and customer type, which can be consider as a customer's importance weight for product i, which follows a uniform distribution of $[0, 1]$. Therefore, a customer of type θ gets a value θV_i from buying a unit of product i, and the highest price that he is willing to pay for a unit of product i is θV_i [22, 23]. A customer only buys a unit of one product, which provides the highest positive customer surplus (the margin between what he is willing to pay and the selling price, $\theta V_i - S_i$). If the highest customer surplus is less than zero, he will not buy any products. The quality and selling price of products i, V_i, and S_i, as well as the distribution of customer type θ are common knowledge, that is, the information is known to all the players, including the ATO manufacturer and all customers, and each player knows that all the players know it, each player knows that all the players know that all the players know it, and so forth ad infinitum.

The process of the ATO manufacturer makes its product selection and components replenishment policies as follows. As V_i, S_i, and θ are all common knowledge, the manufacturer can shortlist the preliminary set of products with positive market segmentation, m_i, for final selection. Then, it deters the final set of products with positive optimal selling duration, T_i, according to the market segmentation of products, the demand substitution among the preliminary set of products, as well as the purchase price, P_i and P_c, and unit holding cost of components, H_i and H_c. Subsequently, it deters the purchase quantity of every kind of components, Q_i and Q_c, and replenish it according to its optimal selling duration, and will make no replenishment during the whole selling season. Finally, it assembles and sales product after receiving customer order, therefore, it has no product inventory but components inventory.

The process of customers making purchase decision is as follows. When a potential customer arrives, he makes the decision of whether and which product should he buy according to products' qualities, selling price, and his heterogeneous preference on products' qualities with the goal of maximizing his surplus. If he decides to buy, the potential demand becomes actual demand. If the product he decides to buy is in short supply, he makes a new decision of whether and which substitution should he buy among the spot products to maximize his surplus. Denote the substitution product for product i, $i = 1, 2, \ldots, n$, as product j, where $j = 1, 2, \ldots, n$, $j \neq i$, as all the other products except product i may be the substitution product. If no product provides a positive customer surplus, he gives up purchase.

2.2. Notation. The other notations in this paper are as follows, where $i, j = 1, 2, \ldots, n$, $j \neq i$:

ρ_{ji}: substitution parameter of product i, namely, when product i is in short supply, the proportion of customer who wants to buy it chooses to buy product j, $\sum \rho_{ji} \leq 1$;

T_i: selling duration of product i, decision variable;

Q_i: selling quantity of product i, and replenishment quantity of special component i, decision variable;

P_i: purchase price of special component i;

H_i: holding cost per unit per time of special component i;

Q_c: replenishment quantity of common component c, $Q_c = \sum Q_i$, decision variable;

P_c: purchase price of common component c;

H_c: holding cost per unit per time of common component c.

2.3. The ATO Inventory Model. Without losing generalization, renumbering all products in order of their selling duration, we get $T_{I-1} < T_I$, $I = 1, 2, \ldots, n$. When product I is sold out, the percentage of buying the substitutions is $\sum_{j=I+1}^{n} \rho_{jI}$, and then percentage of giving up purchase is $1 - \sum_{j=I+1}^{n} \rho_{jI}$. Letting $T_0 = 0$, we can get the demand per time of special component I at t time of kth period ($t \in [T_{k-1}, T_k]$) as

$$D_I^k = \begin{cases} D\left(m_I + \sum_{j=1}^{k-1} \rho_{ij} m_j\right), & k \leq I; \\ 0, & k > I. \end{cases} \tag{1}$$

And the demand per time of common component c at t time of kth duration as

$$D_c^k = D \sum_{I=k}^{n} \left[m_I + \sum_{j=1}^{k-1} \left(\rho_{Ij} m_j\right) \right], \quad k = 1, 2, \ldots, n. \tag{2}$$

