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Research Article

GF-Regular Modules

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We introduced and studied GF-regular modules as a generalization of π-regular rings to modules as well as regular modules (in the sense of Fieldhouse). An R-module M is called GF-regular if for each $x \in M$ and $r \in R$, there exist $t \in R$ and a positive integer n such that $r^n t r^n x = r^n x$. The notion of G-pure submodules was introduced to generalize pure submodules and proved that an R-module M is GF-regular if and only if every submodule of M is G-pure iff $M_{\mathfrak{M}}$ is a GF-regular $R_{\mathfrak{M}}$-module for each maximal ideal $\mathfrak{M}$ of R. Many characterizations and properties of GF-regular modules were given. An R-module M is GF-regular iff $R/\text{ann}(x)$ is a π-regular ring for each $0 \neq x \in M$ iff $R/\text{ann}(M)$ is a π-regular ring for finitely generated module M. If M is a GF-regular module, then $J(M) = 0$.

1. Introduction

Throughout this paper, unless otherwise stated, R is a commutative ring with nonzero identity and all modules are left unitary. For an R-module M, the annihilator of $x \in M$ in R is $\text{ann}_R(x) = \{r \in R : rx = 0\}$. The symbol □ stands for the end of the proof if the proof is given or the end of the statement when the proof is not given.

Recall that a ring R is said to be regular (in the sense of von Neumann) if for each $r \in R$, there exists $t \in R$ such that $rtr = r$ [1]. The concept of regular rings was extended firstly to π-regular rings by McCoy [2], recall that a ring R is π-regular if for each $r \in R$, there exist $t \in R$ and a positive integer n such that $r^n t r^n = r^n$ [2] and secondly to modules in several nonequivalent ways considered by Fieldhouse [3], Ware [4], Zelmanowitz [5], and Ramamurthi and Rangaswamy [6]. In [7], Jayaraman and Vanaja have studied generalizations of regular modules (in the sense of Zelmanowitz) by Ramamurthi [8] and Mabuchi [9]. Following [10], we denoted Fieldhouse' regular modules by F-regular. An R-module M is called F-regular if each submodule of M is pure [3].

Dissimilar to the generalizations that have been studied in [7, 9] and [8], in this paper a new generalization of π-regular rings to modules and F-regular modules was introduced, called GF-reular (generalized F-regular) modules. An R-module M is called GF-regular if for each $x \in M$ and $r \in R$, there exist $t \in R$ and a positive integer n such that $r^n t r^n x = r^n x$. A ring R is called GF-regular if R is GF-regular as an R-module. On the other hand, GF-regular modules are also a generalization of π-regular rings. Thus, R is a π-regular ring if and only if R is a GF-regular R-module. Furthermore, we introduced a new class of submodules, named, G-pure submodules as a generalization of pure submodules. A submodule P of an R-module M is said to be G-pure if for each $r \in R$, there exists a positive integer n such that $P \cap r^n M = r^n P$. Recall that a submodule P of an R-module M is pure if $P \cap IM = IP$ for each ideal I of R [11]. We find that the relationship between GF-regular modules and G-pure submodules is an analogous relationship between F-regular modules and pure submodules.

In Section 3.1 of this paper, after the concept of GF-regular modules was introduced, we obtained several characteristic properties of GF-regular modules. For instance, it was proved that the following are equivalent for an R-module M: (1) M is GF-regular; (2) every submodule of M is G-pure; (3) $R/\text{ann}(x)$ is a π-regular ring for each $0 \neq x \in M$; (4) and for each $x \in M$ and $r \in R$, there exist $t \in R$ and a positive integer n such that $r^{n+1} t x = r^n x$. It is also shown that if M

is a finitely generated R-module, then M is GF-regular if and only if $R/\text{ann}(M)$ is a π-regular ring.

