Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 706496, 6 pages
http://dx.doi.org/10.1155/2013/706496

Research Article

Hyperbolic Tessellation and Colorings of Trees

Dong Han Kim[1] and Seonhee Lim[2]

[1] *Department of Mathematics Education, Dongguk University-Seoul, Seoul 100-715, Republic of Korea*
[2] *Department of Mathematical Sciences, Seoul National University, Seoul 151-747, Republic of Korea*

Correspondence should be addressed to Seonhee Lim; slim@snu.ac.kr

Received 22 February 2013; Accepted 3 May 2013

Academic Editor: Baodong Zheng

We study colorings of a tree induced from isometries of the hyperbolic plane given an ideal tessellation. We show that, for a given tessellation of the hyperbolic plane by ideal polygons, a coloring can be associated with any element of $\text{Isom}(\mathbb{H}^2)$, and the element is a commensurator of Γ if and only if its associated coloring is periodic, generalizing a result of Hedlund and Morse.

1. Introduction

Let T be a locally finite tree, VT its vertex set, and ET the set of oriented edges of T. Let $\mathscr{A}$ be a countable set which will be called the alphabet. Let ϕ be a coloring of T, that is, a map $\phi : VT \to \mathscr{A}$. Let $\text{Aut}(T)$ be the automorphism group of T. A periodic coloring is a coloring which is Γ-invariant for some cocompact subgroup $\Gamma \subset \text{Aut}(T)$.

In this paper, we study colorings of regular trees induced from some tessellations of the hyperbolic plane.

There is a well-known family of sequences coming from rotations of circle as follows. Consider the tiling of the real line $\mathbb{R}$ by unit length intervals $\{[n, n+1) : n \in \mathbb{Z}\}$ and a map $t \mapsto at + b$ from $\mathbb{R}$ to itself. There exists an integer j such that each interval $[n, n+1)$ is partitioned into j or $j+1$ subintervals of the form $\{[an+b, a(n+1)+b) : n \in \mathbb{Z}\} \cap \{[n, n+1) : n \in \mathbb{Z}\}$. Consider the sequence $(u_n)_{n\in\mathbb{N}}$ with $u_n \in \mathscr{A} = \{j, j+1\}$, which is given by the number of such subintervals of $[n, n+1)$. It is well known that this two-sided sequence u_n is periodic if only if a is rational [1].

As a generalization, we associate a coloring ϕ_g of a k-regular tree ($k \geq 3$) for any isometry g of the hyperbolic plane, given a specific hyperbolic tessellation $\mathscr{D}$ generated by a discrete subgroup Γ of the group of isometries on the hyperbolic plane $\mathbb{H}^2$. Suppose that each vertex of elements of $\mathscr{D}$ lies on the boundary of the hyperbolic plane so that the dual graph of $\mathscr{D}$ is a tree.

For such a tessellation $\mathscr{D}$, we show that the coloring ϕ_g is periodic if and only if g is a commensurator of Γ in $\text{Isom}(\mathbb{H}^2)$. Recall that an element $g \in \text{Isom}(\mathbb{H}^2)$ is called a *commensurator of* Γ if and only if $g\Gamma g^{-1} \cap \Gamma$ is a subgroup of Γ and of $g\Gamma g^{-1}$ of finite index. Let us denote the group of commensurators of Γ by $\text{Comm}(\Gamma)$. Commensurator subgroup $\text{Comm}(\Gamma)$ plays an important role in the study of rigidity of locally symmetric spaces and more generally in geometric group theory ([2–4]).

This is a result analogous to the rotation case in the sense that the group of commensurators of $\text{SL}_2(\mathbb{Z})$ is a group containing $\text{SL}_2(\mathbb{Q})$ with finite index [5].

After showing the main theorem (Theorem 3), we show that our construction is an analogue of sequences induced from a rotation of circle only when the multiplicative constant a of $t \mapsto at + b$ is rational.