Now, we can get the total holding cost of all components as

$$\begin{aligned} & H_c \sum_{k=1}^{n} \left\{ \frac{D_c^k \left(T_k - T_{k-1}\right)}{2} \right. \\ & \qquad \left. + \left(T_k - T_{k-1}\right) \sum_{j=k}^{n} \left[D_c^{j+1} \left(T_{j+1} - T_j\right) \right] \right\} \\ & + \sum_{I=1}^{n} \sum_{k=1}^{I} \left\{ \frac{H_I D_I^k \left(T_k - T_{k-1}\right)^2}{2} \right. \\ & \qquad \left. + H_I \left(T_k - T_{k-1}\right) \sum_{j=k}^{I-1} \left[D_I^{j+1} \left(T_{j+1} - T_j\right) \right] \right\}, \end{aligned} \tag{3}$$

the sales income of products as

$$\sum_{I=1}^{n} \left(S_I Q_I\right), \tag{4}$$

the purchase cost of all components as

$$P_c \sum_{I=1}^{n} Q_I + \sum_{I=1}^{n} \left(P_I Q_I\right), \tag{5}$$

and the profit of ATO manufacturer as

$$\begin{aligned} \pi = & \sum_{i=I}^{n} \left(S_I Q_I\right) \\ & - H_c \sum_{k=1}^{n} \left\{ \frac{D_c^k \left(T_k - T_{k-1}\right)^2}{2} \right. \\ & \qquad \left. + \left(T_k - T_{k-1}\right) \sum_{j=k}^{n} \left[D_c^{j+1} \left(T_{j+1} - T_j\right) \right] \right\} \\ & - \sum_{I=1}^{n} \sum_{k=1}^{I} \left\{ \frac{H_I D_I^k \left(T_k - T_{k-1}\right)^2}{2} \right. \\ & \qquad \left. + H_I \left(T_k - T_{k-1}\right) \sum_{j=k}^{I-1} \left[D_I^{j+1} \left(T_{j+1} - T_j\right) \right] \right\} \\ & - P_c \sum_{I=1}^{n} Q_I - \sum_{I=1}^{n} \left(P_I Q_I\right). \end{aligned} \tag{6}$$

3. Model Analysis

For a better analysis and understanding of the model, we introduce two fictitious products labeled as product 0 and product $n + 1$. The qualities and selling prices of product 0 and $n + 1$ are $S_0 = V_0 = 0$, $S_{n+1} = V_{n+1} = +\propto$. Then, there are $n + 2$ kinds of products for a customer's choosing (i.e., $j = 1, 2 \ldots, n$, changes into $j = 0, 1, \ldots, n+1$), but nobody will choose these two fictitious products as the quality of product 0 is $V_0 = 0$ while the selling price of product $n + 1$ is $S + +\propto$. Therefore, introducing these two fictitious products has no effect on the result of the model.

Letting $\gamma_j^i = (S_i - S_j)/(V_i - V_j)$, $i = 1, 2, \ldots, n$, $j = 0, 1, \ldots, n + 1$, and $j \neq i$, we can get Lemma 1 as follows.

Lemma 1. *A customer of type* $\theta_i \in [\mathrm{Max}[\gamma_j^i, j = 0, 1, \ldots, i - 1], \mathrm{Min}[\gamma_j^i, j = i + 1, i + 2, \ldots, n + 1]]$ *is willing to buy product* i, $i = 1, 2, \ldots, n$.

Proof. Only if customer surplus gotten from buying product i, $i = 1, 2, \ldots, n$, is nonnegative; that is, $\theta_i V_i - S_i \geq 0$, and no less than that from product j, $j = 0, 1, \ldots, n+1$ and $j \neq i$; that is, $\theta_i V_i - S_i \geq \theta_i V_j - S_j$, will a customer of type θ_i buy product i.

Form $\theta_i V_i - S_i \geq 0$, we get $\theta_i \geq (S_i/V_i)$, so a customer of type $\theta_i \in [S_i/V_i, 1]$ is willing to buy product i if there is no other products. From $S_0 = V_0 = 0$ and $S_{n+1} = V_{n+1} = +\propto$, we get $\gamma_0^i = ((S_i - S_0)/(V_i - V_0)) = (S_i/V_i)$, $\gamma_{n+1}^i = ((S_i - S_{n+1})/(V_i - V_{n+1})) = 1$, so $\theta_i \in [S_i/V_i, 1]$ can be denoted as $\theta_i[\gamma_0^i, \gamma_{n+1}^i]$.

From $\theta_i V_i - S_i \geq \theta_i V_j - S_j$ and $V_1 < V_2 < \cdots < V_n$, we get $\theta_i \geq (S_i - S_j)/(V_i - V_j)$ when $j < i$ and $\theta_i \leq (S_i - S_j)/(V_i - V_j)$ when $j > i$.

