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Research Article

Stability Switches and Hopf Bifurcation in a Kaleckian Model of Business Cycle

Luca Vincenzo Ballestra,[1] Luca Guerrini,[2] and Graziella Pacelli[2]

[1] *Department of Economics, Second University of Naples, 81043 Capua, Italy*
[2] *Department of Management, Polytechnic University of Marche, 60121 Ancona, Italy*

Correspondence should be addressed to Graziella Pacelli; g.pacelli@univpm.it

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This paper considers a Kaleckian type model of business cycle based on a nonlinear delay differential equation, whose associated characteristic equation is a transcendental equation with delay dependent coefficients. Using the conventional analysis introduced by Beretta and Kuang (2002), we show that the unique equilibrium can be destabilized through a Hopf bifurcation and stability switches may occur. Then some properties of Hopf bifurcation such as direction, stability, and period are determined by the normal form theory and the center manifold theorem.

1. Introduction

The fact that transformation of inputs into outputs does not occur instantaneously but exhibits a gestation lag was first approached formally and systematically by Kalecki [1], who introduced a time delay between the moment when an investment decision is made and the delivering of the finished real investment. In his analysis, Kalecki showed that a gestation period can cause cycles in the economy. The mathematical description of Kalecki's model of business cycle leads to the following differential equation with delay, which describes the change in net investment:

$$\dot{K}(t) = \frac{a}{\tau}K(t) - \left(\frac{a}{\tau} + m\right)K(t-\tau), \tag{1}$$

where t is time, K represents the net level of investment, τ is the delay between taking the decision to invest and carrying it out, and m is the degree of effect of the capital stock on profit rates. The term a is the percentage of profits reinvested, usually ranging from 0.8 to 0.95. Henceforth, we assume $a < 1$. Delay differential equations have been widely employed for economic modeling in the last twenty years. In particular, it has been shown by many authors that the introduction of a delay has the advantage of producing more complex dynamics. Among others, we recall Asea and Zak [2], Bianca et al. [3, 4], Bianca and Guerrini [5], Szydłowski and Krawiec [6], Matsumoto and Szidarovszky [7], Szydłowski [8, 9], and Zak [10]. Their results indicate that the time delay is an important control parameter in economic processes and can often cause an otherwise stable system to oscillate.

Recently, Casal and Dibeh [11] made the linear model in (1) nonlinear by assuming that the negative effect of the capital stock, which is represented in (1) by $mK(t-\tau)$, increases disproportionally as the capital stock builds up. Specifically, they considered the nonlinear effect of K on profitability and, hence, on investment to be represented by the function

$$G(K(t), K(t-\tau)) = mK(t-\tau) + nK(t) + \varepsilon K(t)^3, \tag{2}$$

where n is the degree of responsiveness of investment to the level of capital stock, and ε is a small parameter. Under assumption (2), the economy happens to be described by the following delay differential equation:

$$\dot{K}(t) = \left(\frac{a}{\tau} - n\right)K(t) - \left(\frac{a}{\tau} + m\right)K(t-\tau) - \varepsilon K(t)^3. \tag{3}$$

Notice that when $n = \varepsilon = 0$, (3) reduces to the Kalecki's model (1).

Casal and Dibeh [11] solved (3) by using approximation methods developed by Kryloff and Bogoliuboff [12] and applied to delay differential equations by Clark and Sebasta [13]. Moreover, Deeba et al. [14] worked out (3) using the Adomian decomposition algorithm. Limit cycle solutions, taking positive, zero, and negative values, were found for a wide range of the model's parameters. However, we remark that the dynamic analysis of (3) is still far from being complete.

The purpose of this paper is to perform a thorough analysis on the stability switches and Hopf bifurcation of the nonlinear model (3) when the time delay is taken as the bifurcation parameter. First of all, the linear stability about the trivial equilibrium is investigated by considering the characteristic transcendental equation associated with the linearized Kaleckian model. Unfortunately, the coefficients of this equation (and not only the transcendental term in it) depend on the delay, which makes the stability analysis rather complicated from the mathematical standpoint. However, a method that combines graphical information with analytical work has been developed by Beretta and Kuang [15] in order to determine stability switches for delay equations, which do not lend themselves to classical methods because of the aforementioned difficulties. Such an approach allows us to show that stability switches and Hopf bifurcations may occur as the delay increases. In addition, using the normal form reduction and the center manifold theory, suitable formulas for determining the direction of Hopf bifurcation and the stability of bifurcation periodic solution are obtained. Finally, some numerical simulations are performed to illustrate our qualitative results.

2. Local Stability and Existence of Hopf Bifurcation

Equation (3) is a delay differential equation with delay dependent coefficients. To investigate the local asymptotic stability of its unique steady state $K_* = 0$, we consider the linear part of (3) at this equilibrium. This gives

$$\dot{K}(t) = \left(\frac{a}{\tau} - n\right) K(t) - \left(\frac{a}{\tau} + m\right) K(t-\tau). \tag{4}$$

The corresponding characteristic equation, obtained by substituting $K(t) = e^{\lambda t}$ into (4), is

$$P(\lambda, \tau) = \lambda - \left(\frac{a}{\tau} - n\right) + \left(\frac{a}{\tau} + m\right) e^{-\lambda\tau} = 0. \tag{5}$$

Equation (5) is a transcendental equation having, in general, an infinite number of complex roots. It is well known that the equilibrium of (4) is locally asymptotically stable if each of the characteristic roots of (5) has negative real part. Thus, the marginal stability is determined by $\lambda = 0$ and $\lambda = i\omega$ $(\omega > 0)$. We start our analysis with the case of no delay.