We show that, in the case of an isometry of $\mathbb{H}^2$ which is not a commensurator, we obtain colorings of unbounded alphabet, in contrast with the motivating example where irrational rotations correspond to Sturmian sequences, which are in particular sequences with a finite alphabet (see Section 3 for details).

We then explain in a heuristic way how to obtain eventually periodic colorings and colorings of "low complexity" by disregarding some information of the induced colorings.

2. Periodic Tree Colorings from Hyperbolic Tessellations

We first reformulate the classical example of two-sided sequences mentioned in Section 1. Consider the tessellation $\mathscr{D}$ of the hyperbolic plane (upper-half plane) $\mathbb{H}^2$ given by the group Γ_r generated by the reflections about the lines $x = 0$ and $x = 1$. More precisely, elements of $\mathscr{D}$ are of the form $\{z \in \mathbb{C} : n \leq \mathrm{Re}(z) < n+1\}$. Then $\Gamma_r = \langle z \mapsto -\overline{z}, z \mapsto 2-\overline{z}\rangle$ is isomorphic to the infinite dihedral group, and its dual graph T is a 2-regular tree. Let $g = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{PSL}_2(\mathbb{R})$, which sends $z \in \mathbb{H}^2$ to $(az+b)/(cz+d)$. Then it is not difficult to check that g is a commensurator of Γ_r if and only if $c = 0$, and a/d is rational.

For each vertex $x \in VT$, denote by $D_x \in \mathscr{D}$ the element of $\mathscr{D}$ dual to x. Let $\phi_g^{\#}$ be the coloring given by

$$\phi_g^{\#}(x) = \#\{gD : gD \cap D_x \neq \emptyset, D \in \mathscr{D}\}. \tag{1}$$

Then $\phi_g^{\#}$ is periodic if a/d is rational, $c = 0$, and Sturmian if a/d is irrational, $c = 0$ (e.g., [1], [9, Chapter 6]).

Let us generalize the above construction. Let us fix an ideal polygon D in the hyperbolic plane $\mathbb{H}^2$. Consider the group Γ_r generated by the reflections in the edges of D, which is a discrete subgroup of finite covolume in the isometry group of $\mathbb{H}^2$. By Poincaré's theorem on fundamental polygons, there exists a tessellation $\mathscr{D}$ of $\mathbb{H}^2$ by the images of D by the elements of Γ_r.

Let T be the dual graph of the tessellation $\mathscr{D}$, which is a tree since D is an ideal polygon. The tree T is the Cayley graph of the group Γ_r.

More generally, we will also consider the case when Γ_r is generated by the reflections in the edges of a *generalized ideal polygon*, by which we mean a polygon in $\mathbb{H}^2 \cup \partial\mathbb{H}^2$ such that all vertices are on the boundary $\partial\mathbb{H}^2$. Note that such a generalized ideal polygon may have infinite volume.

For any given $g \in \mathrm{Isom}(\mathbb{H}^2)$, we associate a coloring to g as follows. Consider $g\mathscr{D} = \{gD : D \in \mathscr{D}\}$. For any vertex x of the dual graph T, the polygon D_x dual to x is a union of subsets of the form $D_x \cap gD'$, for $D' \in \mathscr{D}$, with mutually disjoint interiors. We call by the *partition of D by $\mathscr{D} \vee g\mathscr{D}$* the collection $\{D \cap gD' : D' \in \mathscr{D}\}$ just described (disregarding intersections on the boundary).

Definition 1. Let $\mathscr{A}$ be the set of equivalence classes of partitions of elements of $\mathscr{D}$, where two partitions of D and D', respectively, are equivalent if there exists an isometry from D to D' which sends elements of the partition of D bijectively to elements of the partition of D'. *The coloring ϕ_g associated* with g is the map $\phi_g : VT \to \mathscr{A}$ sending x to the class of the partition of D_x by $\mathscr{D} \vee g\mathscr{D}$.

