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Research Article

On the Largest Disc Mapped by Sum of Convex and Starlike Functions

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For a normalized analytic function f defined on the unit disc $\mathbb{D}$, let $\phi(f, f', f''; z)$ be a function of positive real part in $\mathbb{D}$, $\psi(f, f', f''; z)$ need not have that property in $\mathbb{D}$, and $\chi = \phi + \psi$. For certain choices of ϕ and ψ, a sharp radius constant ρ is determined, $0 < \rho < 1$, so that $\chi(\rho z)/\rho$ maps $\mathbb{D}$ onto a specified region in the right half-plane.

1. Introduction

Let $\mathcal{A}$ be the class of functions f analytic in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by $f(0) = 0 = f'(0) - 1$. Let $\mathcal{S}$ be its subclass consisting of univalent functions. For two analytic functions f and g, the function f is subordinate to g, written $f(z) \prec g(z)$, if there is an analytic self-map $w : \mathbb{D} \to \mathbb{D}$ with $w(0) = 0$ satisfying $f(z) = g(w(z))$. Given an analytic function p with $p(0) = 1$ and $\operatorname{Re} p(z) > 0$ in $\mathbb{D}$, denote by $\mathcal{ST}(p)$ and $\mathcal{CV}(p)$ the subclasses of $\mathcal{A}$ consisting, respectively, of f satisfying $zf'(z)/f(z) \prec p(z)$ and $1 + zf''(z)/f'(z) \prec p(z)$.

For various choices of p, these classes reduce to well-known subclasses of starlike and convex functions. For instance, with $p(z) = (1 + (1 - 2\alpha)z)/(1 - z)$, $0 \le \alpha < 1$, then $\mathcal{ST}(\alpha)$ and $\mathcal{CV}(\alpha)$ are, respectively, the subclasses consisting of *starlike functions of order* α and *convex functions of order* α. The classes $\mathcal{ST} := \mathcal{ST}(0)$ and $\mathcal{CV} := \mathcal{CV}(0)$ are the familiar subclasses of $\mathcal{S}$ of starlike and convex functions. For $p(z) = (1 + (1 - 2\beta)z)/(1 - z)$, $\beta > 1$, $\mathcal{M}(\beta) = \mathcal{ST}(p)$ is the class of functions $f \in \mathcal{A}$ satisfying

$$\mathcal{M}(\beta) := \left\{ f \in \mathcal{A} : \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) < \beta \ (z \in \mathbb{D}) \right\} \quad (1)$$

studied by Uralegaddi et al. [1]. Various subclasses of $\mathcal{M}(\beta)$ have been investigated in [2–5]. For $p(z) = ((1 + z)/(1 - z))^{\alpha}$, $0 < \gamma \le 1$, the class $\mathcal{SST}(\gamma) := \mathcal{ST}(p)$ is the class of *strongly starlike functions of order* γ. The class $\mathcal{S}_{\mathcal{L}} := \mathcal{ST}(\sqrt{1 + z})$ introduced by Sokół and Stankiewicz [6] consists of functions $f \in \mathcal{A}$ satisfying

$$\left| \left(\frac{zf'(z)}{f(z)} \right)^2 - 1 \right| < 1 \quad (z \in \mathbb{D}). \quad (2)$$

Thus, a function $f \in \mathcal{A}$ is in the class $\mathcal{S}_{\mathcal{L}}$ if $zf'(z)/f(z)$ lies in the region bounded by the right-half of the lemniscate of Bernoulli given by $|w^2 - 1| < 1$. Results related to the class $\mathcal{S}_{\mathcal{L}}$ can be found in [3, 7–9].

In investigating the class $\mathcal{UCV}$ of uniformly convex functions, Rønning [10] introduced a class $\mathcal{S}_{\mathcal{P}}$ of *parabolic starlike functions*. These are functions $f \in \mathcal{A}$ satisfying

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \left| \frac{zf'(z)}{f(z)} - 1 \right| \quad (z \in \mathbb{D}). \quad (3)$$

It is important to keep in mind that the qualifier "parabolic" refers to the geometry of the image of $\mathbb{D}$ under the map $zf'(z)/f(z)$; that is, the domain necessarily lies in a parabolic region of the w-plane. It does not convey the interpretation that the function f maps the disk $\mathbb{D}$ onto a parabolic region. This terminology of *parabolic starlike functions* is however widely accepted and used by authors. Ali and Ravichandran [11] recently surveyed works on uniformly convex and *parabolic starlike functions*.

This paper finds radius estimates for classes of functions in $\mathcal{A}$. The radius of a property P in a given set of functions $\mathcal{M}$ [12, page 119] is the largest number R such that every function in the set $\mathcal{M}$ has the property P in each disc $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$ for every $r < R$. For example, the Koebe function $k(z) = z/(1-z)^2$, which maps $\mathbb{D}$ onto the domain $\mathbb{C} \setminus \{w \in \mathbb{R} : w \le -1/4\}$, is starlike but not convex. However, k maps the disc $\mathbb{D}_r$ onto a convex domain for every $r \le 2-\sqrt{3}$. Indeed, every univalent function $f \in \mathcal{S}$ maps $\mathbb{D}_r$ onto a convex domain for $r \le 2-\sqrt{3}$ [13, Theorem 2.13, page 44]. This number is known as the radius of convexity for $\mathcal{S}$.

It is known that $\mathcal{CV} \subseteq \mathcal{ST}(1/2) \subseteq \{f \in \mathcal{A} : \operatorname{Re}(f(z)/z) > 1/2, z \in \mathbb{D}\}$. The function $g(z) = z/(1-z)$ is convex and therefore starlike of order $1/2$; it is clear that the function

$$\phi(z) := \frac{zg'(z)}{g(z)} \tag{4}$$

has real part greater than 1/2. Now the function

$$\psi(z) := \frac{z^2 g''(z)}{g(z)} \tag{5}$$

takes values in $\mathbb{C} \setminus \{w \in \mathbb{R} : w \le -1/2\}$, and therefore does not have positive real part for all $z \in \mathbb{D}$. Their sum

$$\begin{aligned}\phi(z) + \psi(z) &= \frac{zg'(z)}{g(z)} + \frac{z^2 g''(z)}{g(z)} \\ &= \frac{zg'(z)}{g(z)}\left(1 + \frac{zg''(z)}{g'(z)}\right)\end{aligned} \tag{6}$$

takes values in $\{w := x+iy \in \mathbb{C} : y^2 > -x/2\}$ and therefore the sum $\phi+\psi$ does not have positive real part in $\mathbb{D}$. This motivates us to determine the largest radius ρ such that

$$\operatorname{Re}\left(\frac{z^2 g''(z)}{g(z)} + \frac{zg'(z)}{g(z)}\right) > \alpha \quad (|z| \le \rho). \tag{7}$$

More generally, let $\phi = \phi(f, f', f''; z)$ and $\psi = \psi(f, f', f''; z)$ be functions satisfying $\operatorname{Re}\phi > 0$ in $\mathbb{D}$, while $\operatorname{Re}\psi$ need not necessarily be positive in the whole unit disc $\mathbb{D}$. For certain choices of ϕ and ψ, a sharp radius constant ρ is determined, $0 \le \rho < 1$, so that whenever $|z| < \rho$, the sum $\phi+\psi$ takes values in specified regions in the complex plane. The results obtained are shown to reduce those of Singh and Paul [14] in certain special cases.

2. Main Results

For $\phi(z) := \phi(f, f', f''; z) = zf'(z)/f(z)$ and $\psi(z) := \psi(f, f', f''; z) = z^2 f''(z)/f(z)$, with $f \in \mathcal{ST}(1/2)$, several radius results for the sum $\phi+\psi$ to be in certain regions in the complex plane are obtained in the following result.

