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Research Article

Permanence and Global Attractivity of a Discrete Logistic Model with Impulses

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By piecewise Euler method, we construct a discrete logistic equation with impulses. The constructed model is more easily implemented at computer and is a better analogue of the continuous-time dynamic system. The dynamic behaviors of the constructed model are investigated. Sufficient conditions which guarantee the permanence and the global attractivity of positive solutions of the model are obtained. Numerical simulations show the feasibility of the main results.

1. Introduction

It is well known that impulsive differential equations have been considered by many authors (see, e.g., [1–7]). Such equations may exhibit several real world phenomena, such as rhythmical beating, merging of solutions, and noncontinuity of solutions. Since it is much richer than the corresponding theory of differential equations without impulsive effects, the theory of impulsive differential equations is emerging as an important area of investigation of model.

At all times, continuous-time dynamic systems play an important role in control theory, networks design, and so on. However, with the development of computer techniques, when implementing the continuous-time dynamic systems for computer simulation, for experimental or computational purposes, it is essential to formulate discrete-time dynamic systems which are an analogue of the continuous-time dynamic systems. These discrete-time systems, which are described by difference equations, inherit the similar dynamical characteristics. Because of that, more and more researchers pay their attention to the dynamical behaviors of difference equations (see [8–12]). However, few papers investigate the discrete model with impulses (see [13–17]). The main difficulty of constructing discrete model with impulses is how to describe impulsive moment. The main purpose of this paper is to construct the discrete model with impulses and investigate the permanence and global stability of the discrete model with impulses.

By piecewise Euler method, we construct the following discrete logistic model with impulses:

$$\begin{gathered} x_{m_k+l+1} = x_{m_k+l} \exp\left\{r\left(1 - \frac{x_{m_k+l}}{K}\right)\right\}, \\ x_{m_k+0} = \left(1+b_k\right) x_{m_k}, \\ l = 0, 1, 2, \ldots, \left(m_{k+1} - m_k - 2\right), \left(m_{k+1} - m_k - 1\right), \\ k = 0, 1, 2, \ldots, \end{gathered} \tag{1}$$

where the fixed moments of time m_k satisfy $m_0 = 0$, $m_k < m_{k+1}$, and $\lim_{k\to\infty} m_k = +\infty$. r is the intrinsic rate and K is the carrying capacity of the system; $x_0 = x_{0+0}$ is the initial value.

In this paper, we always assume that $0 < B_1 < 1 + b_k < B_2$, $k = 1, 2, 3, \ldots$.

We simply explain model (1).

It is obvious that $x_{m_k+l} > 0$, $k = 0, 1, 2, \ldots$, $l = 0, 1, 2, \ldots, (m_{k+1} - m_k - 2), (m_{k+1} - m_k - 1)$, for $x_0 > 0$.

When $k = 0$, $x_{l+1} = x_l \exp\{r(1 - (x_l/K))\}$, $l = 0, 1, 2, \ldots, m_1 - 1$. We can obtain that $x_1, x_2, \ldots, x_{m_1}$ and $x_{m_1+0} = (1 + b_1)x_{m_1}$.

When $k = 1$, $x_{m_1+l+1} = x_{m_1+l}\exp\{r(1-(x_{m_1+l}/K))\}$, $l = 0, 1, 2, \ldots, m_2 - m_1 - 1$. We can obtain that $x_{m_1+1}, x_{m_1+2}, \ldots, x_{m_2}$, and $x_{m_2+0} = (1+b_2)x_{m_2}$, and so on.

In some papers, authors use m_k^+ to denote impulsive moment (see [15]). It is obvious that describing the impulsive moment of model (1) is easily realized at computer. In addition, some authors use $m_k + 1$ to denote impulsive moment (see [13, 16]). Compared with it, model (1) is a better analogue of the continuous-time dynamic system.

The organization of this paper is as follows. In the next two sections, we give sufficient conditions on permanence and global stability of system (1), respectively. In Section 4, two examples are given. To conclude this paper, a discussion follows in Section 5.

2. Permanence

For convenience, let $n = m_k + l$.

Definition 1. The species x_n of system (1) is said to be permanence if there exist positive constants M and m such that each positive solution x_n of the system satisfies

$$m \le \liminf_{n\to\infty} x_n \le \limsup_{n\to\infty} x_n \le M. \tag{2}$$

If the species of the system is permanence, then the system is called permanent.

Lemma 2. *There exists a positive constant M, such that for every solution x_n of system (1), one has*

$$x_n \le M, \quad \text{for } n > 0. \tag{3}$$

Proof. To prove (3), we have two cases.