Lemma 1. *The equilibrium point is locally asymptotically stable when the delay is very small.*

Proof. An application of L'Hopital's rule gives

$$\begin{aligned}\lim_{\tau\to 0^+} P(\lambda,\tau) &= \lambda + n + \lim_{\tau\to 0^+} \frac{a\left(-1 + e^{-\lambda\tau}\right)}{\tau} + m \lim_{\tau\to 0^+} e^{-\lambda\tau} \\ &= 0 \Rightarrow \lambda = -\frac{m+n}{1-a} < 0.\end{aligned} \tag{6}$$

Thus, the statement holds. □

Let $\tau > 0$. As τ varies, stability switches may occur in correspondence with any critical value of τ at which a root of (5) transitions from having negative to having positive or null real part. If this occurs, then there must be a critical value of τ such that the characteristic equation has a purely imaginary root. Notice that the case $\lambda = 0$ does not need to be considered since it is not a root for (5). In fact, $P(0,\tau) = n + m \neq 0$. We begin by looking for a purely imaginary root of (5). Let $\lambda = i\omega$, with $\omega > 0$, be a solution of (5). Then

$$P(i\omega, \tau) = i\omega - \left(\frac{a}{\tau} - n\right) + \left(\frac{a}{\tau} + m\right) e^{-i\omega\tau} = 0. \tag{7}$$

Breaking the polynomial up into its real and imaginary parts, and writing the exponential in (7) in terms of trigonometric functions, we get

$$\omega = \left(\frac{a}{\tau} + m\right)\sin\omega\tau, \qquad \frac{a}{\tau} - n = \left(\frac{a}{\tau} + m\right)\cos\omega\tau. \tag{8}$$

Squaring each equation in (8) and summing the results yields

$$F(\omega,\tau) \equiv \omega^2 - \left(\frac{a}{\tau} + m\right)^2 + \left(\frac{a}{\tau} - n\right)^2 = 0. \tag{9}$$

Lemma 2. *Let*

$$I = \begin{cases} (0, +\infty), & \text{if } n \le m, \\ (0, \overline{\tau}), & \text{if } n > m, \end{cases} \tag{10}$$

where

$$\overline{\tau} = \frac{2a}{n-m}. \tag{11}$$

For $\tau \in I$, (9) has a unique positive root given by

$$\omega(\tau) = \sqrt{(m+n)\left(m - n + \frac{2a}{\tau}\right)}. \tag{12}$$

Proof. If $F(\omega,\tau) = 0$ has positive roots, then one has

$$\begin{aligned}\left(\frac{a}{\tau} + m\right)^2 - \left(\frac{a}{\tau} - n\right)^2 &= (m+n)\left(m - n + \frac{2a}{\tau}\right) > 0 \\ &\Rightarrow m - n + \frac{2a}{\tau} > 0,\end{aligned} \tag{13}$$

completing the proof. □

We have shown that, for $\tau \in I$, there exists an $\omega = \omega(\tau) > 0$, given by (12), such that $F(\omega, \tau) = 0$. Let $\theta(\tau) \in (0, \pi)$ be defined for $\tau \in I$ by

$$\sin(\theta(\tau)) = \frac{\omega(\tau)}{(a/\tau) + m}, \qquad \cos(\theta(\tau)) = \frac{(a/\tau) - n}{(a/\tau) + m}. \tag{14}$$

Then, $\omega(\tau)\tau = \theta(\tau) + 2j\pi$, $j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Following Beretta and Kuang [15], we introduce the maps $\tau_j : I \to \mathbb{R}_0^+$,

$$\tau_j(\tau) = \frac{\theta(\tau) + 2j\pi}{\omega(\tau)}, \quad j \in \mathbb{N}_0, \ \tau \in I. \tag{15}$$

The expression of $\tau_j(\tau)$ can be obtained from (14) as follows:

$$\tau_j(\tau) = \begin{cases} \dfrac{1}{\omega(\tau)} \left\{ \tan^{-1} \left[\dfrac{\omega(\tau)}{(a/\tau) - n} \right] + 2j\pi \right\}, & \text{if } \tau \le \dfrac{a}{n}, \\ \dfrac{1}{\omega(\tau)} \left\{ \tan^{-1} \left[\dfrac{\omega(\tau)}{(a/\tau) - n} \right] + (2j+1)\pi \right\}, & \text{if } \tau > \dfrac{a}{n}. \end{cases} \tag{16}$$

Remark 3. When $n > m$, $\overline{\tau} = 2a/(n-m) > a/n$, so that (16) leads to the cases $\tau < a/n$ and $a/n < \tau < \overline{\tau}$.