Theorem 1. *Let $f \in \mathcal{ST}(1/2)$; let $\chi : \mathbb{D} \to \mathbb{C}$ be defined by*

$$\begin{aligned}\chi(z) &= \frac{z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)}, \\ \chi_i(z) &= \chi(\rho_i z), \quad i = 1, 2, \ldots, 6.\end{aligned} \tag{8}$$

Then

(a) $\operatorname{Re}\chi_1(z) > \alpha$, $0 \le \alpha < 1$, *where ρ_1 is given by*

$$\rho_1 = \begin{cases} \sqrt{\dfrac{7-16\alpha}{11-16\alpha+8\sqrt{2-4\alpha}}}, & 0 \le \alpha \le \dfrac{2+\sqrt{13}}{18} \\ \dfrac{2(1-\alpha)}{1+2\alpha+\sqrt{1+8\alpha}}, & \dfrac{2+\sqrt{13}}{18} < \alpha < 1. \end{cases} \tag{9}$$

(b) $|\chi_2^2(z) - 1| < 1$, *where ρ_2 is given by*

$$\rho_2 = \frac{2\left(\sqrt{2}-1\right)}{1+2\sqrt{2}+\sqrt{1+8\sqrt{2}}} \simeq 0.112903. \tag{10}$$

(c) $\operatorname{Re}\chi_3(z) < \beta$, $\beta > 1$, *where ρ_3 is given by*

$$\rho_3 = \frac{2(\beta-1)}{1+2\beta+\sqrt{1+8\beta}}. \tag{11}$$

(d) $|\chi_4(z) - 1| < 1-\alpha$, $0 \le \alpha < 1$, *where ρ_4 is given by*

$$\rho_4 = \frac{2(1-\alpha)}{5-2\alpha+\sqrt{17-8\alpha}}. \tag{12}$$

(e) $|\arg(\chi_5(z))| < \gamma\pi/2$, $0 < \gamma \le 1$ *where $\rho_5 = \rho_5(\gamma) \in (0,1)$ is the root of the equation in r:*

$$\begin{aligned}&\left(1+2\left(1-r^2\right)t_0\right)\sqrt{4t_0 - \left(1+t_0-r^2 t_0\right)^2} \\ &\quad - \left(1 + t_0\left(-1-3r^2+2\left(1-r^2\right)^2 t_0\right)\right)\tan\left(\frac{\pi\gamma}{2}\right) = 0, \\ &t_0 = \frac{5-r^2+\sqrt{9-10r^2+17r^4}}{8\left(1-r^4\right)}.\end{aligned} \tag{13}$$

In particular,

$$\begin{aligned}&\rho_5\left(\frac{1}{4}\right) \simeq 0.131522, \qquad \rho_5\left(\frac{1}{2}\right) \simeq 0.266747, \\ &\rho_5\left(\frac{3}{4}\right) \simeq 0.409049, \qquad \rho_5(1) \simeq 0.560097.\end{aligned} \tag{14}$$

(f) *Also* $|\chi_6(z) - 1| < \operatorname{Re}\chi_6(z)$, *where* $\rho_6 \simeq 0.23605 \in (0,1)$ *is the root of the following equation in r:*

$$\begin{aligned}&2\left(1+9r^2\right)t_0 - 1 + \left(-5+26r^2-21r^4\right)t_0^2 \\ &\quad + 4\left(1-r^2\right)^2\left(1+3r^2\right)t_0^3 - 4\left(1-r^2\right)^4 t_0^4 = 0,\end{aligned} \tag{15}$$

and $t_0 \in (1/(1+r)^2, 1/(1-r)^2)$ *is the root of the equation in* t:

$$\begin{aligned} &2\left(1+9r^2\right)+2\left(-5+26r^2-21r^4\right)t \\ &\quad+12\left(1-r^2\right)^2\left(1+3r^2\right)t^2-16\left(1-r^2\right)^4t^3=0. \end{aligned} \tag{16}$$

Each radius constant ρ_i *is sharp.*

For two analytic functions $f, g \in \mathcal{A}$, their *convolution or Hadamard product*, denoted by $f * g$, is defined by $(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n$. The following results are needed in the sequel.

Lemma 2 ([15, Lemma 2.7, page 126; Lemma 3.5, page 130]). *If* $f \in \mathcal{CV}$ *and* $g \in \mathcal{ST}$, *or* f *and* g *belong to* $\mathcal{ST}(1/2)$, *then*

$$\frac{f * gF}{f * g}(\mathbb{D}) \subset \overline{\mathrm{co}}(F(\mathbb{D})) \tag{17}$$

for any function F *analytic in* $\mathbb{D}$, *where* $\overline{\mathrm{co}}(F(\mathbb{D}))$ *denotes the closed convex hull of* $F(\mathbb{D})$.

Lemma 3 ([7, Lemma 2.2, page 6559]). *For* $0 < a < \sqrt{2}$, *let* r_a *be given by*

$$r_a = \begin{cases} \left(\sqrt{1-a^2}-\left(1-a^2\right)\right)^{1/2}, & 0 < a \le 2\sqrt{2}/3 \\ \sqrt{2}-a, & 2\sqrt{2}/3 \le a < \sqrt{2}, \end{cases} \tag{18}$$

and for $a > 0$, *let* R_a *be given by*

$$R_a = \begin{cases} \sqrt{2}-a, & 0 < a \le \dfrac{1}{\sqrt{2}}, \\ a, & \dfrac{1}{\sqrt{2}} \le a. \end{cases} \tag{19}$$

Then,

$$\begin{aligned} \{w : |w-a| < r_a\} &\subseteq \left\{w : \left|w^2-1\right| < 1\right\} \\ &\subseteq \{w : |w-a| < R_a\}. \end{aligned} \tag{20}$$

Proof of Theorem 1. Let $h : \mathbb{D} \to \mathbb{C}$ be defined by

$$h(z) = \frac{2}{(1-z)^2} - \frac{1}{1-z}. \tag{21}$$

First, for each $i = 1, 2, \ldots, 6$, $h_i(z) = h(\rho_i z)$ will be shown to, respectively, satisfy $\operatorname{Re} h_1(z) > \alpha$, $|h_2(z)-1| < 1$, $\operatorname{Re} h_3(z) < \beta$, $|h_4(z)-1| < 1-\alpha$, $|\arg h_5(z)| < \gamma\pi/2$, and $|h_6(z)-1| < \operatorname{Re} h_6(z)$. Then, using Lemma 2, χ_i is deduced to satisfy the required condition.

As in [14], let

$$\frac{1}{1-z} = Re^{i\theta}, \tag{22}$$

so that

$$\begin{aligned} \frac{1}{1+r} \le R \le \frac{1}{1-r} \quad (|z| = r), \\ \cos\theta = \frac{1+R^2-r^2R^2}{2R}. \end{aligned} \tag{23}$$

(a) By (21), (22), and (23), it follows that

$$\begin{aligned} \operatorname{Re} h(z) &= 2R^2\cos 2\theta - R\cos\theta \\ &= \frac{1}{2} - \frac{1}{2}\left(1+3r^2\right)t + \left(1-r^2\right)^2 t^2 \quad \left(t := R^2\right) \\ &=: \phi(t). \end{aligned} \tag{24}$$

Case (i). Suppose that $0 \le \alpha \le (2+\sqrt{13})/18$. We assert that $\min \phi(t) > \alpha$ for $|z| < \rho_1$, where the minimum is taken over all $t \in (1/(1+r)^2, 1/(1-r)^2)$. Let $r < \rho_1$. Then

$$\frac{\partial \phi(t)}{\partial t} = -\frac{1}{2}\left(1+3r^2\right) + 2\left(1-r^2\right)^2 t = 0 \tag{25}$$

if $t = t_0 := (1+3r^2)/(4(1-r^2)^2)$, $\partial^2\phi(t_0)/\partial t^2 > 0$, and that for $r \ge 4-\sqrt{13}$,

$$\frac{1}{(1+r)^2} \le t_0 \le \frac{1}{(1-r)^2}. \tag{26}$$

Thus, for $4-\sqrt{13} \le r < \rho_1$,

$$\min \phi(t) = \phi(t_0) = \frac{7-22r^2-r^4}{16\left(1-r^2\right)^2} > \alpha. \tag{27}$$

On the other hand, if $r < 4-\sqrt{13}$, then it can be shown that

$$\begin{aligned} \min \phi(t) &= \phi\left(\frac{1}{(1+r)^2}\right) \\ &= \frac{1-r}{(1+r)^2} > \frac{\sqrt{13}-3}{\left(5-\sqrt{13}\right)^2} \\ &= \frac{2+\sqrt{13}}{18} > \alpha. \end{aligned} \tag{28}$$

Case (ii). For $(2+\sqrt{13})/18 < \alpha < 1$, then $\min \phi(t) > \alpha$ in $|z| < \rho_1$, $t \in (1/(1+r)^2, 1/(1-r)^2)$. Indeed for $r < \rho_1 < 4-\sqrt{13}$, as in the case (i),

$$\min \phi(t) = \phi\left(\frac{1}{(1+r)^2}\right) = \frac{1-r}{(1+r)^2} > \alpha. \tag{29}$$

The previously mentioned two cases show that $\operatorname{Re} h_1(z) > \alpha$ in $\mathbb{D}$. Figure 1 illustrates sharpness of the radius $\rho_1 = \sqrt{5}-2$ in the case $\alpha = 0.5$.