Therefore, a customer of type $\theta_i \in [\text{Max}[\gamma_j^i, j = 0, 1, \ldots, i-1], \text{Min}[\gamma_j^i, j = i+1, i+2, \ldots, n+1]]$ is willing to buy product i. □

Letting $\gamma_L^i = \text{Max}[\gamma_j^i, j = 0, 1, \ldots, i-1]$, $L \in [0, i-1]$, $\gamma_U^i = \text{Min}[\gamma_j^i, j = i+1, i+2, \ldots, n+1]$, and $U \in [i+1, n+1]$, we can get Proposition 2 as follows.

Proposition 2. *The feasible set of products that an ATO manufacturer should assemble consists of products with qualities* $V_i > ((S_i - S_L) + (S_U - S_i)V_L)/(S_U - S_L)$, $i = 1, 2, \ldots, n$.

Proof. From Lemma 1, we know that a customer of type $\theta_i \in [\gamma_L^i, \gamma_U^i]$ is willing to buy product i, $i = 1, 2, \ldots, n$. Therefore, only when $\gamma_L^i < \gamma_U^i$, is there realistic demand on product i. Solving $\gamma_L^i = (S_i - S_L)/(V_i - V_L) < (S_i - S_L)/(V_i - V_L) = \gamma_U^i$, we can get $V_i > ((S_i - S_L) + (S_U - S_i)V_L)/(S_U - S_L)$, $i = 1, 2, \ldots, n$. Therefore, the feasible set of products that an ATO manufacturer should assemble consists of products with qualities $V_i > ((S_i - S_L) + (S_U - S_i)V_L)/(S_U - S_L)$, $i = 1, 2, \ldots, n$. □

Among the n kinds of products, there are N kinds of products with realistic demand, where $N = 1$ when there is only one kind of product with realistic demand, $N = n$ when all the n kinds of products with realistic demand. Therefore, we denote the number of product with realistic demand by N, $N = 1, 2, \ldots, n$.

As some of the products with realistic demand may be less profitable than other products of the same product family, or the holding cost per unit per time of their special component are too high, these products should not be produced or their selling duration should be shorter than other products. Therefore, after getting the feasible set of products with realistic demand, an ATO manufacturer will determine the final set of products which consists of products with positive selling duration in order to maximize its profit.

We rearrange these N kinds of products with realistic demand in order of their qualities as $V_1 < V_2 < \cdots < V_N$, and denote these products by product i, $i = 1, 2, \ldots, N$. Meanwhile, denote these N kinds of products in order of selling duration by product I. Now, we can get the new profit function of an ATO manufacturer as follows

$$\begin{aligned}\pi = {} & \sum_{I=1}^{N} (S_I Q_I) \\ & - H_c \sum_{k=1}^{n} \left\{ \frac{D_c^k (T_k - T_{k-1})^2}{2} \right. \\ & \left. + (T_k - T_{k-1}) \sum_{j=k}^{n} \left[D_c^{j+1} (T_{j+1} - T_j) \right] \right\} \\ & - \sum_{I=1}^{n} \sum_{k=1}^{I} \left\{ \frac{H_I D_I^k (T_k - T_{k-1})^2}{2} \right. \\ & \left. + H_I (T_k - T_{k-1}) \sum_{j=k}^{I-1} \left[D_I^{j+1} (T_{j+1} - T_j) \right] \right\} \\ & - P_c \sum_{I=1}^{N} Q_I - \sum_{I=1}^{N} (P_I Q_I) . \end{aligned} \tag{7}$$

Before analyzing the manufacturer's optimal product selection and components replenishment policies, we determine the market segmentation and substitution parameter of product i, m_i and ρ_{ji}, $i, j = 1, 2, \ldots, N$, $j \neq i$.

Lemma 3. *Customers of type* $\theta_i \in [\gamma_{i-1}^i, \gamma_{i+1}^i]$ *buy product i, so the market segmentation of product i is* $m_i = [\gamma_{i-1}^i - \gamma_{i+1}^i]$, $i = 1, 2, \ldots, N$.

Proof. Firstly, we apply reduction to absurdity to prove $\gamma_{i-1}^i = \text{Max}[\gamma_j^i, j = 1, 2, \ldots, i-1]$, $i = 1, 2, \ldots, N$.