Section 3.2 was devoted to investigate the relationship between GF-regular modules with the localization property and semisimple modules. For example, we proved that M is a GF-regular R-module if and only if $M_{\mathfrak{M}}$ is a GF-regular $R_{\mathfrak{M}}$-module for every maximal ideal $\mathfrak{M}$ of R if and only if $M_{\mathfrak{M}}$ is a semisimple $R_{\mathfrak{M}}$-module for every maximal ideal $\mathfrak{M}$ of R.

Finally, in Section 3.3 we studied some properties of the Jacobson radical, $J(M)$, of GF-regular modules. Thus we proved that if M is a GF-regular R-module, then $J(M) = 0$, and also we get that if $J(R)$ is a reduced ideal of a ring R and M is a GF-regular R-module, then $J(R) \cdot M = 0$.

2. The Notion of *GF*-Regular Modules and General Results

We start by recalling that an R-module M is F-regular if each submodule of M is pure [3], and a ring R is π-regular if for each $r \in R$, there exist $t \in R$ and a positive integer n such that $r^n t r^n = r^n$ [2].

Definition 1. An R-module M is called GF-regular if for each $x \in M$ and $r \in R$, there exist $t \in R$ and a positive integer n such that $r^n t r^n x = r^n x$. A ring R is GF-regular if and only if R is GF-regular as an R-module.

The following gives another characterization for GF-regular modules.

Proposition 2. *An R-module M is GF-regular if and only if $R/\text{ann}(x)$ is a π-regular ring for each $0 \neq x \in M$.*

Proof. Suppose that M is a GF-regular R-module, so for each $x \in M$ and $r \in R$, there exist $t \in R$ and a positive integer n such that $r^n t r^n x = r^n x$; hence, $(r^n t r^n - r^n) \in \text{ann}(x)$ which means that $\bar{r}^n t \bar{r}^n = \bar{r}^n$; therefore, $R/\text{ann}(x)$ is a π-regular ring. Conversely, suppose that $R/\text{ann}(x)$ is a π-regular ring for each $0 \neq x \in M$, thus for each $\bar{r} \in R/\text{ann}(x)$, there exist $\bar{t} \in R/\text{ann}(x)$ and a positive integer n such that $\bar{r}^n \bar{t} \bar{r}^n = \bar{r}^n$; hence, $r^n t r^n - r^n \in \text{ann}(x)$ which implies that $(r^n t r^n - r^n)x = 0$; therefore, M is a GF-regular R-module. □

It is clear that every F-regular module is GF-regular, but the converse may not be true in general; for example, by applying Proposition 2 to the Z-module Z_4, we can easily see that it is GF-regular; however, Z_4 is not an F-regular Z-module. In fact, the Z-module Z_n is GF-regular for each positive integer n [12], while it is not F-regular for some positive integer n. On the other hand, the Z-module Q is not GF-regular because for each $0 \neq x \in Q$ we have that $\text{ann}_Z(x) = 0$, but $Z/\text{ann}_Z \simeq Z$ which is not a π-regular ring [12].

Remark 3.

(1) If R is a π-regular ring, then every R-module is GF-regular.

(2) Every module over Artinian ring R is GF-regular (because every Artinian ring is π-regular [12]).

(3) A ring R is π-regular if and only if R is GF-regular as an R-module.

(4) Every submodule of a GF-regular module is GF-regular module. In particular, every ideal of a π-regular ring R is GF-regular R-module. Furthermore, it follows from (1) that if I is an ideal of a π-regular ring R, then the R-module R/I is GF-regular.

(5) The converse of (1) is true if the module is free, that is, any free R-module M is GF-regular if and only if R is a π-regular ring. For if, M is a free R-module, then $\text{ann}(x) = 0$ for each $0 \neq x \in M$, so $R \simeq R/\text{ann}(x)$ is a π-regular ring.

(6) If an R-module M is GF-regular and it contains a nontorsion element, then R is a π-regular ring. In particular, if M is a GF-regular R-module and R is not a π-regular ring, then M is a torsion R-module.

Now from Proposition 2 and Remark 3(3), we conclude the following.