(b) For h given by (21), a calculation shows that

$$\begin{aligned} |h(z)-1| &= \left|\frac{2}{(1-z)^2} - \frac{1}{1-z} - 1\right| \\ &= \left|\frac{z}{1-z} + \frac{2z}{(1-z)^2}\right| \\ &\le \frac{r}{1-r} + \frac{2r}{(1-r)^2}. \end{aligned} \tag{30}$$

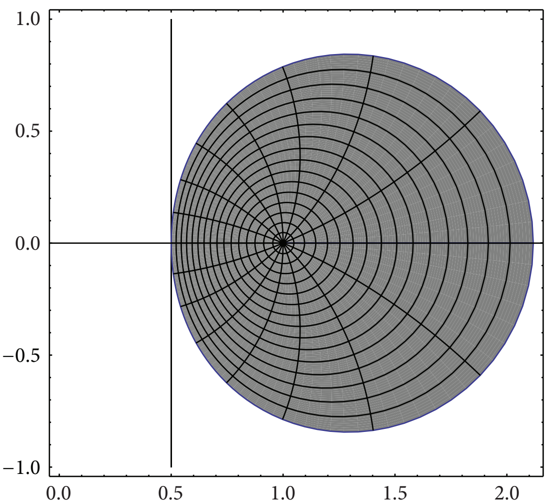

Figure 1: Image of $|z| \le \sqrt{5}-2$ touches Re $w = 0.5$.

By Lemma 3, the function h satisfies

$$\left|h^2(z)-1\right| < 1 \tag{31}$$

provided

$$\frac{r}{1-r}+\frac{2r}{(1-r)^2} \le \sqrt{2}-1; \tag{32}$$

that is,

$$\sqrt{2}r^2-\left(1+2\sqrt{2}\right)r+\left(\sqrt{2}-1\right) \ge 0. \tag{33}$$

This inequality holds if $r \le \rho_2$. Figure 2 illustrates sharpness of the radius $\rho_2 \simeq 0.1129$.

(c) From (30), it follows that

$$\operatorname{Re} h(z) \le 1+\frac{r}{1-r}+\frac{2r}{(1-r)^2} \le \beta \tag{34}$$

provided

$$\beta r^2-(1+2\beta)r-(1-\beta) \ge 0 \tag{35}$$

holds, which occurs whenever $r \le \rho_3$. Sharpness of the radius $\rho_3 = (4-\sqrt{13})/3$ in the case $\beta = 1.5$ is illustrated in Figure 3.

(d) Inequality (30) also yields

$$|h(z)-1| \le \frac{r}{1-r}+\frac{2r}{(1-r)^2} \le 1-\alpha \tag{36}$$

provided

$$r^2(\alpha-2)+r(5-2\alpha)+\alpha-1 \le 0, \tag{37}$$

that is, when $r \le \rho_4$. Figure 4 illustrates sharpness of the radius $\rho_4 = (4-\sqrt{13})/3$ in the case $\alpha = 0.5$.

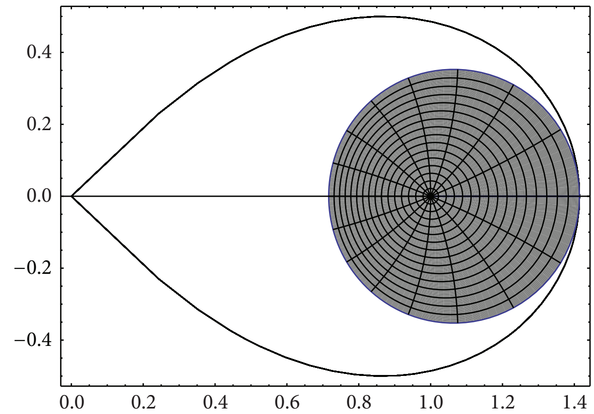

Figure 2: Image of $|z| \le 0.1129$ touches $|w^2-1| = 1$.

(e) For the function h given by (21), it follows from (22) and (23) that

$$\begin{aligned}
\arg h(z) &= \arctan\left(\frac{2R\sin 2\theta-\sin\theta}{2R\cos 2\theta-\cos\theta}\right)\\
&= \arctan\left(\frac{(4R\cos\theta-1)\sin\theta}{2R\left(2\cos^2\theta-1\right)-\cos\theta}\right)\\
&= \arctan\left(\frac{\left(1+2\left(1-r^2\right)R^2\right)\sqrt{4R^2-\left(1+\left(1-r^2\right)R^2\right)^2}}{1-\left(1+3r^2\right)R^2+2\left(1-r^2\right)^2R^4}\right)\\
&= \arctan\left(\frac{\left(1+2\left(1-r^2\right)t\right)\sqrt{4t-\left(1+\left(1-r^2\right)t\right)^2}}{1-\left(1+3r^2\right)t+2\left(1-r^2\right)^2t^2}\right)\\
&:= \arctan(\phi(t)),
\end{aligned} \tag{38}$$

$t \in (1/(1+r)^2, 1/(1-r)^2)$. A calculation shows that $\phi'(t) = 0$ where

$$t = t_0 = \frac{5-r^2+\sqrt{9-10r^2+17r^4}}{8\left(1-r^4\right)} \in \left(\frac{1}{(1+r)^2}, \frac{1}{(1-r)^2}\right). \tag{39}$$

Now $\phi'(t) > 0$ for $t < t_0$, $\phi'(t) < 0$ if $t > t_0$, and

$$\phi\left(\frac{1}{(1+r)^2}\right) = \phi\left(\frac{1}{(1-r)^2}\right) = 0. \tag{40}$$

Thus

$$\max\phi(t) = \phi(t_0) = \frac{\left(1+2\left(1-r^2\right)t_0\right)\sqrt{4t_0-\left(1+\left(1-r^2\right)t_0\right)^2}}{1-\left(1+3r^2\right)t_0+2\left(1-r^2\right)^2t_0^2}. \tag{41}$$

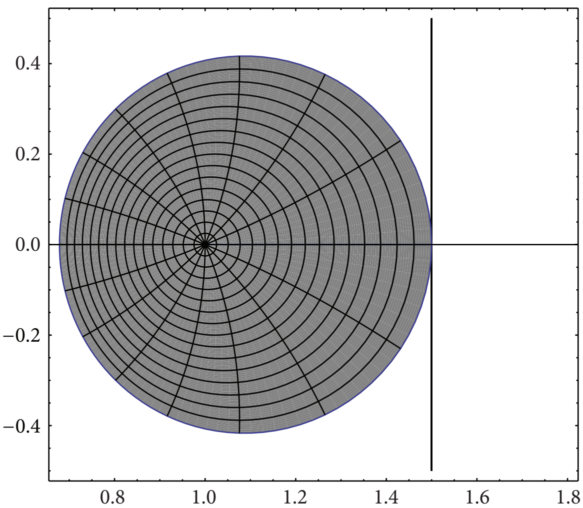

FIGURE 3: Image of $|z| \le (4-\sqrt{13})/3$ touches Re $w = 1.5$.