Assuming that $\gamma_{i-k}^i = \text{Max}[\gamma_j^i, j = 1, 2, \ldots, i-1]$, where, $k = 2, 3, \ldots, i-1$; that is, $\gamma_{i-1}^i < \gamma_{i-k}^i$, we get that the surplus of customers of type $\theta_i \in [\gamma_{i-1}^i, \gamma_{i-k}^i]$ obtained from product i is more than that from product $i-1$ while less than that from product $i-k$, so they will buy product $i-k$. Form the fact that the upper limit of customer type θ_{i-1} is $\gamma_U^{i-1} = \text{Min}[\gamma_j^{i-1}, j = 1, 2, \ldots, n+1] \leq \gamma_i^{i-1} = \gamma_{i-1}^i$, we get $\theta_{i-1} \leq \theta_{i-k}$, while from the fact that $\theta_{i-1} \leq \gamma_{i+1}^i$, $\theta_{i+1} \geq \gamma_i^{i+1}$ and $\gamma_{i+1}^i = (S_i - S_{i+1})/(V_i - V_{i-1}) = (S_{i+1} - S_i)/(V_{i+1} - V_i) = \gamma_i^{i+1}$, we get $\theta_i \leq \theta_{i+1}$. Therefore, the assumption that $\gamma_{i-k}^i = \text{Max}[\gamma_j^i, j = 1, 2, \ldots, j-1]$ is wrong, which means that $\gamma_{i-1}^i = \text{Max}[\gamma_j^i, j = 1, 2, \ldots, i-1]$, $i = 1, 2, \ldots, N$.

Now, we prove that $\gamma_{i-1}^i = \text{Max}[\gamma_0^i, \gamma_{i-1}^i]$, $i = 1, 2, \ldots, N$.

When $i = 1$, $\gamma_0^i = \gamma_{i-1}^i$, so $\gamma_{i-1}^i = \text{Max}[\gamma_0^i, \gamma_{i-1}^i]$. When $i = 2, 3, \ldots, N$, the surplus of customers of type $\theta \geq \gamma_{i-1}^i$ obtained from buying product i is positive and bigger than that obtained from product $i-1$, or else product i should not be assembled, so $\gamma_{i-1}^i > \gamma_0^i$; that is, $\gamma_{i-1}^i = \text{Max}[\gamma_0^i, \gamma_{i-1}^i]$. Now, we can make the conclusion that $\gamma_{i-1}^i = \text{Max}[\gamma_0^i, \gamma_{i-1}^i]$, $i = 1, 2, \ldots, N$; that is, the lower limit of θ_i is γ_{i-1}^i.

Similarly as above, we can prove that if the upper limit of θ_i was not γ_{i+1}^i but γ_{i+k}^i, $k = 2, 3, \ldots, n+1-i$, the customers of type $\theta \in [\gamma_{i+1}^i, \gamma_{i+k}^i]$ should buy product $i+k$; that is, $\theta_{i+k} \leq \theta_{i+1}$, which contradicts $\theta_i \leq \theta_{i+1}$. Therefore, the upper limit of θ_i is γ_{i+1}^i.

From what was discussed above, we can make the conclusion that customers of type $\theta \in [\gamma_{i-1}^i, \gamma_{i+1}^i]$ are willing to buy product i, whose market segmentation is $m_i = (\gamma_{i+1}^i - \gamma_{i-1}^i)/(1-0) = \gamma_{i+1}^i - \gamma_{i-1}^i$, $i = 1, 2, \ldots, N$. □

Lemma 4. *When product i,* $i = 1, 2, \ldots, N$, *is in short supply, and products from i to* $i-k$ *and from i to* $i+l$, $k = 1, 2, \ldots, i-1$, $l = 0, 1, \ldots, N-i$, *are all in short supply too, the percentage of* *customers who were willing to buy product i, that is, customers of type* $\theta_i \in [\gamma_{i-1}^i, \gamma_{i+1}^i]$ *change to buy product j,* $j = 1, 2, \ldots, i-k-1, i+l+1, \ldots, N$*, as follows:*

$$\rho_j^i = \begin{cases} \dfrac{\gamma_{i+1}^i - \gamma_{i+l+1}^{i-l-1}}{m_i}, & j = i+l+1, \ j \neq N+1; \\ \dfrac{\gamma_{i+l+1}^{i-l-1} - \gamma_{i+1}^i}{m_i}, & j = i-k-1, \ j \neq 0; \\ 0, & other. \end{cases} \tag{8}$$