Corollary 4. *The following statements are equivalent for a ring:*

(1) *R is a π-regular ring;*

(2) *$R/\text{ann}(r)$ is a π-regular ring for each $0 \neq r \in R$.*

We have seen previously that every F-regular R-module is GF-regular. In the following we consider some conditions such that the converse is true.

Remark 5.

(1) Let R be a reduced ring. An R-module M is F-regular if and only if M is a GF-regular R-module.

(2) An R-module M is F-regular if and only if M is a GF-regular R-module and $L(R/\text{ann}(x)) = 0$ for each $0 \neq x \in M$, where $L(R/\text{ann}(x))$ is the prime radical of the ring $R/\text{ann}(x)$.

Now, we describe GF-regular modules over the ring of integers Z.

Proposition 6. *A Z-module M is GF-regular if and only if M is a torsion Z-module.*

Proof. If M is a GF-regular Z-module, then by Remark 3(6) M is a torsion Z-module. Conversely, if M is a torsion Z-module, then $\text{ann}_Z(x) = nZ$ for some positive integer n; hence, $Z/\text{ann}_Z(x) \simeq Z_n$ is a π-regular ring for each positive integer n [12], which implies that M is a GF-regular Z-module. □

Proposition 7. *Every homomorphic image of a GF-regular R-module is GF-regular.*

Proof. Let M, M' be two R-modules such that M is GF-regular and let $f : M \to M'$ be an R-epimorphism. For every $y \in M'$, there exists $x \in M$ such that $f(x) = y$. It is clear that $\text{ann}(x) \subseteq \text{ann}(y)$. Define $\alpha : R/\text{ann}(x) \to R/\text{ann}(y)$ by

$\alpha(r+\text{ann}(x)) = r+\text{ann}(y)$ for each $r \in R$. It is an easy matter to check that α is well defined R-epimorphism. Since $R/\text{ann}(x)$ is a π-regular ring, then $R/\text{ann}(y)$ is also a π-regular ring [12]. Therefore, M' is a GF-regular R-module. □

Corollary 8. *The following statements are equivalent for an R-module M:*

(1) *M/N is a GF-regular R-module for every nonzero submodule N of M.*

(2) *M/Rx is a GF-regular R-module for every $0 \neq x \in M$.*

Another characterization of a GF-regular R-module is given in the next result.

Proposition 9. *An R-module M is GF-regular if and only if for each $x \in M$ and $r \in R$, there exist $t \in R$ and a positive integer n such that $r^{n+1}tx = r^n x$.*

Proof. Suppose that M is a GF-regular R-module, so for each $x \in M$ and $r \in R$, there exist $s \in R$ and a positive integer n such that $r^n s r^n x = r^n x$, then we can take $t = sr^{n-1} \in R$ and hence $r^{n+1}tx = r^n x$. Conversely, for each $x \in M$ and $r \in R$, there exist $s \in R$ and a positive integer n such that $r^{n+1}sx = r^n x$. Now, $r^n s^n r^n x = r^{n+1} s s^{n-1} r^{n-1} x = r^n s^{n-1} r^{n-1} x = r^{n+1} s s^{n-2} r^{n-2} x = r^n s^{n-2} r^{n-2} x = \cdots = r^{n+1} sx = r^n x$ (after n times), thus $r^n t r^n x = r^n x$ where $t = s^n$ which implies that M is a GF-regular R-module. □

3. Main Results

3.1. GF-Regular Modules and Purity. Recall that a submodule P of an R-module M is pure in M if each finite system of equations

$$P_i = \sum_j r_{ij} x_j, \quad r_{ij} \in R, \ P_j \in P, \ 1 \leq j \leq m, \tag{1}$$

which is solvable in M, is solvable in P [13]. It is not difficult to prove that P is pure in M if and only if for each ideal I of R, $P \cap IM = IP$ [11]. This motivates us to introduce the following definition as a generalization of pure submodules.

Definition 10. A submodule P of an R-module M is called G-pure if for each $r \in R$, there exists a positive integer n such that $P \cap r^n M = r^n P$.