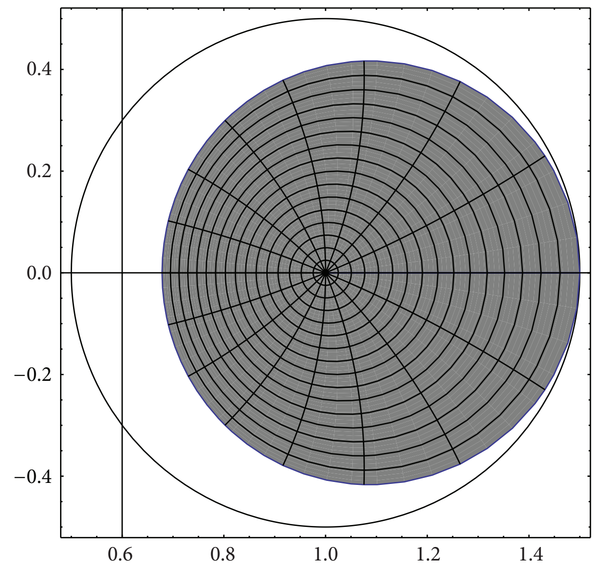

FIGURE 4: Image of $|z| \le (4-\sqrt{13})/3$ touches $|w-1| = 0.5$.

Evidently, (38) and (41) give

$$\begin{aligned} &\left|\arg h(z)\right| \\ &\quad \le \left|\arctan\left(\frac{\left(1+2\left(1-r^2\right)t_0\right)\sqrt{4t_0-\left(1+\left(1-r^2\right)t_0\right)^2}}{1-\left(1+3r^2\right)t_0+2\left(1-r^2\right)^2 t_0^2}\right)\right| \\ &\quad \le \frac{\gamma\pi}{2} \end{aligned} \tag{42}$$

provided

$$\begin{aligned} &\left(1+2\left(1-r^2\right)t_0\right)\sqrt{4t_0-\left(1+\left(1-r^2\right)t_0\right)^2} \\ &\quad -\left(1+t_0\left(-1-3r^2+2\left(1-r^2\right)^2 t_0\right)\right)\tan\left(\frac{\pi\gamma}{2}\right) \\ &\quad \le 0. \end{aligned} \tag{43}$$

Figure 5 illustrates sharpness of the radius $\rho_5 \simeq 0.266747$ in the case $\gamma = 0.5$.

(f) The inequality

$$|h(z)-1| < \operatorname{Re} h(z) \tag{44}$$

holds if

$$\begin{aligned} &\left(2R^2\cos 2\theta - R\cos\theta - 1\right)^2 + \left(2R^2\sin 2\theta - R\sin\theta\right)^2 \\ &\quad < \left(2R^2\cos 2\theta - R\cos\theta\right)^2, \end{aligned} \tag{45}$$

or, with $t = R^2$,

$$\begin{aligned} \phi(t) :=\ & 2\left(1+9r^2\right)t - 1 + \left(-5+26r^2-21r^4\right)t^2 \\ & + 4\left(1-r^2\right)^2\left(1+3r^2\right)t^3 - 4\left(1-r^2\right)^4 t^4 < 0, \end{aligned} \tag{46}$$

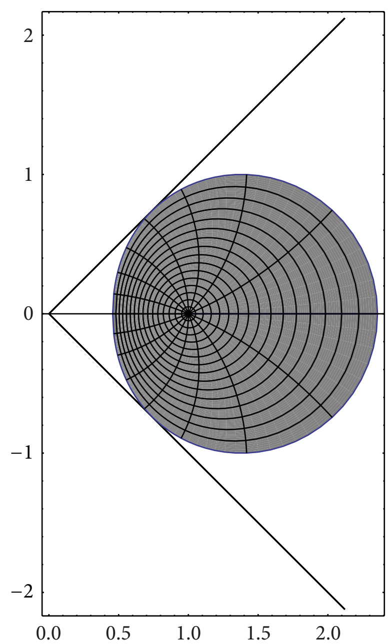

FIGURE 5: Image of $|z| \le 0.266747$ touches $|\arg w| = \pi/4$.

$t \in (1/(1+r)^2, 1/(1-r)^2)$. Then,

$$\begin{aligned} \phi'(t) =\ & 2\left(1+9r^2\right) + 2\left(-5+26r^2-21r^4\right)t \\ & + 12\left(1-r^2\right)^2\left(1+3r^2\right)t^2 - 16\left(1-r^2\right)^4 t^3. \end{aligned} \tag{47}$$

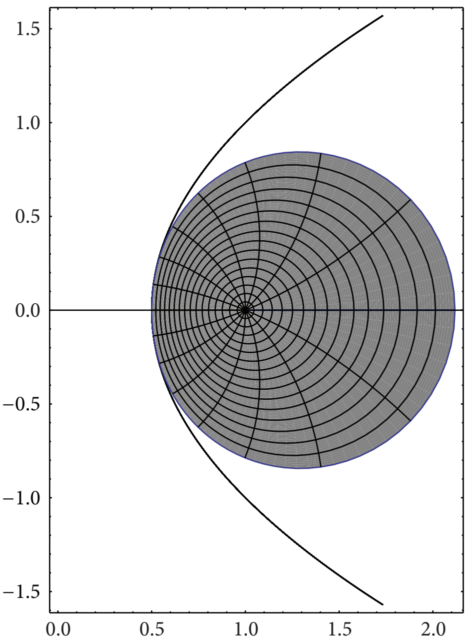

FIGURE 6: Image of $|z| \le 0.23605$ touches $|w-1| = \operatorname{Re} w$.

Let $r < \rho_6$. Since

$$\begin{aligned}\phi'\left(\frac{1}{(1+r)^2}\right) &= \frac{4\left(-3+11r+6r^2+7r^3-r^4\right)}{(1+r)^2} > 0 \quad \text{if } r > 0.234722,\\ \phi'\left(\frac{1}{(1-r)^2}\right) &= -\frac{4\left(3+11r-6r^2+7r^3+r^4\right)}{(1-r)^2} < 0,\end{aligned} \tag{48}$$

there exists a unique $t_0 \in (1/(1+r)^2, 1/(1-r)^2)$ such that $\phi'(t_0) = 0$ and $\max \phi(t) = \phi(t_0)$.

Thus, $\max \phi(t) = \phi(t_0) < 0$ for $0.234722 < r < \rho_6$. When $r \le 0.234722$,

$$\max \phi(t) = \phi\left(\frac{1}{(1+r)^2}\right) = \frac{4\left(4r-1+r^2\right)}{(1+r)^2} < 0; \tag{49}$$

hence, $\phi(t) < 0$ for $r < \rho_6$. Figure 6 illustrates sharpness of the radius $\rho_6 \simeq 0.23605$.

Next, consider $g_i(z) = z/(1-\rho_i z) \in \mathcal{CV} \subset \mathcal{ST}(1/2)$, $i = 1, 2, \ldots, 6$. Then,

$$\begin{aligned}\frac{f(z) * g_i(z) h_i(z)}{f(z) * g_i(z)} &= \frac{f(z) * (z/(1-\rho_i z))\left(2\rho_i z/(1-\rho_i z)^2 + 1/(1-\rho_i z)\right)}{f(z) * (z/(1-\rho_i z))}\\ &= \frac{f(z) * \left(2\rho_i z^2/(1-\rho_i z)^3 + z/(1-\rho_i z)^2\right)}{f(z) * (z/(1-\rho_i z))}\\ &= \frac{\rho_i^2 z^2 f''(\rho_i z) + \rho_i z f'(\rho_i z)}{f(\rho_i z)}\\ &= \chi(\rho_i z) = \chi_i(z).\end{aligned} \tag{50}$$

Lemma 2, together with (50) and the corresponding inequality for the function h_i, shows that each function χ_i satisfies the required condition. For sharpness, consider the function $f_0(z) = z/(1-z) \in \mathcal{CV} \subset \mathcal{ST}(1/2)$. Then,

$$\frac{z^2 f_0''(z) + z f_0'(z)}{f_0(z)} = \frac{2}{(1-z)^2} - \frac{1}{1-z} = h(z). \tag{51}$$

Sharpness of the numbers ρ_i is now evident in view of the definition h. □

For $\alpha = 0$, Theorem 1(a) reduces to the following corollary.