As a result, $i\omega_*$, with $\omega_* = \omega(\tau_*) > 0$, is a purely imaginary root of (7) if and only if τ_* is a zero of the function $S_j(\tau)$, for some $j \in \mathbb{N}_0$, where $S_j(\tau)$ is defined by

$$S_j(\tau) = \tau - \tau_j(\tau), \quad j \in \mathbb{N}_0, \ \tau \in I. \tag{17}$$

Proposition 4. *Assume that the equation $S_j(\tau) = 0$ has a positive root $\tau_* \in I$ for some $j \in \mathbb{N}_0$. Then there exists a pair of simple purely imaginary roots $\lambda = \pm i\omega(\tau_*)$ of (5) at $\tau = \tau_*$, crossing the imaginary axis from left to right if $\Gamma(\tau_*) > 0$ and from right to left if $\Gamma(\tau_*) < 0$, where*

$$\Gamma(\tau_*) = \operatorname{sign} \left\{ \left. \frac{d(\operatorname{Re}\lambda)}{d\tau} \right|_{\lambda = i\omega_*} \right\} = \operatorname{sign} \left\{ \left. \frac{dS_j(\tau)}{d\tau} \right|_{\tau = \tau_*} \right\}. \tag{18}$$

Proof. We start proving that $\lambda = i\omega_*$ is a simple root for (5). If $\lambda = i\omega_*$ has repeated roots, then $dP(i\omega_*, \tau_*)/d\lambda = 0$ gives rise to the contradiction $\omega_* = 0$. Differentiating (5) with respect to τ we find that

$$\frac{d\lambda}{d\tau} = \left(-\frac{a}{\tau^2} - \frac{a}{\tau^2} \left(\frac{\lambda - (a/\tau) + n}{(a/\tau) + m} \right) - \lambda \left(\tau - \frac{a}{\tau} + n \right) \right) \left(1 + \tau \left(\lambda - \frac{a}{\tau} + n \right) \right)^{-1}. \tag{19}$$

We remark that

$$\operatorname{sign} \left\{ \left. \frac{d(\operatorname{Re}\lambda)}{d\tau} \right|_{\lambda = i\omega_*} \right\} = \operatorname{sign} \left\{ \operatorname{Re} \left. \left(\frac{d\lambda}{d\tau} \right)^{-1} \right|_{\lambda = i\omega_*} \right\}. \tag{20}$$

From (12) we have

$$\omega(\tau)\omega'(\tau) = -(m+n)\frac{a}{\tau^2}, \tag{21}$$

so that (19) yields

$$\begin{aligned} \operatorname{Re} \left. \left(\frac{d\lambda}{d\tau} \right)^{-1} \right|_{\lambda = i\omega_*} &= \operatorname{Re} \left\{ \frac{(a/\tau_*) - n - \tau_* ((a/\tau_*) + m)^2 + i\omega_*}{-\omega_*\omega_*' + i\omega_* \left[-(a/\tau_*^2) + ((a/\tau_*) + m)^2 \right]} \right\} \\ &= \left(-\omega_*\omega_*' \left[\frac{a}{\tau_*} - n - \tau_* \left(\frac{a}{\tau_*} + m \right)^2 \right] + \omega_*^2 \left[-\frac{a}{\tau_*^2} + \left(\frac{a}{\tau_*} + m \right)^2 \right] \right) \\ &\quad \times \left((\omega_*\omega_*')^2 + \omega_*^2 \left[-\frac{a}{\tau_*^2} + \left(\frac{a}{\tau_*} + m \right)^2 \right]^2 \right)^{-1}. \end{aligned} \tag{22}$$

Next, differentiating (16) and (17) with respect to τ gives

$$\begin{aligned} \frac{dS_j(\tau_*)}{d\tau} &= 1 - \frac{d\tau_j(\tau_*)}{d\tau} \\ &= 1 - \left(\frac{a}{\tau_*^2}\omega_*^2 - \left[-\left(\frac{a}{\tau_*} - n \right) + \left(\frac{a}{\tau_*} + m \right)^2 \tau_* \right] \times \omega_*\omega_*' \right) \left(\left(\frac{a}{\tau_*} + m \right) \omega_*^2 \right)^{-1} \\ &= \left(\left(\frac{a}{\tau_*} + m \right)^2 \omega_*^2 - \frac{a}{\tau_*^2}\omega_*^2 + \left[-\left(\frac{a}{\tau_*} - n \right) + \left(\frac{a}{\tau_*} + m \right)^2 \tau_* \right] \omega_*\omega_*' \right) \\ &\quad \times \left(\left(\frac{a}{\tau_*} + m \right)^2 \omega_*^2 \right)^{-1}. \end{aligned} \tag{23}$$

A comparison of (22) and (23) leads to the conclusion. □

According to the previous result, the knowledge of the geometric shape of the functions $S_j(\tau)$, $\tau \in I$, the location of their zeros τ_*, and the sign of $\Gamma(\tau_*)$ allow us to determine at which delay values stability switching of the equilibrium eventually occur.

The following properties of $S_j(\tau)$ can be verified using (17):

(1) $S_j(\tau) > S_{j+1}(\tau)$, $j \in \mathbb{N}_0$. Hence, if S_0 has no zeros in I, nor does S_j for all $j > 0$.