It is clear that every pure module is G-pure.

The following theorem gives another characterization of GF-regular modules in terms of G-pure submodules.

Theorem 11. *An R-module M is GF-regular if and only if every submodule of M is G-pure.*

Proof. Suppose that M is a GF-regular R-module and let P be any submodule of M. For each $r \in R$ and for some positive integer n, let $x \in P \cap r^n M$, then there exists $y \in M$ such that $x = r^n y$. Since M is GF-regular, then there exists $t \in R$ such that $r^n y = r^n t r^n y$. Put $e = tr^n$, then $r^n y = e r^n y$ which implies that $x = ex$, but $x \in P$, so $x = ex \in r^n P$ and hence $P \cap r^n M \subseteq r^n P$. On the other hand, it is clear that $r^n P \subseteq P \cap r^n M$, thus $P \bigcap r^n M = r^n P$ which means that P is a G-pure submodule.

Conversely, assume that every submodule is G-pure and let $x \in M$ and $p \in R$ such that $Rp^n x = P$ which is a G-pure submodule of M for some positive integer n, then $P \cap r^n M = r^n P$ for each $r \in R$. In particular, if $r = p$ we get $r^n x \in P \cap r^n M \subseteq r^n P = r^n R r^n x$ which implies that there exists $t \in R$ such that $r^n t r^n x = r^n x$, so M is a GF-regular R-module. □

Corollary 12. *An R-module M is GF-regular if and only if for each $x \in M$, there exist $p \in R$ and a positive integer n such that $Rp^n x$ is a G-pure submodule.*

Remark 13. Fieldhouse in [11] proved that for a submodule P of an R-module M, if M/P is a flat R-module, then P is pure. On the other hand, if M is flat and P is pure, then M/P is flat. So, immediately we have that for a flat R-module, if M/P is a flat R-module for each submodule P of M, then M is GF-regular R-module. It is not difficult to prove that in case of F-regular modules the converse of the latest statement is true; however, we do not know whether it is true for GF-regular modules or not.

Remark 14. In [14], Mao proved that a right R-module N is GP-flat if and only if there exists an exact sequence $0 \to K \to M \to N \to 0$ with M free such that for any $r \in R$, there exists a positive integer n satisfying $K \cap Mr^n = Kr^n$, where (1) a right R-module N is said to be generalized P-flat (GP-flat for short) if for any $r \in R$, there exists a positive integer n (depending on r) such that the sequence $0 \to N \otimes Rr^n \to N \otimes R$ is exact [15], (2) a right R-module N is P-flat [16] or torsion-free [15] if for any $r \in R$, the sequence $0 \to N \otimes Rr \to N \otimes R$ is exact. Obviously, every flat module is P-flat [16] and every P-flat module is GP-flat [14].

According to the above remark we get the following.

Corollary 15. *An R-module N is GP-flat if and only if there exists an exact sequence $0 \to P \to M \to N \to 0$ with P is a submodule of a free R-module M such that P is a G-pure submodule.*

Corollary 16. *For every submodule P of a free R-module M, if there exists an exact sequence $0 \to P \to M \to N \to 0$ such that P is a G-pure submodule in M, then N is a GP-flat R-module if and only if M is GF-regular.*

Now, we recall that (1) an R-module M is p-injective if for every principal ideal I of R, every R-homomorphism of I into M extends to one of R into M [17]. A ring R is called p-injective if R is p-injective as an R-module. (2) An R-module M is called YJ-injective if for any $0 \neq r \in R$, there exists a positive integer n such that $r^n \neq 0$ and any R-homomorphism of Rr^n into M extends to one of R into M. A ring R is called YJ-injective if R is YJ-injective as an R-module [18]. YJ-injective modules are called GP-injective modules by some other authors [19–22]. (3) An R-module M is called WGP-injective (weak GP-injective) if for any $r \in R$, there exists a positive integer n such that every R-homomorphism of Rr^n into M extends to one of R into M (r^n may be zero). A ring R is called *WGP*-injective if R is *WGP*-injective as an R-module [23–25]. (4) A ring R is called *p.p.* if every principal ideal of R is projective. And R is called *GPP*-ring if for any $r \in R$, there exists a positive integer n (depending on r) such that Rr^n is projective [26, 27].