Corollary 4 ([14, Theorem 5, page 724]). *If $f \in \mathcal{ST}(1/2)$, then*

$$\operatorname{Re}\left(\frac{z^2 f''(z)}{f(z)} + \frac{z f'(z)}{f(z)}\right) > 0 \tag{52}$$

in $|z| < \rho = \sqrt{8\sqrt{2}-11} = 0.56$. The number ρ is sharp.

Theorem 5. *Let $f \in \mathcal{ST}(1/2)$ and $\chi : \mathbb{D} \to \mathbb{C}$ be defined by*

$$\chi(z) = \frac{f(z)}{z} + f'(z), \qquad \chi_i(z) = \chi(\rho_i z), \quad i = 1, 2, 3. \tag{53}$$

Then

(a) $\operatorname{Re} \chi_1(z) > \alpha$, $0 \le \alpha < 1$, *where ρ_1 is given by*

$$\rho_1 = \begin{cases} \sqrt{\dfrac{7-8\alpha}{5-8\alpha+4\sqrt{2}\sqrt{1-\alpha}}}, & 0 \le \alpha < \dfrac{7+4\sqrt{7}}{18},\\ \dfrac{4-2\alpha}{2\alpha-1+\sqrt{1+4\alpha}}, & \dfrac{7+4\sqrt{7}}{18} \le \alpha < 1.\end{cases} \tag{54}$$

(b) $|\arg \chi_2(z)| < \gamma\pi/2$, $0 < \gamma \le 1$, *where $\rho_2 = \rho_2(\gamma) \in (0,1)$ is the root of the equation*

$$\begin{aligned}&\left(\sqrt{9+5r^2(-2+r^2)} - 2r^2\right)\\ &\quad \times \sqrt{16r^2 + \left(3+3r^2-\sqrt{9+5r^2(r^2-2)}\right)^2}\\ &\quad + \left(\sqrt{9+5r^2(r^2-2)} - 11 - 11r^4 + 5r^2\right.\\ &\quad \left.\times\left(2+\sqrt{9+5r^2(r^2-2)}\right)\right)\tan\left(\frac{\pi\gamma}{2}\right) = 0.\end{aligned} \tag{55}$$

In particular,

$$\rho_2\left(\frac{1}{4}\right) \simeq 0.257136, \qquad \rho_2\left(\frac{1}{2}\right) \simeq 0.487998,$$
$$\rho_2\left(\frac{3}{4}\right) \simeq 0.674274, \qquad \rho_2(1) = \sqrt{\frac{7}{5+4\sqrt{2}}} \simeq 0.810465. \tag{56}$$

(c) *Also* $|\chi_3(z) - 1| < \operatorname{Re}\chi_3(z)$, *where* $\rho_3 \simeq 0.44915$ *is given by the equation in* r

$$24r^2t_0 - 8 + \left(-1 + 18r^2 - 17r^4\right)t_0^2 + 2\left(1-r^2\right)^2\left(-1+3r^2\right)t_0^3 - \left(1-r^2\right)^4 t_0^4 = 0 \tag{57}$$

and t_0 *is given by the equation in* t

$$24r^2 + 2\left(-1 + 18r^2 - 17r^4\right)t + 6\left(1-r^2\right)^2\left(-1+3r^2\right)t^2 - 4\left(1-r^2\right)^4 t^3 = 0. \tag{58}$$

Each radius constant ρ_i *is sharp.*

Proof. Let

$$h(z) = \frac{1}{1-z} + \frac{1}{(1-z)^2} \quad (z \in \mathbb{D}). \tag{59}$$

Each $h_i(z) = h(\rho_i z)$, $i = 1, 2, 3$, is shown to, respectively, satisfy $\operatorname{Re} h_1(z) > \alpha$, $|\arg h_2(z)| < \gamma\pi/2$, and $|h_3(z) - 1| < \operatorname{Re} h_3(z)$. Then, it follows from Lemma 2 that χ_i satisfies the required condition.

(a) We claim that $\operatorname{Re} h(z) > \alpha$ in $|z| < \rho_1$. By (22) and (23),

$$\begin{aligned}\operatorname{Re} h(z) &= R\cos\theta + R^2\cos 2\theta \\ &= 1 + \frac{1}{2}\left(\left(1-3r^2\right)t + \left(1-r^2\right)^2 t^2\right) \quad \left(t := R^2\right) \\ &:= \phi(t).\end{aligned} \tag{60}$$

Case (i). Suppose $0 \le \alpha < (7+4\sqrt{7})/18$. In this case, it is shown that $\min\phi(t) > \alpha$ for $|z| < \rho_1$ over all t in $(1/(1+r)^2, 1/(1-r)^2)$. Let $r < \rho_1$. It can be verified that

$$\frac{\partial\phi(t)}{\partial t} = \frac{1}{2}\left(1 - 3r^2 + 2\left(1-r^2\right)^2 t\right) = 0 \tag{61}$$

if $t = t_0 = (3r^2-1)/(2(1-r^2)^2)$, $\partial^2\phi(t_0)/\partial t^2 = (1-r^2)^2 > 0$, and that for $r \ge \sqrt{7}-2$,

$$\frac{1}{(1+r)^2} \le t_0 \le \frac{1}{(1-r)^2}. \tag{62}$$

Thus for $\sqrt{7}-2 \le r < \rho_1$,

$$\min\phi(t) = \phi(t_0) = \frac{7 - 10r^2 - r^4}{8(1-r^2)^2} > \alpha. \tag{63}$$

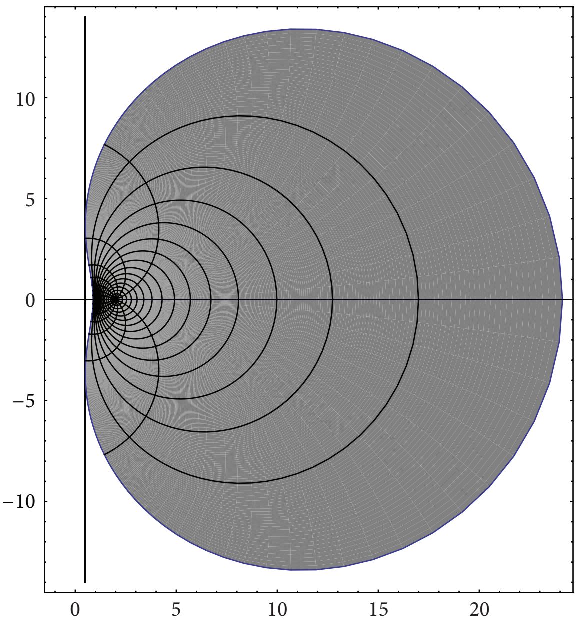

FIGURE 7: Image of $|z| \le \sqrt{3/5}$ touches $\operatorname{Re} w = 0.5$.

On the other hand, if $r < \sqrt{7}-2$, then

$$\min\phi(t) = \phi\left(\frac{1}{(1+r)^2}\right) = \frac{2+r}{(1+r)^2}. \tag{64}$$

Since $g(r) = (2+r)/(1+r)^2$ is a decreasing function in $(0, \sqrt{7}-2)$,

$$\begin{aligned}\min\phi(t) &= \frac{(2+r)}{(1+r)^2} > \frac{2+\sqrt{7}-2}{\left(1+\sqrt{7}-2\right)^2} \\ &= \frac{\sqrt{7}}{\left(\sqrt{7}-1\right)^2} = \frac{4\sqrt{7}+7}{18} > \alpha.\end{aligned} \tag{65}$$

Case (ii). For $(7+4\sqrt{7})/18 \le \alpha < 1$, we prove that $\min\phi(t) > \alpha$ in $|z| < \rho_1$, $t \in (1/(1+r)^2, 1/(1-r)^2)$. Let $r < \rho_1 < \sqrt{7}-2$. As in Case (i), then

$$\min\phi(t) = \phi\left(\frac{1}{(1+r)^2}\right) = \frac{(2+r)}{(1+r)^2} > \alpha. \tag{66}$$

It is evident from the previous two cases that $\operatorname{Re} h_1(z) > \alpha$ in $\mathbb{D}$. Figure 7 shows that, for $\alpha = 0.5$, the radius $\rho_1 = \sqrt{3/5}$ is sharp.