Let $\Gamma_{\mathscr{D}} \subset \mathrm{Isom}(\mathbb{H}^2)$ be the set of isometries leaving $\mathscr{D}$ invariant. Since every $D \in \mathscr{D}$ is a generalized ideal polygon with finitely many sides, Γ_r is a finite index subgroup of $\Gamma_{\mathscr{D}}$. Thus $g \in \mathrm{Isom}(\mathbb{H}^2)$ is a commensurator of Γ_r if and only if g is a commensurator of $\Gamma_{\mathscr{D}}$.

Lemma 2. *For each $x, y \in VT$, one has $\phi_g(x) = \phi_g(y)$ if and only if there exists $\gamma \in \Gamma_{\mathscr{D}} \cap g\Gamma_{\mathscr{D}}g^{-1} \subset \mathrm{Isom}(\mathbb{H}^2)$ such that $\gamma D_x = D_y$, where D_x, D_y are the elements of $\mathscr{D}$ associated with $x, y \in VT$.*

Proof. Suppose that $\gamma D_x = D_y$ for some $\gamma \in \Gamma_{\mathscr{D}} \cap g\Gamma_{\mathscr{D}}g^{-1}$. Let the partition of D_x by $\mathscr{D} \vee g\mathscr{D}$ be

$$D_x = \bigcup_{i\in I}(D_x \cap gE_i), \tag{2}$$

where $\{E_i\}_{i\in I} \subset \mathscr{D}$. Then

$$\gamma D_x = \bigcup_{i\in I}(\gamma D_x \cap \gamma gE_i). \tag{3}$$

Since $\gamma \in \Gamma_{\mathscr{D}} \cap g\Gamma_{\mathscr{D}}g^{-1}$, we have $\gamma = g\gamma' g^{-1}$ for some $\gamma' \in \Gamma_{\mathscr{D}}$. Thus

$$\gamma D_x = \bigcup_{i\in I}\left(\gamma D_x \cap g\gamma' E_i\right). \tag{4}$$

Since $g\gamma' E_i$ are all elements of $g\mathscr{D}$, the above partition is a partition of γD_x by $\mathscr{D} \vee g\mathscr{D}$. Therefore, the colorings ϕ_g on D_x and $D_y = \gamma D_x$ are the same.

Conversely, any isometry from D_x to D_y extends to an isometry of $\mathbb{H}^2$ leaving $\mathscr{D}$ invariant. Thus if $\phi_g(x) = \phi_g(y)$, then there exists $\gamma \in \Gamma_{\mathscr{D}}$ such that $\gamma D_x = D_y$ which sends elements of the partition of D_x by $\mathscr{D} \vee g\mathscr{D}$ bijectively to those of D_y. Let us denote the partitions of D_x and $D_y = \gamma D_x$ by $\mathscr{D} \vee g\mathscr{D}$ by

$$\begin{aligned} D_x &= \bigcup_{i\in I}(D_x \cap gE_i), \\ D_y = \gamma D_x &= \bigcup_{j\in J}\left(\gamma D_x \cap gF_j\right), \end{aligned} \tag{5}$$

for some $\{E_i\}_{i\in I}, \{F_j\}_{j\in J} \subset \mathscr{D}$. Since D_x and D_y have the same coloring, the above partition is equal to