Note that p-injectivity implies YJ-injectivity (or *GP*-injectivity) and *WGP*-injectivity, as well as the concept of *p.p.* rings implies the concept of *GPP*-rings. However, the notion of YJ-injective (or *GP*-injective) modules is not the same notion of *WGP*-injective modules.

It is known that a ring R is π-regular if and only if every R-module is *WGP*-injective [12, 22], so from all the above we conclude the following theorem.

Theorem 17. *The following statements are equivalent for a ring R.*

(1) *R is a π-regular ring.*

(2) *$R/ann(r)$ is a π-regular ring for each $0 \neq r \in R$.*

(3) *Any free R-module is GF-regular.*

(4) *Every R-module is WGP-injective.*

We end this section by the following two related results.

Proposition 18. *Let M be an R-module. If $R/ann(M)$ is a π-regular ring, then M is a GF-regular R-module.*

Proof. We have that $\operatorname{ann}(M) \subseteq \operatorname{ann}(x)$ for each $x \in M$, so there exists an obvious R-epimorphism $\varphi : R/\operatorname{ann}(M) \to R/\operatorname{ann}(x)$ defined by $\varphi(r + \operatorname{ann}(M)) = r + \operatorname{ann}(x)$. Since $R/\operatorname{ann}(M)$ is a π-regular ring, then $R/\operatorname{ann}(x)$ is a π-regular ring [12]; therefore, M is a *GF*-regular R-module. □

In case of finitely generated modules, the converse of Proposition 18 is true.

Proposition 19. *Let M be an R-module. If M is a finitely generated GF-regular R-module, then $R/ann(M)$ is a π-regular ring.*

Proof. Let $\{x_1, x_2, \ldots, x_n\}$ be a finite set of generators of M. Put $N = \operatorname{ann}(M)$, and $N_i = \operatorname{ann}(x_i)$, $1 \leq i \leq k$, then $N = \cap_i N_i$. Now define $\varphi : R/N \to \oplus\sum_{i=1}^{n} R/N_i$ by $\varphi(r + N) = (r + N_1, r + N_2, \ldots, r + N_n)$ for each $r + N \in R/N$. It is easily checked that φ is a ring monomorphism. Thus, R/N can be identified with a subring T of $\oplus\sum_{i=1}^{n} R/N_i$. In fact

$$T = \{(r + N_1, r + N_2, \ldots, r + N_n) : r \in R\}. \quad (2)$$

We will show now that T, and hence R/N is a π-regular ring. Since M is a *GF*-regular R-module, then R/N_i is a π-regular ring, thus for each $r \in R$ and $1 \leq i \leq k$, there exist $t_i \in R$ and a positive integer n such that $r^n t_i r^n + N_i = r^n + N_i$; this means that $r^n t_i r^n x_i = r^n x_i$. Define t by the relation $1 - tr^n = \prod_{i=1}^{k}(1 - t_i r^n)$, then $r^n(1 - tr^n)x_i = r^n\prod_{i=1}^{k}(1 - t_i r^n)x_i = \prod_{i=1}^{k}(r^n - r^n t_i r^n)x_i = 0$ which implies that for each i, $r^n + N_i = r^n t r^n + N_i$, so T is a π-regular ring and hence R/N is a π-regular ring. □

3.2. GF-Regular Modules and Localization. In this section we study the localization property and semisimple modules with *GF*-regular modules and we give some characterizations of *GF*-regular modules in the sense of them.