(b) Let $h(re^{it}) = u + iv$. Then,

$$\begin{aligned}u &= \frac{2\left(1+r^2+r^2\cos 2t\right) - r\left(5+r^2\right)\cos t}{\left(1+r^2-2r\cos t\right)^2}, \\ v &= \frac{r\left(3+r^2-4r\cos t\right)\sin t}{\left(1+r^2-2r\cos t\right)^2}.\end{aligned} \tag{67}$$

By (67), it follows that

$$\arg h\left(re^{it}\right) = \arctan\left(\frac{r\left(3+r^2-4r\cos t\right)\sin t}{2\left(1+r^2+r^2\cos 2t\right)-r\left(5+r^2\right)\cos t}\right). \quad (68)$$

Let $g:[-1,1]\to\mathbb{R}$ be defined by

$$g(x)=\frac{r\left(3+r^2-4rx\right)\sqrt{1-x^2}}{2-r\left(5+r^2\right)x+4r^2x^2}. \quad (69)$$

(The case $-\sqrt{1-x^2}$ is similar.) A calculation shows that

$$g'(x)=\frac{r\left(1+r^2-2rx\right)\left(r\left(7+r^2\right)-6x\left(1+r^2\right)+4rx^2\right)}{\sqrt{1-x^2}\left(2-r\left(5+r^2\right)x+4r^2x^2\right)^2}. \quad (70)$$

Let

$$x_0=\frac{3+3r^2-\sqrt{9-10r^2+5r^4}}{4r}\in(0,1]. \quad (71)$$

Then, $g'(x_0)=0$, $g'(x)>0$ for $x<x_0$, and $g'(x)<0$ for $x>x_0$. Thus,

$$g(x)\le\max_{x\in[-1,1]}g(x)=g(x_0). \quad (72)$$

Now (68) and (72) show that

$$\left|\arg h\left(re^{it}\right)\right|\le\frac{\gamma\pi}{2} \quad (73)$$

provided

$$\frac{r\left(3+r^2-4rx_0\right)\sqrt{1-x_0^2}}{2-r\left(5+r^2\right)x_0+4r^2x_0^2}\le\tan\left(\frac{\gamma\pi}{2}\right); \quad (74)$$

that is,

$$\tan\left(\frac{\gamma\pi}{2}\right)-\frac{r\left(3+r^2-4rx_0\right)\sqrt{1-x_0^2}}{2-r\left(5+r^2\right)x_0+4r^2x_0^2}\ge 0. \quad (75)$$

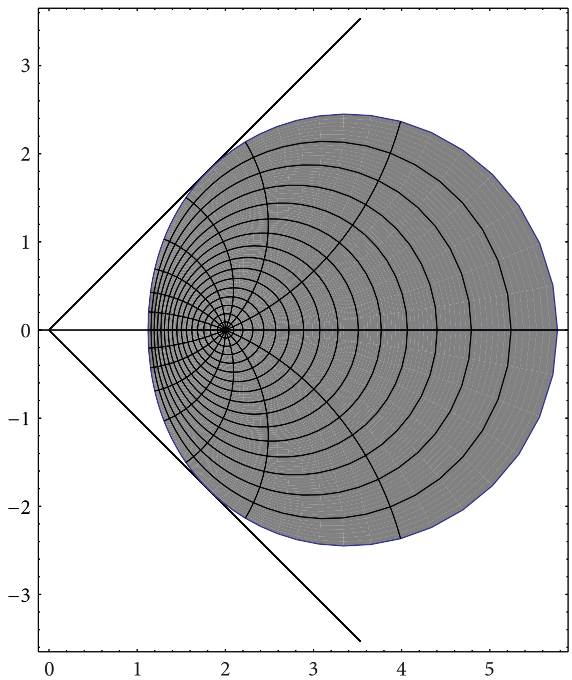

Figure 8: Image of $|z|\le 0.487998$ touches $|\arg w|=\pi/4$.

Thus, $|\arg h_2(z)|<\gamma\pi/2$ in $\mathbb{D}$. Figure 8 shows that, for $\gamma=0.5$, the radius $\rho_2=0.487998$ is sharp.

(c) Proceeding similarly as in part (a),

$$|h(z)-1|<\operatorname{Re} h(z), \quad (76)$$

provided

$$24r^2R^2-8+\left(18r^2-1-17r^4\right)R^4 -2\left(1-r^2\right)^2\left(1-3r^2\right)R^6-\left(1-r^2\right)^4R^8<0. \quad (77)$$

Let $\phi:(1/(1+r)^2,1/(1-r)^2)\to\mathbb{R}$ be defined by

$$\phi(t)=24r^2t-8+\left(18r^2-1-17r^4\right)t^2 -2\left(1-r^2\right)^2\left(1-3r^2\right)t^3-\left(1-r^2\right)^4t^4. \quad (78)$$

Now

$$\phi'(t)=24r^2+2\left(18r^2-1-17r^4\right)t -6\left(1-r^2\right)^2\left(1-3r^2\right)t^2-4\left(1-r^2\right)^4t^3. \quad (79)$$

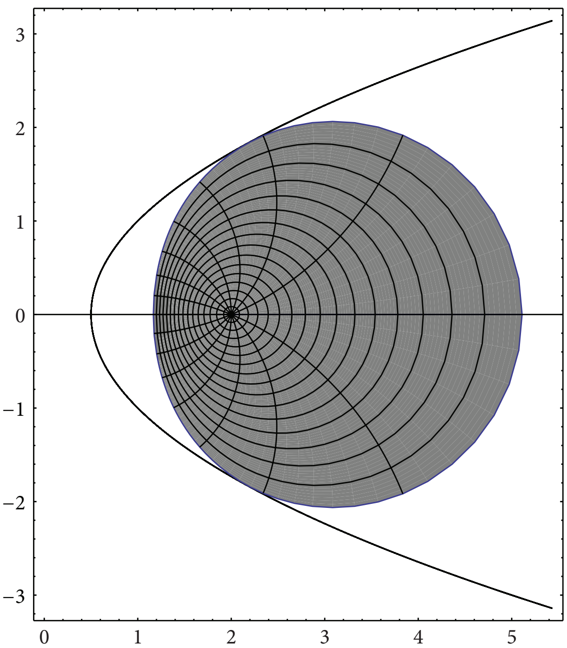

FIGURE 9: Image of $|z| \le 0.44915$ touches $|w-1| = \operatorname{Re} w$.

Let $r < \rho_3$. Since

$$\begin{aligned}\phi'\left(\frac{1}{(1+r)^2}\right) &= \frac{4\left(-3+7r+12r^2+7r^3+r^4\right)}{(1+r)^2} > 0 \quad \text{if } r > 0.27606,\\ \phi'\left(\frac{1}{(1-r)^2}\right) &= \frac{4\left(-3-7r+12r^2-7r^3+r^4\right)}{(1-r)^2} < 0,\end{aligned} \tag{80}$$

there exists a unique $t_0 \in (1/(1+r)^2, 1/(1-r)^2)$ such that $\phi'(t_0) = 0$ and $\max \phi(t) = \phi(t_0)$.

Then, for $0.27606 < r < \rho_3$,

$$\max \phi(t) = \phi(t_0) < 0. \tag{81}$$

When $r \le 0.27606$,

$$\max \phi(t) = \phi\left(\frac{1}{(1+r)^2}\right) = \frac{4\left(-3+r^2\right)}{(1+r)^2} < 0. \tag{82}$$

Evidently, $\phi(t) < 0$ for $r < \rho_3$ and hence $|h_3(z) - 1| < \operatorname{Re} h_3(z)$ in $\mathbb{D}$. Figure 9 shows that the radius $\rho_3 \simeq 0.44915$ is sharp.