$$\gamma D_x = \bigcup_{i\in I}(\gamma D_x \cap \gamma gE_i). \tag{6}$$

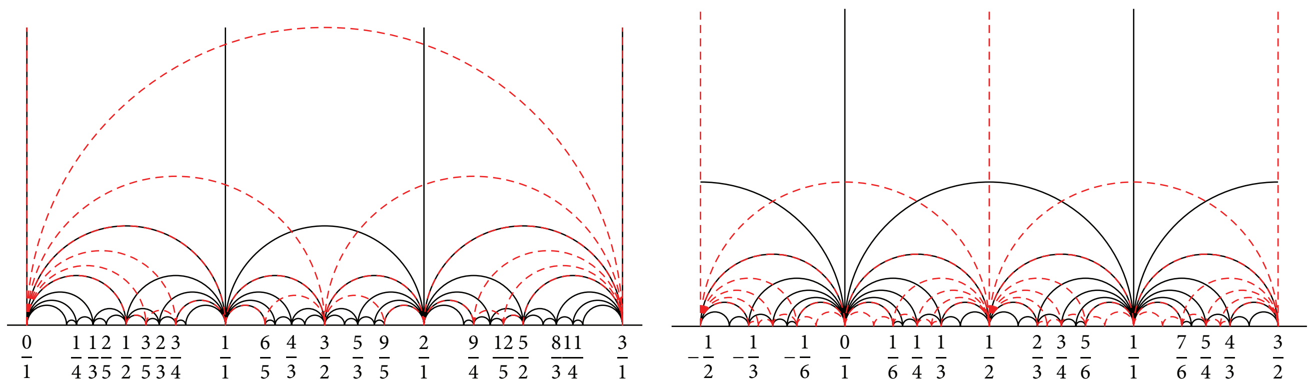
Thus $I = J$, and by rearranging F_j if necessary, we have $D_y \cap gF_i = D_y \cap \gamma gE_i$ for each i. Since $\mathscr{D}$ is a tessellation by ideal polygons, this implies that $gF_i = \gamma gE_i$. As F_i and E_i are elements of $\mathscr{D}$, there exists $\gamma_i \in \Gamma_{\mathscr{D}}$ such that $F_i = \gamma_i E_i$. Thus $g\gamma_i(\gamma g)^{-1}$ stabilizes E_i; thus it is an element of $\Gamma_{\mathscr{D}}$, say γ'. We conclude that $\gamma'\gamma = g\gamma_i g^{-1} \in g\Gamma_{\mathscr{D}}g^{-1}$ satisfies the statement of the lemma. □

Now let us formulate our theorem.

Theorem 3. *Let Γ_r be a group generated by the reflections in the edges of a generalized ideal polygon. An isometry $g \in \mathrm{Isom}(\mathbb{H}^2)$ is a commensurator of Γ_r if and only if its associated coloring ϕ_g is periodic.*

Proof. As we mentioned earlier, $g \in \mathrm{Comm}(\Gamma_r)$ if and only if $g \in \mathrm{Comm}(\Gamma_{\mathscr{D}})$. Suppose that $\Gamma' = g\Gamma_{\mathscr{D}}g^{-1} \cap \Gamma_{\mathscr{D}}$ is a finite-indexed subgroup of $\Gamma_{\mathscr{D}}$. By Lemma 2, we know that ϕ_g is Γ'-invariant. Since $\Gamma_{\mathscr{D}}$ is cocompact in $\mathrm{Aut}(T)$, Γ' is also a cocompact discrete subgroup of $\mathrm{Aut}(T)$. Thus ϕ_g is periodic.

Conversely, suppose that ϕ_g is periodic. Let Γ be a cocompact subgroup of $\mathrm{Aut}(T)$ preserving ϕ_g. For any $x \in VT$ and $\gamma \in \Gamma$ we have $\phi_g(x) = \phi_g(\gamma x)$; thus, by Lemma 2, there exists $\gamma' \in \Gamma' = \Gamma_{\mathscr{D}} \cap g\Gamma_{\mathscr{D}}g^{-1}$ such that $\gamma'(D_x) = D_{\gamma x}$. Letting



(a) $z \mapsto 3z$ (b) $z \mapsto z + (1/2)$

FIGURE 1: Examples of isometry g associated with periodic colorings.

$\Gamma'' = \{\gamma' \in \Gamma' : \gamma \in \Gamma\}$, it follows that $\Gamma'' \setminus T \simeq \Gamma' \setminus T$ is finite since $\Gamma \setminus T$ is finite. Since Γ_r is a finite index subgroup of $\Gamma_{\mathscr{D}}$ and T is the Cayley graph of Γ_r, $\Gamma' \subset \Gamma_{\mathscr{D}}$ is of finite index. Therefore g is a commensurator of $\Gamma_{\mathscr{D}}$, thus a commensurator of Γ_r. □

Note that the coloring $\phi_g^{\#}$ on a 2-regular tree in (1) is periodic if ϕ_g is periodic.