Proof. As the lower and upper limit of customer type θ_i is individually γ_{i-1}^i and γ_{i+1}^i, $i = 1, 2, \ldots, N$, when product i is in short supply, customers who were willing to buy it will just buy product $i-1$ or $i+1$. As a matter of course, if product $i+1$ is in short supply too, they will buy product $i-1$ or $i+2$, generally, if products from i to $i-k$ and from i to $i+l$, $k = 0, 1, \ldots, i-1$, $l = 0, 1, \ldots, N-i$, are all in short supply too, they will buy product $i-k-1$ or $i+l+1$. Therefore, $\rho_{ji} = 0$ when $j \neq i-k-1, i+l+1$.

When products from $i-k$ to $i+l$ are all in short supply, the upper limit of customer type θ_{i-k-1} and the lower limit of customer type θ_{i+l+1} change into γ_{i+l+1}^{i-k-1}, therefore, among the customers of type $\theta_i \in [\gamma_{i-1}^i, \gamma_{i+1}^i]$, those of type $\theta_i \in [\gamma_{i+l+1}^{i-k-1}, \gamma_{i+1}^i]$ will buy product $i+l+1$, and those of type $\theta_i \in [\gamma_{i-1}^i, \gamma_{i+l+1}^{i-k-1}]$ will buy product $i-l-1$, in other words, $\rho_{ji} = (\gamma_{i+1}^i - \gamma_{i+l+1}^{i-k-l})/m_i$, where $j = i+l+1$, and $\rho_{ji} = (\gamma_{i+l+1}^{i-k-l} - \gamma_{i-1}^i)/m_i$, where $j = i-l-1$.

As product 0 and $N+1$ are fictitious, we can get that $\rho_{(N+1)i} = \rho_{0i} = 0$. □

There are $N!$ kinds of permutations that products in order of qualities are ranged in order of selling duration, from [product 1, product 2,..., product N] to [product N, product $N-1$,..., product 1]. Therefore, we should analyze the ATO manufacturer's profits under these $N!$ kinds of permutations to obtain the optimal profit and selling duration of all products.

We take the first permutation, [product 1, product 2,..., product N], as an example to show the process of obtaining the optimal solution.

Under the first permutation, product i is product I, $i, I = 1, 2, \ldots, N$; therefore, from (1), we can get the selling quantity of product i and the replenishment quantity of special component i as

$$Q_i = \sum_{k=1}^{i} \left[D_i^k \left(T_k - T_{k-1} \right) \right], \quad i = 1, 2, \ldots, N. \tag{9}$$

Substituting (9) into (7), we can get the ATO manufacturer's profit under the first permutation as

$$\begin{aligned} \pi_1 = {} & \sum_{i=1}^{N} \left\{ \left(S_i - P_c - P_i \right) \sum_{k=1}^{i} \left[D_i^k \left(T_{k-T_{k-1}} \right) \right] \right\} \\ & - H_c \sum_{k=1}^{N} \left\{ \left(T_k - T_{k-1} \right) \sum_{j=k}^{N} \left[D_c^{j+1} \left(T_{j+1} - T_j \right) \right] \right. \\ & \left. + \frac{D_c^k \left(T_k - T_{k-1} \right)^2}{2} \right\} \\ & - \sum_{i=1}^{N} \sum_{k=1}^{i} \left\{ \frac{H_i D_i^k \left(T_k - T_{k-1} \right)^2}{2} \right. \\ & \left. + H_i \left(T_k - T_{k-1} \right) \sum_{j=k}^{i-1} \left[D_i^{j+1} \left(T_{j+1} - T_j \right) \right] \right\}. \end{aligned} \tag{10}$$

The optimal selling duration of product i, T_i^*, $i = 1, 2, \ldots, N$, is the solution of first order derivative of the ATO manufacturer's profit, π_1, with respect to the selling duration, T_i, equaling 0, that is, the solution of the following equation:

$$\begin{aligned} \frac{\partial \pi_1}{\partial T_i} = {} & \sum_{j=i}^{N} \left[\left(S_j - P_j - P_c \right) D_j^i \right] - \sum_{j=i+1}^{N} \left[\left(S_j - P_j - P_c \right) D_j^{i+1} \right] \\ & - H_c \left(D_c^i - D_c^{i+1} \right) T_i - T_i \sum_{j=i+1}^{N} \left[H_j \left(D_j^i - D_j^{i+1} \right) \right] \\ & - H_i D_i^i T_i = 0, \quad i = 1, 2, \ldots, N. \end{aligned} \tag{11}$$