Case I. For any $x_0 > 0$, we have

$$\begin{aligned} x_{n+1} &= x_n \exp\left\{r\left(1-\frac{x_n}{K}\right)\right\} \\ &\le \frac{K\exp\{r-1\}}{r} \triangleq M_1. \end{aligned} \tag{4}$$

Here, we used

$$\max_{x\ge 0}\{x\exp\{b-ax\}\} = \frac{\exp\{b-1\}}{a}. \tag{5}$$

Case II. $n = m_k + 0$, $k = 1, 2, 3, \ldots$. We have

$$\begin{aligned} x_{m_k+0} &= (1+b_k)x_{m_k} \\ &= (1+b_k)x_{m_k-1}\exp\left\{r\left(1-\frac{x_{m_k-1}}{K}\right)\right\} \\ &\le B_2\frac{K\exp\{r-1\}}{r} \triangleq \widehat{M}_1. \end{aligned} \tag{6}$$

Let $M = \max\{M_1, \widehat{M}_1\}$. This completes the proof. □

Lemma 3. *Assume there exists a positive constant B_3, such that $B_3 < \prod_k^\infty (1+b_k)$. Then there exist two positive constants m and n_1, such that for every solution x_n of system (1), one has $x_n > m$ for $n > n_1$.*

Proof

Step I. Let $0 < m' < K$. We will firstly prove that there exists n_1 such that $x_{n_1} \ge m'$. Otherwise, for all n, $x_n < m'$,

$$\begin{aligned} x_{n+1} &= x_n\exp\left\{r\left(1-\frac{x_n}{K}\right)\right\} \\ &\ge x_n\exp\left\{r\left(1-\frac{m'}{K}\right)\right\}. \end{aligned} \tag{7}$$

Therefore, there exists a constant k', such that if $k > k'$,

$$\begin{aligned} x_{m_{k+1}} &\ge x_{m_{k+1}-1}\exp\left\{r\left(1-\frac{m'}{K}\right)\right\} \\ &\ge x_{m_{k+1}-2}\exp\left\{2r\left(1-\frac{m'}{K}\right)\right\} \\ &\vdots \\ &\ge x_{m_k+0}\exp\left\{(m_{k+1}-m_k)r\left(1-\frac{m'}{K}\right)\right\} \\ &= (1+b_k)x_{m_k}\exp\left\{(m_{k+1}-m_k)r\left(1-\frac{m'}{K}\right)\right\} \\ &\ge x_{m_{k'}}\prod_{i=k'}^{k}(1+b_i)\exp\left\{(m_{k+1}-m_{k'})r\left(1-\frac{m'}{K}\right)\right\} \\ &\longrightarrow \infty \end{aligned} \tag{8}$$

as $k \to \infty$, which contradicts the boundedness of x_n. Hence, we conclude that there exists n_1 such that $x_{n_1} \ge m'$.

Step II. If $x_n \ge m'$, for all $n \ge n_1$, then our aim is achieved. Otherwise, $x_n < m'$ for some $n > n_1$. Set $n^* = \min_{n>n_1}\{x_n < m'\}$. We have $x_n \ge m'$ for $n \in [n_1, n^*)$.

Assume that $n^* \in [m_k, m_{k+1})$. It is easy to see that $x_{n^*} < m'$, $x_{n^*-1} \ge m'$.

There are two possible cases for $n \in [n^*, m_{k+1})$.

Case A. There exists a $n'' \in [n^*, m_{k+1})$, such that $x_{n''} \ge m'$. Let $\hat{n} = \min_{n>n^*}\{x_n \ge m'\}$; then $x_n < m'$ for $n \in [n^*, \hat{n})$ and $x_{\hat{n}} \ge m'$. Select an M such that

$$x_n < M, \qquad \max(1, B_1)\exp\left(r\left(1-\frac{M}{K}\right)\right) < 1. \tag{9}$$

If $n^* \ne m_k + 0$, then

$$\begin{aligned} x_{n^*} &= x_{n^*-1}\exp\left(r\left(1-\frac{x_{n^*-1}}{K}\right)\right) \\ &\ge m'\exp\left(r\left(1-\frac{M}{K}\right)\right). \end{aligned} \tag{10}$$

If $n^* = m_k + 0$, then $m_k \ge n_1$.