Now let us provide some examples of isometries of the hyperbolic plane giving periodic colorings. A periodic coloring which is Γ-invariant for some $\Gamma \subset \mathrm{Isom}(\mathbb{H}^2)$ will be expressed on the quotient, denoted by $\Gamma \backslash\backslash T$, which is either a graph (if there is no torsion element) or a graph of groups (if there are some torsion elements, we attach the stabilizers of vertices and edges on the quotient graph). In fact, we will express a coloring on the edge-indexed graph of the quotient graph of groups $\Gamma\backslash\backslash T$, as we only need the edge-indexed graph of a graph of groups to recover T from a graph of groups.

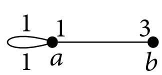
Recall that the edge-indexed graph of a graph of groups is a graph with an index on each oriented edge, where the graph is given by the quotient graph $\Gamma \setminus T$ and the index $i(e)$ of the oriented edge e is given by the index of the edge group G_e in the vertex group $G_{\partial_0(e)}$ of the initial vertex $\partial_0(e)$ of e. (For details on graph of groups and the edge-indexed graph of a graph of groups, see [6–8].)

Example 4. Consider the Farey tessellation $\mathscr{D}$ of the hyperbolic plane, which is the tessellation with D the ideal triangle of vertices ∞, 0, and 1. Then $\Gamma_r = \langle -z,\ z/(2z-1),\ 2-z\rangle$, and the dual graph of Γ_r is a 3-regular tree T. Note that $\Gamma_r \cap \mathrm{PSL}_2(\mathbb{Z})$ is a subgroup of Γ_r of index 2. Γ_r is commensurable to $\mathrm{PSL}_2(\mathbb{Z})$ as $\Gamma_r \cap \mathrm{PSL}_2(\mathbb{Z})$ is a subgroup of $\mathrm{PSL}_2(\mathbb{Z})$ of index six since $\{z,\ 1/(1-z),\ (z+1)/z\}$ is the stabilizer subgroup of D in $\mathrm{PSL}_2(\mathbb{Z})$.

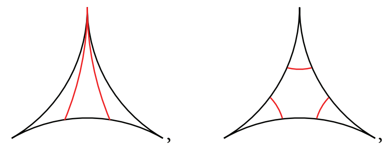
A hyperbolic element $g_1 : z \mapsto 3z$ and a parabolic element $g_2 : z \mapsto z + (1/2)$ is considered in Figure 1. The associated colorings ϕ_{g_1} and ϕ_{g_2} are both periodic.

The periodic coloring of the edge-indexed graph of a graph of groups $\Gamma \setminus T$ for $z \mapsto 3z$ and $z \mapsto z + (1/2)$ is as follows:

$$g_1 : z \longmapsto 3z, \tag{7}$$



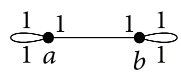
Here, vertices a and b represent the ideal triangle partitioned as



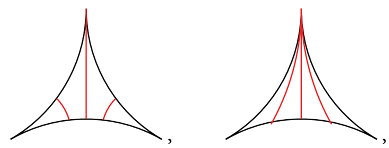
(8)

respectively. On the other hand,

$$g_2 : z \longmapsto z + \frac{1}{2}, \tag{9}$$



In this graph, vertices a and b represent the ideal triangle partitioned as



(10)

respectively (see Figure 1).

Remark also that an elliptic element $z \mapsto (2z-1)/(z+1)$ has a periodic coloring identical to that of $z \mapsto 3z$.

3. Eventually Periodic Colorings and Their Generalizations

Now consider an element of $\mathrm{Isom}(\mathbb{H}^2)$ which is not a commensurator of Γ_r. We know that the associated coloring ϕ_g is not periodic.