Solving (11), we can get

$$\begin{aligned} T_i^* = {} & \left(\sum_{j=i}^{N} \left[\left(S_j - P_j - P_c \right) D_j^i \right] - \sum_{j=i+1}^{N} \left[\left(S_j - P_j - P_c \right) D_j^{i+1} \right] \right) \\ & \times \left(H_c \left(D_c^i - D_c^{i+1} \right) + H_i D_i^i + \sum_{j=i+1}^{N} \left[H_j \left(D_j^i - D_j^{i+1} \right) \right] \right)^{-1}, \\ & i = 1, 2, \ldots, N. \end{aligned} \tag{12}$$

Substituting T_i^* into (9), we get the optimal purchase quantity of product i as

$$Q_i^* = \sum_{k=1}^{i} \left[D_i^k \left(T_k^* - T_{k-1}^* \right) \right], \quad i = 1, 2, \ldots, N. \tag{13}$$

Similarly, we can get the function of the selling quantity of product i and special component i, Q_i, $i = 1, 2, \ldots, N$, as well as the ATO manufacturer's profit, π_j, under the jth permutation, $j = 1, 2, \ldots, N!$. Subsequently, solving $\partial \pi_j / \partial T_I = 0$, we can get the optimal selling duration of product I, T_I^*, $I = 1, 2, \ldots, N$, as well as the optimal selling duration and replenishment quantity of product i, T_i^* and Q_i^*, according to the corresponding relationship between product i and product I under the jth permutation.

Proposition 5. *The optimal replenishment quantity of special component i, Q_i^*, $i = 1, 2, \ldots, N$, increases with the extending of the optimal selling duration of product i, T_i^*, while it is unrelated to the optimal selling duration of product j, T_i^*, $j = 1, 2, \ldots, N$ and $j \neq i$ if product i is in short supply before product j or decreases with the extending of T_j^* if products from j to $i-1$ is in short supply before product i.*

Proof. Solving the first order derivative of Q_i^* with respect to T_i^*, we can get $\partial Q_i^* / \partial T_i^* = D_i^i > 0$, $i = 1, 2, \ldots, N$. Therefore, Q_i^* increases with the extending of T_i^*.

Let product i and product j, $j = 1, 2, \ldots, N$ and $j \neq i$, individually be the $\tilde{k}$th and $\tilde{k}$th product of being in short supply. If $\hat{k} > \tilde{k}$, we can get that Q_i^* is unrelated to form $Q_i^* = \sum_{k=1}^{\tilde{k}} [D_i^k (T_k^* - T_{k-1}^*)]$. From Lemma 4, we can get that $\rho_{ij} > 0$ if products from j to $i-1$ are in short supply before product i. Therefore, we can get that Q_i^* decreases with the extending of T_j^* from $\partial Q_i^* / \partial T_i^* = D_i^{\hat{k}} - D_i^{\hat{k}+1} < 0$. □

Proposition 6. *The optimal selling duration of product i, T_i^*, $i = 1, 2, \ldots, N$, is not related to the potential total demand per time of whole product family, D, but related to the products' selling price, the purchase price, and the holding cost per unit of components.*

Proof. Substituting $Q_I = \sum_{k=1}^{I} [D_I^k (T_k - T_{k-1})]$ into (7), we can get

$$\pi = \sum_{i=I}^{N} \left\{ (S_I - P_I - P_c) \sum_{k=1}^{I} \left[D_I^k (T_k - T_{k-1}) \right] \right\} - H_c \sum_{k=1}^{N} \left\{ (T_k - T_{k-1}) \sum_{j=k}^{N} \left[D_c^{j+1} \left(T_{j+1} - T_j \right) \right] + \frac{D_c^k (T_k - T_{k-1})^2}{2} \right\} - H_I \sum_{I=1}^{N} \sum_{k=1}^{I} \left\{ \frac{D_I^k (T_k - T_{k-1})^2}{2} + (T_k - T_{k-1}) \sum_{j=k}^{I-1} \left[D_I^{j+1} \left(T_{j+1} - T_j \right) \right] \right\}. \tag{14}$$