Theorem 20. *Let M be an R-module. M is a GF-regular R-module if and only if $M_{\mathfrak{M}}$ is a GF-regular $R_{\mathfrak{M}}$-module for each maximal ideal $\mathfrak{M}$ in R.*

Proof. Let M be a *GF*-regular R-module, and let $\mathfrak{M}$ be any maximal ideal in R. Let $x/t \in M_{\mathfrak{M}}$ and $r/t_1 \in R_{\mathfrak{M}}$, where $x \in M$, $r \in R$ and $t, t_1 \in R - \mathfrak{M}$. So there exist $k \in R$ and a positive integer n such that $r^n k r^n x = r^n x$. Hence, $(r/t_1)^n(x/t) = r^n x/t_1^n t = (r^n k r^n x/t_1^n t)(t_1^n/t_1^n) = (r^n/t_1^n)(kt_1^n/1)(r^n/t_1^n)(x/t) = (r/t_1)^n(kt_1^n/1)(r/t_1)^n$, where $kt_1^n/1 \in R_{\mathfrak{M}}$, then $M_{\mathfrak{M}}$ is *GF*-regular $R_{\mathfrak{M}}$-module.

Conversely, suppose that $M_{\mathfrak{M}}$ is a *GF*-regular $R_{\mathfrak{M}}$-module. Let P be a submodule of M and let $\mathfrak{M}$ be a maximal ideal of R. By Theorem 11, $P_{\mathfrak{M}}$ is a G-pure submodule of $M_{\mathfrak{M}}$; therefore, $P_{\mathfrak{M}} \cap (Rr^n)_{\mathfrak{M}} M_{\mathfrak{M}} = (Rr^n)_{\mathfrak{M}} P_{\mathfrak{M}}$ for each $r \in R$ and for some positive integer n. But by [28], we have that $P_{\mathfrak{M}} \cap (Rr^n)_{\mathfrak{M}} M_{\mathfrak{M}} = P_{\mathfrak{M}} \cap (Rr^n M)_{\mathfrak{M}} = (P \cap Rr^n M)_{\mathfrak{M}}$ and $(Rr^n P)_{\mathfrak{M}} = (Rr^n)_{\mathfrak{M}} P_{\mathfrak{M}}$, then $(Rr^n M \cap P)_{\mathfrak{M}} = (Rr^n P)_{\mathfrak{M}}$, again by [28], we get that $Rr^n M \cap P = Rr^n P$, which implies that P is a G-pure submodule of M and by Theorem 11 M, is a *GF*-regular R-module. □

Recall that an R-module M is simple if 0 and M are the only submodules of M, and an R-module M is said to be semisimple if M is a sum of simple modules (may be infinite). A ring R is semisimple if it is semisimple as an R-module [29]. It is known that over any ring R, a semisimple module is F-regular [4, 30], consequently it is *GF*-regular. Furthermore, it is known that over a local ring, every F-regular module is semisimple [31]. We can generalize the latest statement as the following.

Proposition 21. *Every GF-regular module over local ring is semisimple.*

Proof. Let $\mathfrak{M}$ be the only maximal ideal of R. Since M is *GF*-regular, then for each $0 \neq x \in M$ we have that $R/\operatorname{ann}(x)$ is *GF*-regular local ring which implies that $R/\operatorname{ann}(x)$ is a field [12]; hence, $\operatorname{ann}(x)$ is a maximal ideal, so $\mathfrak{M} = \operatorname{ann}(x)$ for each $0 \neq x \in M$. Therefore, $\mathfrak{M} = \operatorname{ann}(x) = \operatorname{ann}(M)$. On the other hand, $R/\mathfrak{M} \simeq R/\operatorname{ann}(M)$ is a field, which implies that M is a vector space over the field $R/\operatorname{ann}(M)$ which is a simple ring. Then M is a semisimple module over the ring $R/\operatorname{ann}(M)$. Thus, M is a semismple R-module [29]. □

As an immediate result from Theorem 20 and Proposition 21, we get the following.

Corollary 22. *Let M be an R-module. M is GF-regular if and only if $M_{\mathfrak{M}}$ is a semisimple $R_{\mathfrak{M}}$-module for each maximal ideal $\mathfrak{M}$ of R.*

We mentioned before that every F-regular R-module is *GF*-regular; the following gives us another condition such that the converse is true.