Corollary 5. *Let ϕ_g be a coloring associated with an element $g \in \mathrm{Isom}(\mathbb{H}^2)$ which is not a commensurator of Γ_r. Then its associated coloring has infinite alphabet.*

Proof. Let $\Gamma' = \Gamma_{\mathscr{D}} \cap g\Gamma_{\mathscr{D}}g^{-1}$ and $\Gamma = \Gamma_r \cap \Gamma'$. By Lemma 2, $\phi_g(x) = \phi_g(y)$ implies that x, y are in the same right coset of Γ. Therefore the coloring ϕ_g has a finite alphabet if and only if Γ is a finite index subgroup of Γ_r. By Theorem 3, finiteness of the coloring alphabet is equivalent to the fact that g is a commensurator of Γ_r. □

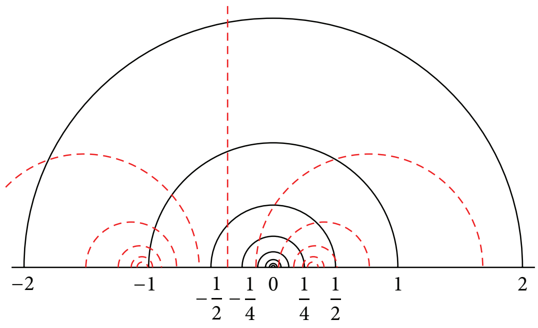


FIGURE 2: $z \mapsto (z - \sqrt{11})/(\sqrt{10}(z+1))$.

This phenomenon is in contrast to the motivating example of circle rotation explained in the beginning of the last section. In that case, irrational rotations correspond to nonperiodic colorings. However they are defined on a finite set of alphabets, and the corresponding sequences are Sturmian, that is, sequences with subword complexity $p(n) = n + 1$. See [9] for Sturmian sequences.

Now let us explain how to obtain colorings of "low complexity" with a finite set of alphabets from hyperbolic tessellations by disregarding some information, as the coloring on a 2-regular tree $\phi_g^{\#}$ in (1) for noncommensurable g is Sturmian.

Definition 6. One calls a coloring ϕ *eventually periodic* if there exists a subtree K of finite number of vertices such that $T - K = \cup T_i$ is a finite union of subtrees T_i such that ϕ on each T_i can be extended to a periodic coloring on T.

In the next examples, let us denote a geodesic in $\mathbb{H}^2$ between points $x, y \in \partial\mathbb{H}^2$ by (x, y), and let us call the edges in $\mathbb{R} = \partial\mathbb{H}^2$ boundary edges.

Example 7. Let $\mathscr{D}$ be the tessellation of $\mathbb{H}^2$ with D a generalized ideal polygon whose edges are two geodesics $(-2, 2), (-1, 1)$ and two boundary edges $[-2, -1]$, $[1, 2]$. Let $e_n = (-2^n, 2^n)$. An element of $\mathscr{D}$ is a generalized ideal polygon which is the region bounded by e_n, e_{n+1}, for some $n \in \mathbb{Z}$, which we denote by D_n. The dual graph is a 2-regular tree T, and we can naturally denote the element of VT dual to D_n by $n \in \mathbb{Z}$.

In this case, the commensurator of Γ_r is of the form $z \mapsto az$ and $z \mapsto a/z$ for $a \in \mathbb{R}$. If g is considered as a map on $\mathbb{H}^2 \cup \partial\mathbb{H}^2$, then $g \in \mathrm{Comm}(\Gamma_r)$ if and only if $g(\{0, \infty\}) = \{0, \infty\}$.

Let $g \notin \mathrm{Comm}(\Gamma_r)$ and ϕ_g^0 be a coloring given by

$$\phi_g^0(m) = \begin{cases} a, & \text{if there is } D \in \mathscr{D} \text{ such that } gD \subset D_m, \\ b, & \text{otherwise.} \end{cases} \tag{11}$$

If the boundary edge $[2^m, 2^{m+1}]$ or $[-2^{m+1}, -2^m]$ contains $g(0)$ (or $g(\infty)$) in its interior, then D_m contains $gD_n \in g\mathscr{D}$, for sufficiently small (or large, resp.) n. This is the case when all vertices of gD_n are contained in one boundary edge of D_m.