(2) Let $n > m$. Then $I = (0, \overline{\tau})$. As $\tau \to \overline{\tau}^-$, $\omega(\tau) \to 0$, while $\sin(\vartheta(\tau)) \to 0$ and $\cos(\vartheta(\tau)) \to -1$. As a consequence, $\vartheta(\tau) \to \pi$ and $S_j(\tau) \to -\infty$ as $\tau \to \overline{\tau}^-$.

(3) Let $n \le m$. Then $I = (0, +\infty)$. As $\tau \to +\infty$, $\omega(\tau) \to \sqrt{m^2 - n^2} \ge 0$, while $\tau_j(\tau)$ tends to a positive number. Hence, $S_j(\tau) \to +\infty$ as $\tau \to +\infty$.

(4) Let $\tau \in I$. As $\tau \to 0^+$, $\omega(\tau) \to +\infty$ yielding $S_j(\tau) \to 0$ for $j \in \mathbb{N}_0$.

(5) Let $\tau \in I$ be small, with $\tau \le a/n$. An application of L'Hopital's rule implies $\omega(\tau)/(a/\tau - n) \to 0$ as $\tau \to 0^+$. Hence, a Taylor series expansion gives $\tau_0(\tau) = 1/(a/\tau - n)$. From this we derive that $dS_0(\tau)/d\tau < 0$ if $\tau \le a/n < 2a/n$.

According to the above analysis, we state the following conclusions for (3).

Theorem 5. *(1) Let $I = (0, \overline{\tau})$ (case $n > m$). If the function $S_0(\tau)$ has no zeros in I, then the equilibrium $K_* = 0$ is locally asymptotically stable for all $\tau \ge 0$.*

(2) Let $I = (0, +\infty)$ (case $n \le m$). The function $S_j(\tau)$, $j \in \mathbb{N}_0$, has a simple zero in I.

(3) If the function $S_j(\tau)$ has a finite number of simple zeros in I, for some fixed $j \in \mathbb{N}_0$, then they must be even if $I = (0, \overline{\tau})$, say $\tau_1 < \tau_2 < \cdots < \tau_{2r}$, and odd if $I = (0, +\infty)$, say $\tau_1 < \tau_2 < \cdots < \tau_{2r+1}$. In both cases, the equilibrium $K_* = 0$ is locally asymptotically stable if $\tau \in (\tau_{2k}, \tau_{2k+1})$ $(k = 0, 1, 2, \ldots, r;\ \tau_0 = 0)$ and unstable otherwise. Furthermore, a Hopf bifurcation occurs at $\tau_1, \tau_2, \ldots, \tau_{2r+1}$.

3. Properties of Hopf Bifurcation

In the previous section, we have obtained the conditions under which a family of periodic solutions bifurcates from the equilibrium $K_* = 0$ at the critical value τ_*. In this section, we wills study the direction, stability, and period of the Hopf bifurcation. Following the ideas of Hassard et al. [16], we derive the explicit formulae for determining the properties of the Hopf bifurcation at the critical value of τ_* by using the normal form and the center manifold theory. Throughout this section, we always assume that (3) undergoes Hopf bifurcation at the equilibrium $K_* = 0$ for $\tau = \tau_*$, and then $\pm i\omega(\tau_*)$ are the corresponding purely imaginary roots of the characteristic equation at the equilibrium $K_* = 0$. For notational convenience, let $\tau = \tau_* + \mu$, $\mu \in \mathbb{R}$. Then $\mu = 0$ is the Hopf bifurcation point for (3). Define by $C = C([-\tau_*, 0], \mathbb{R})$ the Banach space of continuous real-valued functions that map $[-\tau_*, 0]$ into $\mathbb{R}$. Then (3) can be written as a functional differential equation in the phase space C. Let $\varphi \in C$ and define the linear operator:

$$L_\mu(\varphi) = \left(\frac{a}{\tau} - n\right)\varphi(0) - \left(\frac{a}{\tau} + m\right)\varphi(-\tau), \tag{24}$$

and the nonlinear operator:

$$f(\mu, \varphi) = -\varepsilon\varphi(0)^3. \tag{25}$$

There is a bounded variation function $\eta(\vartheta, \mu)$ with $\vartheta \in [-\tau_*, 0]$ such that

$$L_\mu(\varphi) = \int_{-\tau_*}^{0} d\eta(\vartheta, \mu)\,\varphi(\vartheta). \tag{26}$$

In fact, we can choose $d\eta(\vartheta, \mu) = [(a/\tau - n)\delta(\vartheta) - (a/\tau + m)\delta(\vartheta + \tau)]d\vartheta$, where $\delta(\vartheta)$ is the Dirac delta function. Define operators

$$A(\mu)\varphi = \begin{cases} \dfrac{d\varphi(\vartheta)}{d\vartheta}, & \vartheta \in [-\tau_*, 0), \\ \displaystyle\int_{-\tau_*}^{0} d\eta(s, \mu)\,\varphi(s), & \vartheta = 0, \end{cases}$$
$$R(\mu)\varphi = \begin{cases} 0, & \vartheta \in [-\tau_*, 0), \\ f(\mu, \varphi), & \vartheta = 0. \end{cases} \tag{27}$$