Corollary 23. *Let R be a local ring. An R-module M is F-regular if and only if M is a GF-regular R-module.*

Corollary 24. *An R-module $M = N \bigoplus K$ is GF-regular if and only if N and K are GF-regular R-modules.*

Proof. Assume that N and K are GF-regular R-modules, then for each maximal ideal $\mathfrak{M}$ in R, each of $N_{\mathfrak{M}}$ and $K_{\mathfrak{M}}$ is a semisimple module (Proposition 21); hence, it is an easy matter to check that $N_{\mathfrak{M}} + K_{\mathfrak{M}}$ is a semisimple module, so $M_{\mathfrak{M}} = N_{\mathfrak{M}} \bigoplus K_{\mathfrak{M}}$ is a GF-regular module. Thus, M is a GF-regular module (Theorem 20). The other direction is obtained directly from Proposition 7. □

Finally we can summarize that the conditions under which F-regular modules coincide with GF-regular modules and the characterizations of GF-regular modules, of Section 2 with those of this section, in the following Proposition 25 and Theorem 26, respectively:

Proposition 25. *An R-module M is GF-regular if and only if M is an F-regular module, if any of the following conditions are satisfied.*

(1) *R is a local ring.*

(2) *R is a reduced ring.*

(3) *The prime radical of the ring $R/ann(x)$ is zero for each $0 \neq x \in M$.*

Theorem 26. *The following statements are equivalent for a ring R.*

(1) *M is a GF-regular R-module.*

(2) *$R/ann(x)$ is a π-regular ring for each $0 \neq x \in M$*

(3) *For each $x \in M$ and $r \in R$, there exist $t \in R$ and positive integer n such that $r^{n+1}x = r^n x$.*

(4) *Every submodule of M is G-pure.*

(5) *For each $x \in M$, there exist $p \in R$ and a positive integer n such that $Rp^n x$ is a G-pure submodule.*

(6) *N is a GP-flat R-module, if for every submodule P of a free R-module M there exists an exact sequence $0 \to P \to M \to N \to 0$ such that P is a G-pure submodule in M.*

(7) *If M is a finitely generated R-module, then $R/ann(M)$ is a π-regular ring.*

(8) *$M_{\mathfrak{M}}$ is a GF-regular $R_{\mathfrak{M}}$-module for each maximal ideal $\mathfrak{M}$ in R.*

(9) *$M_{\mathfrak{M}}$ is a semisimple $R_{\mathfrak{M}}$-module for each maximal ideal $\mathfrak{M}$ of R.*

3.3. The Jacobson Radical of GF-Regular Modules. Let M be an R-module. A submodule N of M is said to be small in M if for each submodule K of M such that $N + K = M$, we have $K = M$ [32]. The Jacobson radical of a ring R will be denoted by $J(R)$. The following submodules of M are equal: (1) the intersection of all maximal submodules of M, (2) the sum of all the small submodules of M, and (3) the sum of all cyclic small submodules of M. This submodule is called the Jacobson radical of M and will be denoted by $J(M)$ [29, 32].

It is appropriate now to note that for each element $r \in R$ it may happen that $r^n = 0$. But some cases demand that r^n must be nonzero element. For this purpose we introduce the following concept.

Definition 27. An R-module M is called SGF-regular if for each $0 \neq x \in M$ and $r \in R$, there exist $t \in R$ and a positive integer n with $r^n \neq 0$ such that $r^n t r^n x = r^n x$. A ring R is called SGF-regular if it is SGF-regular as an R-module.

It is clear that SGF-regularity implies GF-regularity and they are coincide if R is a reduced ring.