From (14), we can find that there are the demand of special component in the kth duration, D_I^k, $k \leq I$, or the demand of common component in the kth duration D_c^k, $k \leq N$, in every formula of the function of π, as well as the first order derivative of π with respect to T_I, $\partial \pi / \partial T_I$. Furthermore, from (1) and (2), we know that both D_I^k and D_c^k are equal to the potential total demand per unit of whole product family, D, multiplied by a certain constant. Therefore, we can eliminate D from equation $\partial \pi / \partial T_I = 0$, and get the optimal selling duration, T_i^*,which is unrelated to D but related to the products' selling price, S_I, the purchase price and holding cost per unit of components, P_I, P_c and H_I, H_c. □

4. The Algorithm

Step 1. Determine the feasible set of products with realistic demand by $V_i > ((S_i - S_L)V_U + (S_U - S_i)V_L)/(S_U - S_L)$, and arrange these products in order of their qualities, that is, $V_1 < V_2 \cdots < V_N$.

Step 2. Determine the market segment of product i, m_i, and its substitution parameter, ρ_{ji}, $i, j = 1, 2, \ldots, N$, $j \neq i$.

Step 3. Determine the function of selling quantity of product i and replenishment quantity of special component i from Lemmas 3 and 4, and the ATO manufacturer's profit π_j under the jth permutation, $j = 1, 2, \ldots, N!$.

Step 4. Solving $\partial \pi_j / \partial T_I = 0$ to get the optimal selling duration of product I, T_I^*, $I = 1, 2, \ldots, N$, as well as the optimal selling duration and replenishment quantity of products i, T_i^*, and Q_i^*, as well as the optimal profit, π_j^*, according to the corresponding relationship between product i and product I under the jth permutation.

Step 5. Comparing the optimal profit, π_j^*, under all $N!$ kinds of permutation to get the ATO manufacturer's optimal profit $\pi^* = \text{Max}\, \pi_j^*$, $j = 1, 2, \ldots, N!$.

Step 6. Export π^* and corresponding optimal selling duration of product i, T_i^*, the optimal replenishment quantity of special component i and common component c, Q_i^*, and Q_c^*, $i = 1, 2, \ldots, N$.

5. Numerical Analysis

In this section, we provide a numerical example to illuminate our model and propositions. We consider that an ATO manufacturer is planning to select product to assemble from product family consisting of three products, among which product i consists of one unit common component c and special component i, $i = 1, 2, 3$. The parameters of the ATO manufacturer are as follows: $D = 10000$, $S_1 = 1450$, $S_2 = 1800$, $S_3 = 2350$, $P_1 = 550$, $P_2 = 850$, $P_3 = 1350$, and $P_c = 600$, $H_1 = 20$, $H_2 = 40$, $H_3 = 75$, and $H_c = 35$.

Firstly, from $V_i > ((S_i - S_L)V_U + (S_U - S_i)V_L)/(S_U - S_L)$, we get the feasible product set consisting of product 1 and 3. Therefore, there are two kinds of permutation [product 1, product 3] and [product 3, product 1].

Secondly, we get the market segmentation of product 1 and 3 as $m_1 = 0.055$ and $m_3 = 0.182$, as well as the substitution parameter $\rho_{31} = 0.633$ and $\rho_{13} = 1$.

Thirdly, we can obtain the optimal profit under the first permutation as $\pi_1^* = 1.53 \times 10^6$ and that under the second permutation as $\pi_2^* = 2.10 \times 10^6$.

Finally, comparing π_1^* with π_2^*, we get the ATO manufacturer's optimal profit as $\pi^* = 2.10 \times 10^6$, and the corresponding optimal solutions as $T_1^* = 5.45$, $T_3^* = 1.82$, $Q_1^* = 9613$, $Q_3^* = 3306$, and $Q_c^* = 12919$.

Table 1: The optimal solutions with respect to S_1.

S_1	T_1^*	T_2^*	T_3^*	m_1	m_2	m_3	π^*
1350	3.64	—	3.64	0.199	—	0.091	1383210
1400	4.55	—	2.73	0.127	—	0.136	1774140
1450	5.45	—	1.82	0.055	—	0.182	2103090
1500	—	4.67	1.42	—	0.003	0.214	1851890

We further show how the selling price of product 1 impacts the optimal solutions, the impact of the selling price of products 2 and 3 is similar.