Then (3) can be rewritten as follows:

$$\dot{u}_t = A(\mu)u_t + R(\mu)u_t, \tag{28}$$

where $u_t(\vartheta) = u(t + \vartheta)$ for $\vartheta \in [-\tau_*, 0]$. Let $\widetilde{C} = C([0, \tau_*], \mathbb{R})$. For $\psi \in \widetilde{C}$, define the operator A^* as

$$A^*(\mu)\psi(s) = \begin{cases} -\dfrac{d\psi(s)}{ds}, & s \in (0, \tau_*], \\ \displaystyle\int_{-\tau_*}^{0} d\eta(r, \mu)\,\psi(-r), & s = 0. \end{cases} \tag{29}$$

Next, define the bilinear inner product of $\varphi \in C$ and $\psi \in \widetilde{C}$ as

$$\langle \psi, \varphi \rangle = \overline{\psi}(0)\varphi(0) - \int_{\vartheta=-\tau_*}^{0} \int_{\xi=0}^{\vartheta} \overline{\psi}(\xi - \vartheta)\,d\eta(\vartheta, 0)\,\varphi(\xi)\,d\xi, \tag{30}$$

where the overbar denotes complex conjugation. Then $A(0)$ and $A^*(0)$ are adjoint operators, whose eigenvalues are $i\omega(\tau_*)$ and $-i\omega(\tau_*)$, respectively. A direct computation shows that the eigenvectors associated with them are

$$q(\vartheta) = e^{i\omega(\tau_*)\vartheta}, \quad \vartheta \in -[\tau_*, 0],$$
$$q^*(s) = Be^{i\omega(\tau_*)s}, \quad s \in [0, \tau_*], \tag{31}$$

respectively, where

$$B = \frac{1}{1 - ((a/\tau) + m)\,\tau_* e^{i\omega(\tau_*)\tau_*}}. \tag{32}$$

Notice that $\langle q^*, \overline{q} \rangle = 0$. In addition, this value of B guarantees that $\langle q^*, q \rangle = 1$. In the remainder of this section, following the theory and the same notations as in Hassard et al. [16], we compute the coordinates to describe the center manifold $\mathscr{C}$ at $\mu = 0$. Let u_t be the solution of (3) when $\mu = 0$. Define

$$z(t) = \langle q^*, u_t \rangle,$$
$$W(t, \vartheta) = u_t(\vartheta) - 2\,\mathrm{Re}[z(t)q(\vartheta)]. \tag{33}$$

On the center manifold $\mathscr{C}$, we have $W(t, \vartheta) = W(z(t), \overline{z}(t), \vartheta)$, where

$$W(z, \overline{z}, \vartheta) = W_{20}(\vartheta)\frac{z^2}{2} + W_{11}(\vartheta)z\overline{z} + W_{02}(\vartheta)\frac{\overline{z}^2}{2} + \cdots. \tag{34}$$

In the above relations z and $\overline{z}$ are local coordinates for the center manifold $\mathscr{C}$ in the direction of q^* and $\overline{q}^*$. From (27) for solutions $u_t \in \mathscr{C}$, since $\mu = 0$, we have

$$\begin{aligned} \dot{z}(t) &= \langle q^*, \dot{u}_t\rangle = \langle q^*, A(\mu) u_t\rangle + \langle q^*, R(\mu) u_t\rangle \\ &= i\omega(\tau_*) z + \overline{q}^*(0) f_0(z,\overline{z}), \end{aligned} \quad (35)$$

with

$$f_0(z,\overline{z}) = f(0, W(z,\overline{z},0) + 2\operatorname{Re}[z(t) q(0)]). \quad (36)$$

Equation (35) can be rewritten as

$$\dot{z}(t) = i\omega(\tau_*) z(t) + g(z,\overline{z}), \quad (37)$$

where

$$\begin{aligned} g(z,\overline{z}) &= \overline{q}^*(0) f(0, W(z,\overline{z},0) + 2\operatorname{Re}[z(t) q(0)]) \\ &= g_{20}\frac{z^2}{2} + g_{11} z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + g_{21}\frac{z^2\overline{z}}{6} + \cdots. \end{aligned} \quad (38)$$

Hence, (35) and (38) yield

$$\begin{aligned} \dot{W} &= \dot{u}_t - \dot{z}q - \dot{\overline{z}}\overline{q} \\ &= \begin{cases} A(0) W - 2\operatorname{Re}[\overline{q}^*(0) f_0 q(\vartheta)], & \vartheta \in [-\tau_*, 0), \\ A(0) W - 2\operatorname{Re}[\overline{q}^*(0) f_0 q(0)] + f_0, & \vartheta = 0, \end{cases} \\ &= AW + H(z,\overline{z},\vartheta), \end{aligned} \quad (39)$$

where

$$H(z,\overline{z},\vartheta) = H_{20}(\vartheta)\frac{z^2}{2} + H_{11}(\vartheta) z\overline{z} + H_{02}(\vartheta)\frac{\overline{z}^2}{2} + \cdots. \quad (40)$$