Otherwise, we claim that there is no $D \in \mathscr{D}$ such that $gD \subset D_m$. Indeed, suppose gD_n is contained in D_m. The only remaining case is when the two boundary edges of gD_n are contained in both of the boundary edges of D_m. Let γ be the element of Γ sending D_m to D_n. Since $g\gamma$ is an isometry of $\mathbb{H}^2$ sending D_m into itself and the boundary edges of $g(D_m)$ are contained in both of the boundary edges of D_m, it sends the geodesic segment ℓ of minimal distance between e_m, e_{m+1}, which is the intersection of the y-axis with D_m, to a geodesic segment of minimal distance between $g\gamma e_m, g\gamma e_{m+1}$. Thus the distance between $g\gamma e_m, g\gamma e_{m+1}$ is bounded above by the length of $\ell \cap g\gamma D_m$, which is strictly less than the length of ℓ, which is the distance between e_m and e_{m+1}. This contradicts the fact that $g\gamma$ is an isometry.

Therefore, all vertices except for one or two are colored by b, and the remaining one or two vertices whose dual generalized ideal polygon contains $g(0)$ or $g(\infty)$ in its interior are colored by a. Hence, by omitting one or two vertices, one obtains a periodic coloring. Thus, ϕ_g^0 is an eventually periodic coloring.

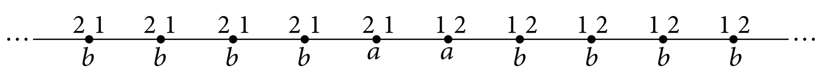
In Figure 2, an example of $g : z \mapsto (z - \sqrt{11})/(\sqrt{10}(z + 1))$ is presented. In this case, there are exactly two vertices colored by a, that is, $\phi_g^0(m) = a$ for $m = -2, 0$ and $\phi_g^0(m) = b$ otherwise.

Now consider the Farey tessellation $\mathscr{D}$ and the corresponding group Γ. The dual graph of Γ is a 3-regular tree T. Let us provide two examples of colorings given by noncommensurable elements of Γ in $\mathrm{Isom}(\mathbb{H}^2)$.

Example 8. Let $g : z \mapsto \alpha z$ with irrational α. Then $g \notin \mathrm{Comm}(\Gamma_r)$. Let ϕ_g^1 be a coloring given by

$$\phi_g^1(x) = \begin{cases} a, & \text{if } \exists D \in \mathscr{D} \text{ such that } D_x \cap gD \text{ contains a geodesic line,} \\ b, & \text{otherwise} \end{cases} \tag{12}$$

for $x \in VT$ and $D_x \in \mathscr{D}$ corresponding to x. A geodesic line is contained in $D_x \cap gD$ if only if the two ideal triangles D_x and gD have two common vertices. Since the only possible rational vertices of gD are 0 and ∞, $\phi_g^1(x) = a$ if and only if D_x corresponds to ideal triangle of vertices $(0, 1, \infty)$ or $(-1, 0, \infty)$. Therefore, ϕ_g^1 is an eventually periodic coloring, and the coloring of the edge-indexed graph of a graph of groups is as follows:



(13)

For example, Figure 3 shows the case $g : z \mapsto \sqrt{3}z$.

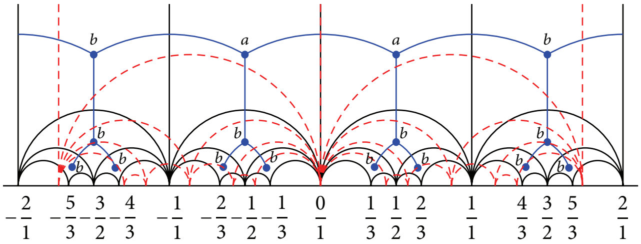


FIGURE 3: $z \mapsto \sqrt{3}z$.