Proposition 28. *Let M be an SGF-regular R-module, then $J(R).M = 0$.*

Proof. For each $0 \neq x \in M$ and for each $0 \neq r \in R$, there exist $t \in R$ and a positive integer n with $r^n \neq 0$ such that $r^n t r^n x = r^n x$, then $r^n x(r^n x - 1) = 0$. If $r \in J(R)$, then $r^n \in J(R)$ and $(r^n t - 1)$ is invertible, so $r^n x = 0$, but we have that $r^n \neq 0$ and $x \neq 0$; hence, $rx = 0$ which implies that $J(R) \cdot M = 0$. □

Recall that an R-module M is faithful if for every $r \in R$ such that $rM = 0$ implies $r = 0$ [29], or equivalently, an R-module M is called faithful if $ann(M) = 0$ [33].

Corollary 29. *If M is a faithful SGF-regular R-module, then $J(R) = 0$.*

Corollary 30. *Let R be a reduced ring and M be a GF-regular R-module, then $J(R) \cdot M = 0$.*

Corollary 31. *Let R be any ring such that $J(R)$ is a reduced ideal of R and let M be a GF-regular R-module, then $J(R) \cdot M = 0$.*

Corollary 32. *Let R be a reduced ring. If M is a faithful GF-regular R-module, then $J(R) = 0$.*

It is suitable to mention that, in general, not every module contains a maximal submodule; for example, Q as Z-module has no maximal submodule. So we have the next two results, but first we need Lemma 33 which is proved in [29].

Lemma 33. *An R-module M is semisimple if and only if each submodule of M is direct summand.*

Proposition 34. *Let M be a GF-regular R-module, then $J(M) = 0$.*

Proof. Since M is a GF-regular R-module, then $M_{\mathfrak{M}}$ is a semisimple $R_{\mathfrak{M}}$-module for each maximal ideal $\mathfrak{M}$ of R (Corollary 22). Since each cyclic submodule of $M_{\mathfrak{M}}$ is direct summand (Lemma 33), then it cannot be small; therefore, the Jacobson radical of a semisimple module is zero, so $J(M_{\mathfrak{M}}) = 0$ for each maximal ideal $\mathfrak{M}$ of R. On the other hand, $J(M)_{\mathfrak{M}} \subseteq J(M_{\mathfrak{M}})$ [28], thus $J(M)_{\mathfrak{M}} = 0$ for each maximal ideal $\mathfrak{M}$ of R, and hence $J(M) = 0$ [28]. □

Corollary 35. *Every nonzero GF-regular R-module M contains a maximal submodule.*

Proof. Suppose not, then $J(M) = M$, but $J(M) = 0$ (Proposition 34), so $M = 0$ which is a contradiction. □

Corollary 36. *Let M be a GF-regular R-module, then for each $0 \neq x \in M$, there exist a maximal submodule $\mathfrak{M}$ such that $x \notin \mathfrak{M}$.*

Proof. If $x \in P$, for each maximal submodule $\mathfrak{M}$ of M, then $x \in J(M) = 0$ which implies that $x = 0$. □

Corollary 37. *Let M be a GF-regular R-module, then every proper submodule of M contained in a maximal submodule.*

Proof. Let N be a proper submodule of M. Since M is a GF-regular R-module, then $M/N \neq 0$ is GF-regular (Proposition 7), so M/N contains a maximal submodule (Corollary 35), which means that there exists a submodule K of M such that $N \subseteq K$, K/N is a maximal submodule of M/N; therefore, K is a maximal submodule of M and contains N. □

Corollary 38. *Every simple submodule of a GF-regular R-module is direct summand.*

Proof. Let N be a simple submodule of a GF-regular R-module M, then N is cyclic; say $N = Rx$, then there exists a maximal submodule $\mathfrak{M}$ of M such that $x \notin \mathfrak{M}$ (Corollary 37). It is clear that $M = \mathfrak{M} + Rx$. Now, if $Rx \cap \mathfrak{M} \neq (0)$, then $Rx \cap \mathfrak{M} = Rx$ because Rx is a simple submodule. Thus, $x \in \mathfrak{M}$ which is a contradiction, so $M = Rx \bigoplus \mathfrak{M}$. □

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