Noticing that $u_t(\vartheta) = W(z,\overline{z},\vartheta) + zq(\vartheta) + \overline{z}\overline{q}(\vartheta)$, we get $u_t(0) = W(z,\overline{z},0) + z + \overline{z}$. Thus,

$$\begin{aligned} f_0(z,\overline{z}) &= f(0, W(z,\overline{z},0) + 2\operatorname{Re}[z(t) q(0)]) \\ &= -\varepsilon[W(z,\overline{z},0) + z + \overline{z}]^3 \\ &= (-3\varepsilon) z^2\overline{z} + (-3\varepsilon) z\overline{z}^2 + \cdots. \end{aligned} \quad (41)$$

On the other hand, we have

$$\begin{aligned} g(z,\overline{z}) &= \overline{q}^*(0) f_0(z,\overline{z}) = \overline{q}^*(0) f(0, u_t) = \overline{B}\left[-\varepsilon u_t(0)^3\right] \\ &= \overline{B}\left[(-\varepsilon) z^2\overline{z} + (-\varepsilon) z\overline{z}^2 + \cdots\right] \\ &= g_{20}\frac{z^2}{2} + g_{11} z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + g_{21}\frac{z^2\overline{z}}{6} + \cdots. \end{aligned} \quad (42)$$

Comparing these coefficients we find

$$\begin{aligned} g_{20} &= g_{11} = g_{02} = 0, \\ g_{21} &= -6\varepsilon\overline{B} = \frac{-6\varepsilon}{1 - ((a/\tau) + m)\tau_* e^{-i\omega(\tau_*)\tau_*}}. \end{aligned} \quad (43)$$

Based on the above analysis, each g_{ij} is computed and we can calculate the following quantities:

$$\begin{aligned} c_1(0) &= \frac{i}{2\omega(\tau_*)}\left[g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right] + \frac{g_{21}}{2}, \\ \mu_2 &= -\frac{\operatorname{Re}\{c_1(0)\}}{\operatorname{Re}\{\lambda'(\tau_*)\}}, \\ \beta_2 &= 2\operatorname{Re}\{c_1(0)\}, \\ T_2 &= -\frac{Im\{c_1(0)\} + \mu_2 Im\{\lambda'(\tau_*)\}}{\omega(\tau_*)}, \end{aligned} \quad (44)$$

which determine the properties of bifurcating periodic solutions. Specifically, μ_2, β_2, and T_2 determine the direction, stability, and period of the corresponding Hopf bifurcation, respectively (see Hassard et al. [16]). A direct computation shows

$$\begin{aligned} c_1(0) &= \frac{g_{21}}{2} \\ &= \frac{-3\varepsilon\{1 - ((a/\tau_*) + m)\tau_* \cos(\omega(\tau_*)\tau_*)\}}{m^2\tau_*^2 + 2(am+n)\tau_* + (1-a)^2} \\ &\quad + i\frac{3\varepsilon((a/\tau_*) + m)\tau_* \cos(\omega(\tau_*)\tau_*)}{m^2\tau_*^2 + 2(am+n)\tau_* + (1-a)^2}, \end{aligned} \quad (45)$$

so that using (8) we get

$$\beta_2 = \frac{-6\varepsilon(1 - a + n\tau_*)}{m^2\tau_*^2 + 2(am+n)\tau_* + (1-a)^2} < 0. \quad (46)$$

Theorem 6. *(1) The direction of the Hopf bifurcation of* (3) *at the trivial equilibrium $K_* = 0$ when $\tau = \tau_*$ is subcritical (resp., supercritical) if $\mu_2 < 0$ (resp., $\mu_2 > 0$); that is, there exists a bifurcating periodic solution for $\tau < \tau_*$ (resp. $\tau > \tau_*$) in the sufficiently small τ_*-neighborhood.*

(2) The bifurcating periodic solution is locally asymptotically stable since $\beta_2 < 0$.

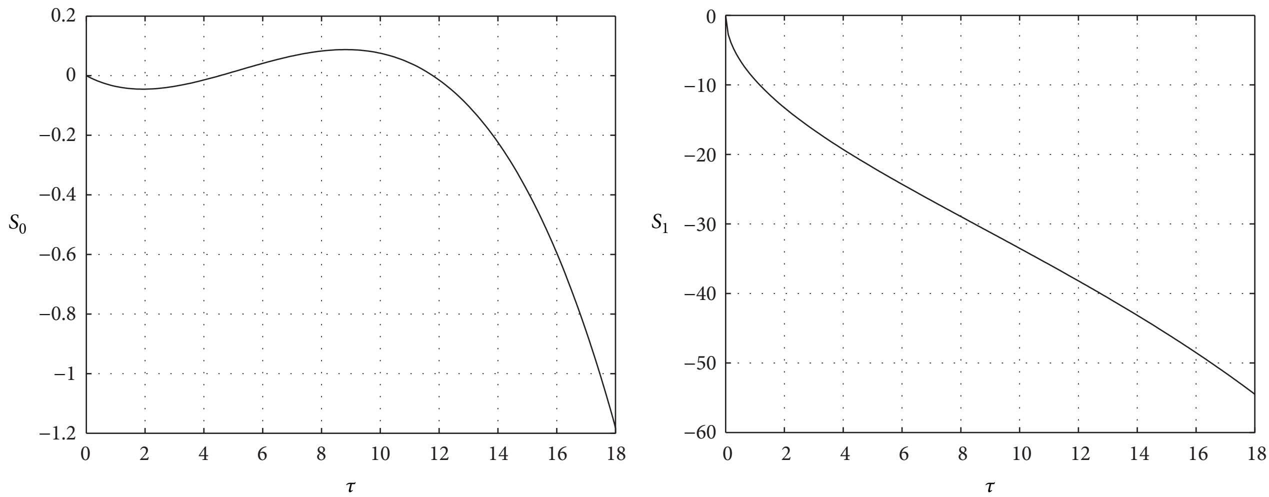
(3) The period of the bifurcating periodic solution decreases (resp., increases) with respect to τ if T_2 <0 (resp., $T_2 > 0$).