From Table 1, we can find that m_1 and T_3^* decrease while T_1^*, m_3, and π^* increase with the increase of S_1 because parts of customers who were planning to buy product 1 give up and parts buy product 3. Meanwhile, as the cost of product 1 remains unchanged, its marginal profit becomes greater and the ATO manufacturer prefers to sell more of product 1. From Lemma 4, we know that when product 3 is in short supply, all its customers change to buy product 1, therefore, the ATO manufacturer should reduce T_3^* while extending T_1^* to earn more profit, π^*. With the further increase of S_1, no customer buys it, parts of customers who were planning to buy it change to buy product 2, and parts buy product 3, so the product set consists of product 2 and 3, and m_3 gets a further expanding.

6. Case Analysis

In this section, we provide a real case of ChongQing JianShe Motorcycle Co., LTD. (shortened as JianShe) to illuminate our model and propositions. JianShe is a state-owned enterprise providing products including motorcycle manufacturing, distribution, related services, and automotive components. We take the series of JS110-B as the subject to study its product selection and components replenishment policies.

The JS110-B series consists of 3 models, including LingYa, LingYing, and BF3, in other words, $n = 3$. The features of JS110-B are as follows. Firstly, it is of fashionable and unique appearance with sporty style. Secondly, low fuel consumption, which is economical; small vibration, which enables you to enjoy the comfort like a car. Thirdly, different configurations are available, including cargo carrier, medium carriers, small rear carriers, burglar alarm, as well as wide and narrow tires, to meet the needs of different users.

As the demand for BF3 model is rare, JianShe has stopped providing BF3, we should remove it from our feasible set of products with realistic demand, in other words, $N = 2$. Now, there are 2 models of products with realistic demand, LingYa and LingYing, for which we should determine the optimal selling duration and components replenishment quantities.

The special components of LingYing model (labeled as product 1) and LingYa model (labeled as product 2) are covers and an engine, the other components are common components. We simplify covers and engine as a special component, and label the one for LingYing model as special component 1 and the one for LingYa model as special component 2. Furthermore, we simplify all common components as a common component labeled as common component c.

As JianShe replenish the components for LingYing and LingYa every two months, we treat a replenishment circle as a selling season to analyze our model. Through a survey of market and JianShe's operation management, we get that the parameters of LingYing model and LingYa model are as follows: $D = 3000$ units per day, $S_1 = 4050$ yuan per unit, $S_2 = 4250$ yuan per unit, $P_1 = 1275$ yuan per unit, $P_2 = 1450$ yuan per unit, $P_c = 1800$ yuan per unit, $H_1 = 7$ yuan per unit per day, $H_2 = 7.5$ yuan per unit per day, and $H_c = 10$ yuan per unit per day.

Following the algorithm put forward in this paper, we get the optimal replenishment policy for LingYing and LingYa as follows. The optimal selling duration and quantity of LingYing are individually $T_1^* = 57.5$ days and $Q_1^* = 10566$ units, and those of LingYa are individually $T_2^* = 50$ days and $Q_2^* = 3659$ units. Therefore, JianShe should replenish special component $Q_1^* = 10566$ units for LingYing, $Q_2^* = 3659$ units for LingYa, as well as component $Q_c^* = 14225$ units for these two models.

According to our survey, the average selling quantity per month in 2009 of LingYing is 5770 units and that of LingYa is 1769 units. Our theoretical result, $Q_1^*/2 = 5283$ units and $Q_2^*/2 = 1830$ units, is close to the reality.

7. Conclusions

We developed an inventory model, in which customers are heterogeneous and their type follows uniform distribution, to study how an ATO manufacturer makes its product selection and components replenishment policies. Furthermore, we proposed an algorithm to obtain the optimal solution, and use a numerical example and a real case to illuminate the model. We found that an ATO manufacturer should make the performance and qualities of products over a certain level, and determine the optimal selling duration of products according to their selling prices and qualities, as well as the purchase prices and holding costs of their components.

Acknowledgments

This research is supported by the National Natural Science Foundation of China (no. 71102178), the Open Fund of Chongqing Key Laboratory of Logistics (no. CQKLL12006), the Fundamental Research Funds for the Central Universities (no. CQDXWL-2012-183).

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