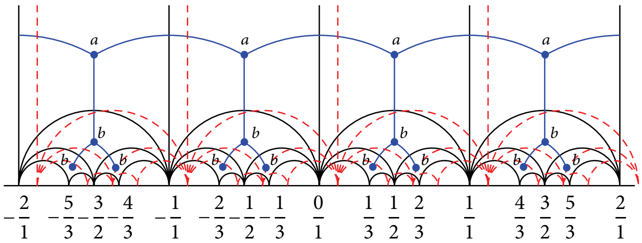
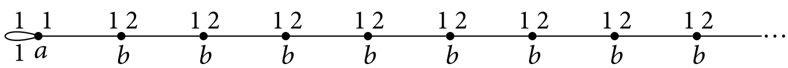


FIGURE 4: $z \mapsto z + \sqrt{17}$.

Example 9. Let $g : z \mapsto z + \beta$ with irrational β. Then we have $g \notin \mathrm{Comm}(\Gamma_r)$. Let ϕ_g^2 be a coloring given by

$$\phi_g^2(x) = \begin{cases} a, & \text{if there is } D \in \mathscr{D} \text{ such that } D_x \cap gD \\ & \text{is not compact,} \\ b, & \text{otherwise} \end{cases} \quad (14)$$

for $x \in VT$ and $D_x \in \mathscr{D}$ corresponding to x. If $D_x \cap gD$ is not compact, then D_x and gD have at least one common vertex. Since all vertices of $g\mathscr{D}$ other than ∞ are irrational, $\phi_g^2(x) = a$ if and only if D_x has the vertex of ∞, which is the only possible common vertex of D_x with $gD \in g\mathscr{D}$. Therefore, ϕ_g^2 is a coloring with two colors whose coloring of the edge-indexed graph of a graph of groups is as follows:



(15)

For example, Figure 4 shows the case $g : z \mapsto z + \sqrt{17}$.

We remark that this last example has the number of colored balls up to isometry equal to $n + 2$. We believe that the colorings of this type (i.e., with the number of isometry classes of colored balls being $n+2$) are the ones corresponding to Sturmian sequences. We leave systematic studies about them for future research.

Remark 10. We can generalize the construction in this paper from torsion-free discrete subgroup to any discrete subgroup with one cusp: in this generality, one should consider the minimal subtree containing vertices not in $\partial\mathbb{H}^2$, which is again a tree.

Acknowledgments

The authors would like to thank the anonymous referee for the valuable comments. The first author is supported by NRF 2012R1A1A2004473. The second author is supported by NRF 2012-000-8829, NRF 2012-000-2388, and TJ Park Science Fellowship.

References

[1] M. Morse and G. A. Hedlund, "Symbolic dynamics II. Sturmian trajectories," *American Journal of Mathematics*, vol. 62, pp. 1–42, 1940.

[2] G. A. Margulis, *Discrete Subgroups of Semisimple Lie Groups*, vol. 17 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*, Springer, Berlin, Germany, 1991.

[3] A. Kubena and A. Thomas, "Density of commensurators for uniform lattices of right-angled buildings," *Journal of Group Theory*, vol. 15, no. 5, pp. 565–611, 2012.

[4] N. Avni, S. Lim, and E. Nevo, "On commensurator growth," *Israel Journal of Mathematics*, vol. 188, pp. 259–279, 2012.

[5] V. Platonov and A. Rapinchuk, *Algebraic Groups and Number Theory*, vol. 139 of *Pure and Applied Mathematics*, Academic Press, Boston, Mass, USA, 1994, translated from the 1991 Russian original by Rachel Rowen.

[6] H. Bass and R. Kulkarni, "Uniform tree lattices," *Journal of the American Mathematical Society*, vol. 3, no. 4, pp. 843–902, 1990.

[7] H. Bass and A. Lubotzky, *Tree Lattices*, vol. 176 of *Progress in Mathematics*, Birkhäuser, Boston, Mass, USA, 2001.

[8] J.-P. Serre, *Trees*, Springer, Berlin, Germany, 1980.

[9] N. P. Fogg, *Substitutions in Dynamics, Arithmetics and Combinatorics*, vol. 1794 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 2002.