4. Numerical Simulations

In this section, we present numerical results to verify the analytical predictions obtained in the previous sections. We set two sets of parameters (a, m, n, and ε).

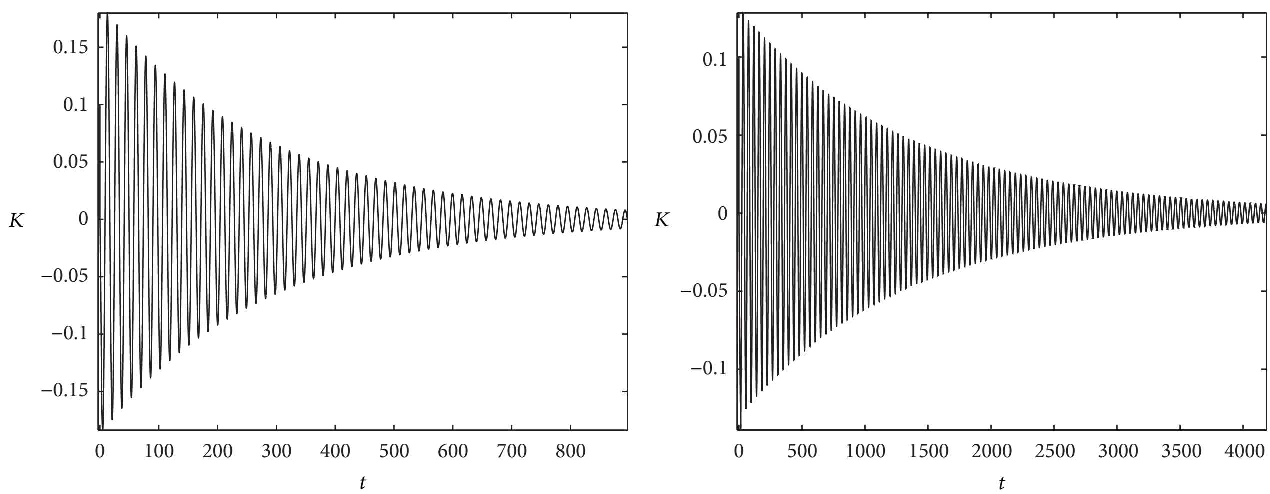
(i) $a = 0.95$, $m = 0.1$, $n = 0.15$, $\varepsilon = 0.01$, and $\overline{\tau} = 38$.

$S_0(\tau)$ has two positive roots in this case (see Figure 1(a)), say $\tau_1 \approx 4.5$ and $\tau_2 \approx 11.8$, while $S_1(\tau)$ is strictly negative (see Figure 1(b)). Hence, all $S_j(\tau)$ functions with $j \geq 1$ have no real roots. The equilibrium is asymptotically stable when $\tau \in [0, \tau_1)$ (see Figure 2(a), where $\tau = 3$) and $\tau \in (\tau_2, \overline{\tau})$ (see Figure 2(b), where $\tau = 14$). When $\tau \in (\tau_1, \tau_2)$, the equilibrium is unstable (see Figure 3, where $\tau = 4.6$). At τ_1 and τ_2, Hopf bifurcation occurs. Furthermore, we know that



(a) (b)

Figure 1



(a) (b)