Now, with $g(z) = z \in \mathcal{ST}(1/2)$, $i = 1, 2, 3$,

$$\begin{aligned}\frac{f(z) * zh_i(z)}{f(z) * z} &= \frac{f(z) * z\left(1/(1-\rho_i z) + 1/(1-\rho_i z)^2\right)}{f(z) * z}\\ &= \frac{f(\rho_i z)}{\rho_i z} + f'(\rho_i z)\\ &= \chi(\rho_i z) = \chi_i(z).\end{aligned} \tag{83}$$

Lemma 2, together with (83) and the corresponding inequality for the function h_i, shows that each function χ_i satisfies the required condition. For sharpness, consider the function $f_0(z) = z/(1-z) \in \mathcal{CV} \subset \mathcal{ST}(1/2)$. Clearly

$$\frac{f_0(z)}{z} + f_0'(z) = \frac{1}{1-z} + \frac{1}{(1-z)^2} = h(z); \tag{84}$$

hence the fact that the number ρ_i is sharp follows from the definition of h. □

For $\alpha = 0$, Theorem 5(a) reduces to the following corollary.

Corollary 6 ([14, Theorem 3, page 722]). *If $f \in \mathcal{ST}(1/2)$, then*

$$\operatorname{Re}\left(\frac{f(z)}{z} + f'(z)\right) > 0 \tag{85}$$

in $|z| < \rho = \sqrt{4\sqrt{2}-5} \simeq 0.81$. The number ρ is sharp.

Theorem 7. *Let $f \in \mathcal{CV}$, $\chi : \mathbb{D} \to \mathbb{C}$ be defined by*

$$\begin{aligned}\chi(z) &= \left(1 + \frac{zf''(z)}{f'(z)}\right) + \frac{1}{f'(z)},\\ \chi_i(z) &= \chi(\rho_i z), \quad i = 1, 2.\end{aligned} \tag{86}$$

Then,

(a) $\operatorname{Re} \chi_1(z) > \alpha$, $0 \le \alpha < 1$, *where* $\rho_1 = \sqrt{3 - \sqrt{5+2\alpha}}$;

(b) $|\arg \chi_2(z)| < \gamma\pi/2$, $0 < \gamma \le 1$, *where* $\rho_2 = \rho(\gamma) \in (0,1)$ *is the root of the equation in* r

$$\begin{aligned}&2r^2\sqrt{1-x_0^2}\left(r - \left(3+r^2\right)x_0 + 2rx_0^2\right) + \tan\left(\frac{\pi\gamma}{2}\right)\\ &\quad\times\left(2 - r^2 - r^4 - 4rx_0 + \left(6r^2 + 2r^4\right)x_0^2 - 4r^3x_0^3\right) = 0,\end{aligned} \tag{87}$$

and $x_0 \in [-1, 1]$ is the root of the equation

$$\begin{aligned}&6 - 5r^2 - 4r^4 - r^6 + \left(-6r + 15r^3 + 7r^5\right)x_0\\ &\quad + \left(-12 - 8r^2 - 16r^4\right)x_0^2 + \left(24r + 16r^3\right)x_0^3\\ &\quad - 16r^2x_0^4 = 0.\end{aligned} \tag{88}$$

In particular,

$$\rho_2\left(\frac{1}{2}\right) \simeq 0.6355, \qquad \rho_2(1) = \sqrt{3-\sqrt{5}}. \tag{89}$$

The radii are sharp.

Proof. Let

$$h(z) = \frac{2}{1-z} + (1-z)^2 - 1 \quad (z \in \mathbb{D}). \tag{90}$$

(a) We claim that $\operatorname{Re} h(z) > \alpha$ in $|z| < \rho_1$. By (22) and (23), it follows that

$$\begin{aligned}\operatorname{Re} h(z) &= 2R\cos\theta - 1 + \frac{\cos 2\theta}{R^2} \\ &= \frac{1+t^2+r^4t^2+2t^3-2r^2\left(t+t^2+t^3\right)}{2t^2} \qquad \left(t := R^2\right) \\ &:= \phi(t).\end{aligned} \tag{91}$$

A calculation shows that $\phi'(t) = 0$ if $t = t_0 = 1 \in (1/(1+r)^2, 1/(1-r)^2)$, $\phi''(t_0) = 3 - 2r^2 > 0$, and that

$$\min \phi(t) = \phi(1) = \frac{1}{2}\left(4 - 6r^2 + r^4\right) > \alpha \tag{92}$$

over all $t \in (1/(1+r)^2, 1/(1-r)^2)$ provided

$$r^4 - 6r^2 + 4 - 2\alpha > 0. \tag{93}$$

This inequality reduces to $r \le \rho_1$. Thus, $\operatorname{Re} h_1(z) > \alpha$ in $\mathbb{D}$. Figure 10 shows that, for $\alpha = 0.5$, the radius $\rho_1 = \sqrt{3-\sqrt{6}}$ is sharp.

(b) Let $h(re^{it}) = u + iv$. Then

$$\begin{aligned}u &= \left(2(1-r\cos t) + \left(r^2\left(\cos^2 t - \sin^2 t\right) - 2r\cos t\right)\right. \\ &\quad \times \left.\left(1 + r^2 - 2r\cos t\right)\right) \\ &\quad \times \left(1 + r^2 - 2r\cos t\right)^{-1}, \\ v &= \frac{2r\sin t\left(1 - (1-r\cos t)\left(1+r^2-2r\cos t\right)\right)}{1+r^2-2r\cos t}.\end{aligned} \tag{94}$$

By (94), it follows that

$$\begin{aligned}&\arg h\left(re^{it}\right) \\ &\quad = \arctan\left(\left(2r^2\left(-r + \cos t\left(3 + r^2 - 2r\cos t\right)\right)\sin t\right)\right. \\ &\qquad \times \left(2 + 2r^2 - r\left(4 + 3r^2\right)\cos t\right. \\ &\qquad \left.\left. + r^2\left(3+r^2\right)\cos 2t - r^3\cos 3t\right)^{-1}\right).\end{aligned} \tag{95}$$

Let

$$\begin{aligned}g(x) &= \left(2r^2\left(-r + x\left(3 + r^2 - 2rx\right)\right)\sqrt{1-x^2}\right) \\ &\quad \times \left(2 + 2r^2 - r\left(4 + 3r^2\right)x\right. \\ &\quad \left. + r^2\left(3+r^2\right)\left(2x^2-1\right) - r^3\left(4x^3 - 3x\right)\right)^{-1}.\end{aligned} \tag{96}$$

A calculation shows that there exists $x_0 \in [0,1]$ such that $g'(x_0) = 0$ and $g''(x_0) < 0$. Thus

$$g(x) \le g(x_0), \quad x \in [-1, 1]. \tag{97}$$

By (95), (96), and (97), evidently

$$\left|\arg h\left(re^{it}\right)\right| \le \frac{\gamma\pi}{2} \tag{98}$$

if

$$\begin{aligned}&\frac{2r^2\left(-r + x_0\left(3 + r^2 - 2rx_0\right)\right)\sqrt{1-x_0^2}}{2 - r^2 - r^4 - 4rx_0 + \left(6r^2 + 2r^4\right)x_0^2 - 4r^3x_0^3} \\ &\quad \le \tan\left(\frac{\gamma\pi}{2}\right);\end{aligned} \tag{99}$$

that is,

$$\begin{aligned}&2r^2\sqrt{1-x_0^2}\left(r - \left(3+r^2\right)x_0 + 2rx_0^2\right) \\ &\quad + \tan\left(\frac{\gamma\pi}{2}\right)\left(2 - r^2 - r^4 - 4rx_0 + \left(6r^2 + 2r^4\right)x_0^2 - 4r^3x_0^3\right) \\ &\quad \ge 0.\end{aligned} \tag{100}$$

Thus, $|\arg h_2(z)| < \gamma\pi/2$ in $\mathbb{D}$. Figure 11 shows that, for $\gamma = 0.5$, the radius $\rho_2 \simeq 0.6335$ is sharp.