Case 1 ($m_k = n_1$). Consider

$$x_{n^*} = (1+b_k)x_{m_k} \ge B_1 m'. \tag{11}$$

Case 2 ($m_k > n_1$). Consider

$$\begin{aligned} x_{n^*} &= (1+b_k)\,x_{m_k} \\ &\geq (1+b_k)\,x_{m_k-1}\exp\left(r\left(1-\frac{x_{m_k-1}}{K}\right)\right) \\ &\geq (1+b_k)\,m'\exp\left(r\left(1-\frac{M}{K}\right)\right) \\ &\geq B_1 m'\exp\left(r\left(1-\frac{M}{K}\right)\right). \end{aligned} \tag{12}$$

For $n \in [n^*, \widehat{n}-1)$,

$$\begin{aligned} x_{n+1} &\geq x_n \exp\left(r\left(1-\frac{m'}{K}\right)\right) \\ &\ \ \vdots \\ &\geq x_{n^*}\exp\left((n+1-n^*)\,r\left(1-\frac{m'}{K}\right)\right). \end{aligned} \tag{13}$$

If $n^* \neq m_k + 0$, then

$$\begin{aligned} x_{n+1} &\geq x_{n^*}\exp\left((n+1-n^*)\,r\left(1-\frac{m'}{K}\right)\right) \\ &\geq m'\exp\left(r\left(1-\frac{M}{K}\right)\right)\exp\left(r\left(1-\frac{m'}{K}\right)\right) \\ &\geq m'\exp\left(r\left(1-\frac{M}{K}\right)\right) \triangleq \overline{m}. \end{aligned} \tag{14}$$

If $n^* = m_k + 0$, then

$$\begin{aligned} x_{n+1} &\geq x_{n^*}\exp\left((n+1-n^*)\,r\left(1-\frac{m'}{K}\right)\right) \\ &\geq (1+b_k)\,m'\exp\left(r\left(1-\frac{M}{K}\right)\right) \\ &\quad\times\exp\left((n+1-n^*)\,r\left(1-\frac{m'}{K}\right)\right) \\ &\geq (1+b_k)\,m'\exp\left(r\left(1-\frac{M}{K}\right)\right)\exp\left(r\left(1-\frac{m'}{K}\right)\right) \\ &\geq B_1 m'\exp\left(r\left(1-\frac{M}{K}\right)\right) \triangleq \widetilde{m}. \end{aligned} \tag{15}$$

The argument of Step II can be continued since $x_{\widehat{n}} \geq m'$, and we have $x_n \geq m = \min(\widetilde{m}, \overline{m})$ for all $n > n_1$.

Case B. $x_n < m'$ for $n \in [n^*, m_{k+1})$. If $x_n \geq m'$ for all $n \geq m_{k+1}$, then our aim is achieved. Otherwise, $x_n < m'$ for some $n > m_{k+1}$. Set $\overline{n} = \min_{n>m_{k+1}}\{x_n < m'\}$.

For $n > \overline{n}$, the argument of step II can be continued since $x_n \geq m'$ for $n \in [m_{k+1}, \overline{n})$; hence $x_n \geq m = \min(\widetilde{m}, \overline{m})$, for all $n > m_{k+1}$. The proof is complete. □

Combining Lemmas 2 and 3, we have proved the main result of this paper, which is stated next.

Theorem 4. *If the condition of Lemma 3 holds, system* (1) *is permanent.*

3. Global Attractivity of the Positive Solution

Next we discuss the global attractivity of the positive solution of system (1). In the following we say a positive solution of system (1) is globally asymptotically stable if it attracts all other positive solutions of the system.

Let $\widetilde{x_n} = \ln(x_n)$. Then system (1) becomes the following form:

$$\begin{aligned} \widetilde{x_{n+1}} &= \widetilde{x_n} + r\left(1-\frac{e^{\widetilde{x_n}}}{K}\right), \\ \widetilde{x_{m_k+0}} &= \ln(1+b_k) + \widetilde{x_{m_k}}. \end{aligned} \tag{16}$$

Theorem 5. *If* $0 < M < 2(K/r)$, *then for any two positive solutions* x_n *and* y_n *of system* (1), *one has*

$$\lim_{n\to\infty}|x_n - y_n| = 0. \tag{17}$$

Proof. Let $\widetilde{x_n} = \ln(x_n)$ and let $\widetilde{y_n} = \ln(y_n)$. Then

$$\begin{aligned} \left|\widetilde{x_{n+1}} - \widetilde{y_{n+1}}\right| &= \left|\widetilde{x_n} - \widetilde{y_n} - \frac{r}{K}\left(e^{\widetilde{x_n}} - e^{\widetilde{y_n}}\right)\right| \\ &= \left|\widetilde{x_n} - \widetilde{y_n} - \frac{r}{K}e^{\xi(n)}\left(\widetilde{x_n} - \widetilde{y_n}\right)\right| \\ &= \left|\left(1-\frac{r}{K}e^{\xi(n)}\right)\right|\left|\left(\widetilde{x_n} - \widetilde{y_n}\right)\right|, \end{aligned} \tag{18}$$

where $e^{\xi(n)}$ is between x_n and y_n and $0 < m < e^{\xi(n)} < M$ for sufficiently large n.