Figure 2

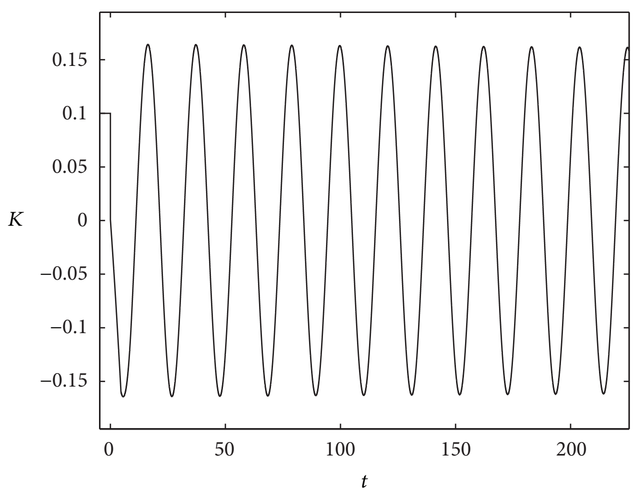
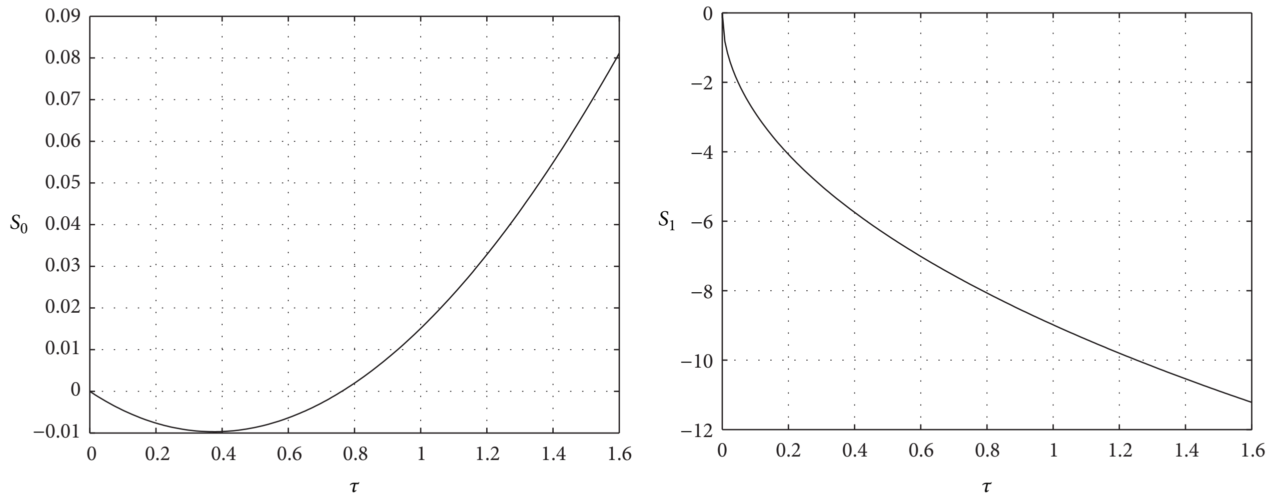
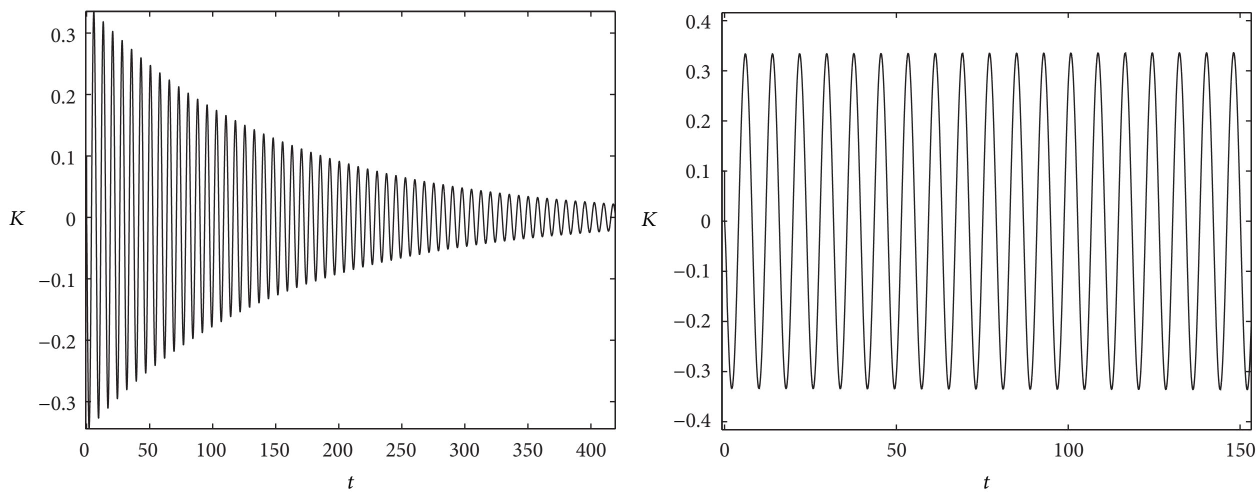


Figure 3



(a) (b)

Figure 4



(a) (b)

Figure 5

$\beta_2 < 0$ and $\Gamma(\tau_*) > 0$. Therefore, the Hopf bifurcation at the equilibrium is supercritical and the bifurcating periodic solutions are orbitally asymptotically stable (see Figure 3, where $\tau = 4.6$).

(ii) $a = 0.95$, $m = 0.15$, $n = 0.1$, and $\varepsilon = 0.01$.

$S_0(\tau)$ does intersect once with the τ-axis at $\tau_1 \approx 0.76$ (see Figure 4(a)) and $S_1(\tau)$ is strictly negative (see Figure 4(b)). Hence, the equilibrium is asymptotically stable for $\tau \in [0, \tau_1)$ (see Figure 5(a), where $\tau = 0.7$), and sustained oscillations occur when $\tau \in (\tau_1, +\infty)$ due to Hopf bifurcations (see Figure 5(b), where $\tau = 0.77$). Also in this case, having $\beta_2 < 0$ and $\Gamma(\tau_*) > 0$, the bifurcation is supercritical and the bifurcating periodic solution is orbitally asymptotically stable (see Figure 5(b), where $\tau = 0.77$).

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