To conclude the proof, let $g(z) = z/(1-\rho_i z)^2 \in \mathcal{ST}$. Then,

$$\begin{aligned}&\frac{f(z) * \left(z/(1-\rho_i z)^2\right) h_i(z)}{f(z) * \left(z/(1-\rho_i z)^2\right)} \\ &\quad = \frac{f(z) * \left(z/(1-\rho_i z)^2\right)\left(2/(1-\rho_i z) - 1 + (1-\rho_i z)^2\right)}{f(z) * \left(z/(1-\rho_i z)^2\right)} \\ &\quad = \frac{f(z) * \left(2z/(1-\rho_i z)^3 + z - z/(1-\rho_i z)^2\right)}{f(z) * (z/(1-\rho_i z))} \\ &\quad = \left(1 + \frac{\rho_i z f''(\rho_i z)}{f'(\rho_i z)}\right) + \frac{1}{f'(\rho_i z)} \\ &\quad = \chi(\rho_i z) = \chi_i(z).\end{aligned} \tag{101}$$

As in the earlier proofs, Lemma 2 together with (101) and the corresponding inequality for the function h_i shows

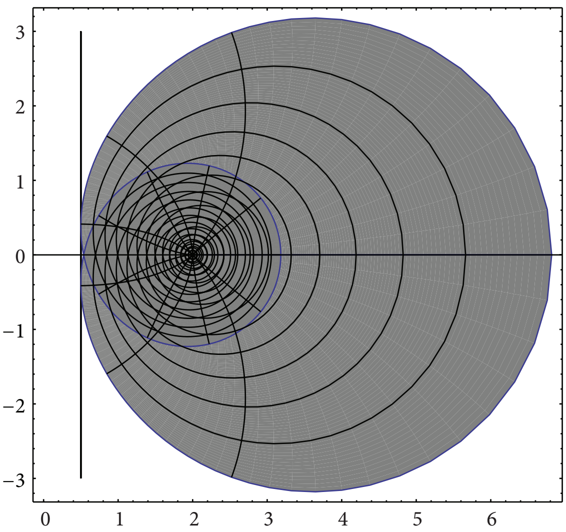

FIGURE 10: Image of $|z| \le \sqrt{3-\sqrt{6}}$ touches Re $w = 0.5$.

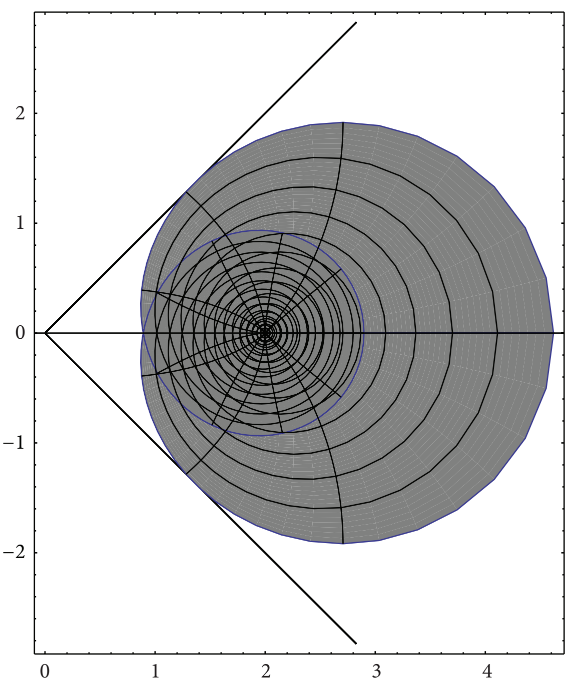

FIGURE 11: Image of $|z| \le 0.6335$ touches $|\arg w| = \pi/4$.

that the function χ_i satisfies the required condition. For sharpness, consider $f_0(z) = z/(1-z) \in \mathcal{CV} \subset \mathcal{ST}(1/2)$. Then,

$$\left(1 + \frac{zf_0''(z)}{f_0'(z)}\right) + \frac{1}{f_0'(z)} = \frac{2}{1-z} - 1 + (1-z)^2 = h(z). \tag{102}$$

□

For $\alpha = 0$, Theorem 7(a) reduces to the following result.

Corollary 8 ([14, Theorem 4, page 723]). *If $f \in \mathcal{CV}$, then*

$$\operatorname{Re}\left(\left(1 + \frac{zf''(z)}{f'(z)}\right) + \frac{1}{f'(z)}\right) > 0 \tag{103}$$

in $|z| < \rho = \sqrt{3-\sqrt{5}} = (\sqrt{5}-1)/\sqrt{2} \simeq 0.874032$. The result is sharp.

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References

[1] B. A. Uralegaddi, M. D. Ganigi, and S. M. Sarangi, "Univalent functions with positive coefficients," *Tamkang Journal of Mathematics*, vol. 25, no. 3, pp. 225–230, 1994.

[2] R. M. Ali, M. Haji Mohd, L. S. Keong, and V. Ravichandran, "Radii of starlikeness, parabolic starlikeness and strong starlikeness for Janowski starlike functions with complex parameters," *Tamsui Oxford Journal of Information and Mathematical Sciences*, vol. 27, no. 3, pp. 253–267, 2011.

[3] R. M. Ali, N. E. Cho, N. K. Jain, and V. Ravichandran, "Radii of starlikeness and convexity for functions with fixed second coefficient defined by subordination," *Filomat*, vol. 26, no. 3, pp. 553–561, 2012.

[4] S. Owa and H. M. Srivastava, "Some generalized convolution properties associated with certain subclasses of analytic functions," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 3, no. 3, article 42, 13 pages, 2002.

[5] V. Ravichandran, M. Hussain Khan, H. Silverman, and K. G. Subramanian, "Radius problems for a class of analytic functions," *Demonstratio Mathematica*, vol. 39, no. 1, pp. 67–74, 2006.

[6] J. Sokół and J. Stankiewicz, "Radius of convexity of some subclasses of strongly starlike functions," *Zeszyty Naukowe Politechniki Rzeszowskiej. Matematyka*, no. 19, pp. 101–105, 1996.

[7] R. M. Ali, N. K. Jain, and V. Ravichandran, "Radii of starlikeness associated with the lemniscate of Bernoulli and the left-half plane," *Applied Mathematics and Computation*, vol. 218, no. 11, pp. 6557–6565, 2012.

[8] J. Sokół, "Coefficient estimates in a class of strongly starlike functions," *Kyungpook Mathematical Journal*, vol. 49, no. 2, pp. 349–353, 2009.

[9] J. Sokół, "Radius problems in the class $\mathcal{SL}^*$," *Applied Mathematics and Computation*, vol. 214, no. 2, pp. 569–573, 2009.

[10] F. Rønning, "Uniformly convex functions and a corresponding class of starlike functions," *Proceedings of the American Mathematical Society*, vol. 118, no. 1, pp. 189–196, 1993.

[11] R. M. Ali and V. Ravichandran, "Uniformly convex and uniformly starlike functions," *Mathematics Newsletter*, vol. 21, no. 1, pp. 16–30, 2011.

[12] A. W. Goodman, *Univalent Functions*, vol. 1, Mariner, Tampa, Fla, USA, 1983.

[13] P. L. Duren, *Univalent Functions*, vol. 259 of *Grundlehren der Mathematischen Wissenschaften*, Springer, New York, NY, USA, 1983.

[14] R. Singh and S. Paul, "Linear sums of certain analytic functions," *Proceedings of the American Mathematical Society*, vol. 99, no. 4, pp. 719–725, 1987.

[15] St. Ruscheweyh and T. Sheil-Small, "Hadamard products of Schlicht functions and the Pólya-Schoenberg conjecture," *Commentarii Mathematici Helvetici*, vol. 48, pp. 119–135, 1973.