In view of $0 < M < 2(K/r)$ such that

$$\left|\left(1-\frac{r}{K}e^{\xi(n)}\right)\right| < 1, \tag{19}$$

hence,

$$\lim_{n\to\infty}\left|\widetilde{x_n} - \widetilde{y_n}\right| = 0, \tag{20}$$

which implies that

$$\lim_{n\to\infty}|x_n - y_n| = 0. \tag{21}$$

This completes the proof. □

4. Examples and Numerical Simulation

Example 1. Corresponding to the system (1), we assume that

$$\begin{gathered} r = 0.2, \qquad K = 100, \qquad m_0 = 0, \\ m_k = 2^k, \quad b_k = \sin k, \quad k = 1, 2, \ldots. \end{gathered} \tag{22}$$

It is easy to see that the conditions in Theorem 4 are verified. Therefore, (1) is permanent (see Figure 1).

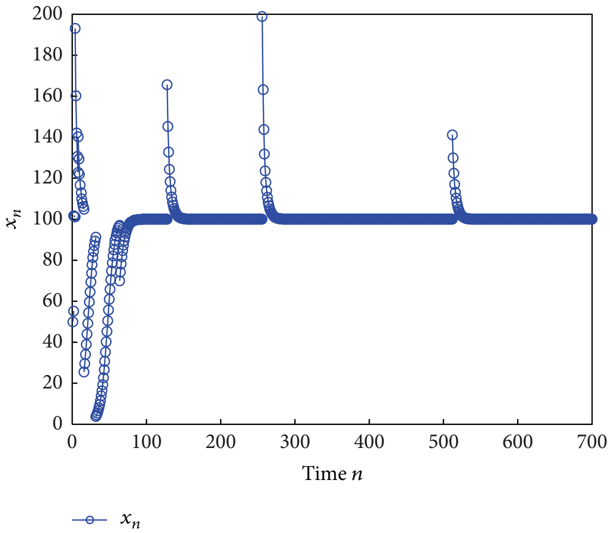

FIGURE 1: Permanence of system (1) with initial condition $x_0 = 50$.

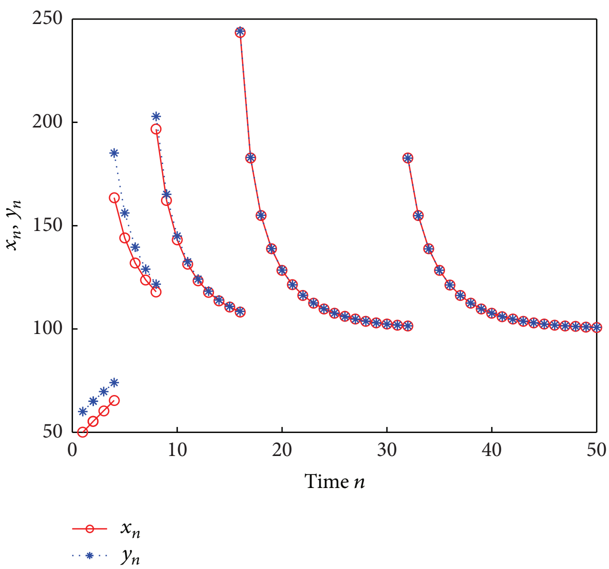

FIGURE 2: Global attractivity of system (1) with initial conditions $x_0 = 50$ and $y_0 = 60$.

Example 2. Corresponding to the system (1), we assume that

$$r = 0.2, \qquad K = 100, \qquad m_0 = 0,$$
$$m_k = 2^k, \quad b_k = \left(1 + (-1)^k \frac{1}{k}\right), \quad k = 1, 2, \ldots. \tag{23}$$

It is easy to see that the conditions in Theorem 5 are verified. Therefore, (1) is global attractivity. Our numerical simulation supports our result (see Figure 2).

5. Conclusion

In this paper, by piecewise Euler method, we construct a discrete logistic equation with impulses. The model gives a new form of describing the impulsive moment, and the model is more easily implemented at computer and is a better analogue of the continuous-time dynamic system. The permanence and the global attractivity of positive solutions of the model are investigated. In our opinion, this discrete idea of the paper can apply to more complex model with impulses, such as model with impulses and delay and multigroup mode with impulses and delay. The other dynamic behaviors of discrete model can be researched, such as stability, periodic solution, and invariant sets.

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