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Research Article

A Lotka-Volterra Competition Model with Cross-Diffusion

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A Lotka-Volterra competition model with cross-diffusions under homogeneous Dirichlet boundary condition is considered, where cross-diffusions are included in such a way that the two species run away from each other because of the competition between them. Using the method of upper and lower solutions, sufficient conditions for the existence of positive solutions are provided when the cross-diffusions are sufficiently small. Furthermore, the investigation of nonexistence of positive solutions is also presented.

1. Introduction

In this paper, we deal with the following Lotka-Volterra competition model with cross-diffusions:

$$\begin{gathered} -\Delta(u+\alpha v)=u(a-u-cv), \quad x\in\Omega, \\ -\Delta(\beta u+v)=v(b-v-du), \quad x\in\Omega, \\ u=v=0, \quad x\in\partial\Omega, \end{gathered} \tag{1}$$

where Ω is a bounded domain in $\mathbf{R}^N$ with smooth boundary $\partial\Omega$ and all parameters $a, b, c, d, \alpha, \beta$ are positive constants. u and v stand for the densities of the two competitors; a and b are the intrinsic growth rates of u and v, respectively; c and d are the competitive parameters between the two species; Here α and β are referred to as cross-diffusions. Cross-diffusions express the two species run away from each other because of the competition between them. In this paper, the boundary condition is under homogeneous Dirichlet boundary condition which in biologically means that the boundary is not suitable for both species and they will all die on the boundary, and this is an ideal case.

In order to describe the meaning of cross-diffusions in this model (1) from the biological point, we give the general model with intrinsic diffusion and cross-diffusion:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \operatorname{div}\{k_{11}(u,v)\nabla u+k_{12}(u,v)\nabla v\}+f(u,v), \\ \frac{\partial v}{\partial t} &= \operatorname{div}\{k_{21}(u,v)\nabla u+k_{22}(u,v)\nabla v\}+g(u,v), \end{aligned} \tag{2}$$

where u and v stand for the densities of the two species, intrinsic diffusion parameters $k_{11}(u,v), k_{22}(u,v) > 0$, cross-diffusion parameters $k_{12}(u,v), k_{21}(u,v)$,

$$\begin{aligned} J_u &= -\{k_{11}(u,v)\nabla u+k_{12}(u,v)\nabla v\}, \\ J_v &= -\{k_{21}(u,v)\nabla u+k_{22}(u,v)\nabla v\} \end{aligned} \tag{3}$$

can be seen as the out-flux vector of u and v at x. The cross-diffusion parameters $k_{12}(u,v), k_{21}(u,v) \geq 0$ imply that the two competitors u and v diffuse in the direction of lower contrary of their competitor to avoid each other. $f(u,v)$, $g(u,v)$ are response function and in this paper the classical Logistic Type is considered and $\alpha, \beta \geq 0$. More biological meaning of the system can be seen in [1–3].

The method of upper and lower solutions is a useful tool to study the existence of solutions of elliptic systems. However, there are many difficulties in investigating the existences of positive solutions of strongly coupled elliptic systems. Recently, by changing general strongly coupled elliptic systems into weakly coupled ones, the author in paper [4] gives the method to judge the solutions existence of elliptic systems by using the Schauder theorem. Furthermore, the method can be used to solve the existence of solutions of strongly coupled elliptic systems. In [5] Ko and Ryu investigate Lotka-Volterra prey-predator model with cross-diffusion:

$$\begin{aligned} -\Delta u &= u\left(a_1 - u - b_{12}v\right), \quad x \in \Omega, \\ -D\Delta u - \Delta v &= v\left(a_2 + b_{21}u - v\right), \quad x \in \Omega, \\ u &= v = 0, \quad x \in \partial\Omega. \end{aligned} \tag{4}$$

Here D may be positive or negative. Using the developing method of upper and lower solutions in [4], the author gave a sufficient conditions for the existence of positive solutions to (4). Inspired by the paper [5], we investigate the existence and nonexistence of positive solutions to (1).

The main goal of this paper is to provide sufficient conditions for the existence of positive solutions to (1) when the cross-diffusions α and β are small. More precisely, we have the following theorem. Let $\lambda_1 > 0$ be the principal eigenvalue of $-\Delta$ under homogeneous Dirichlet boundary condition. It is well known that the principal eigenfunction ϕ corresponding to λ_1 does not change sign in Ω and $\|\phi\|_\infty = 1$.

Theorem 1. *If* $\min\{a-cb, b-da\} > \lambda_1$, *then there exist positive constants* $\overline{\alpha} = \overline{\alpha}(a,b,c,d,\Omega)$, $\overline{\beta} = \overline{\beta}(a,b,c,d,\Omega)$, *when* $\alpha < \overline{\alpha}$, $\beta < \overline{\beta}$, (1) *has at least one positive solution.*

For $\alpha = \beta = 0$, (1) is the Lotka-Volterra competition model under homogeneous Dirichlet boundary condition. In [6, 7], the authors use different methods to prove the existence of positive solutions, a sufficient condition for the existence is $\min\{a - cb, b - da\} > \lambda_1$. The conclusion implies that weakly cross-diffusion does not affect the existence of positive solution.

This paper is organized as follows. In Section 2, the existence theorem of solutions for a general class of strongly coupled elliptic systems is presented using the method of upper and lower solutions. In Section 3, sufficient conditions for the existence and nonexistence of positive solutions of (1) are investigated. Moreover, we give the corresponding results simply if the competitive system only has one cross-diffusion.

2. The Existence Theorem of Solutions for a Class of Strongly Coupled Elliptic Systems

In this section, we presented the existence theorem of solutions for a general class of strongly coupled elliptic systems:

$$\begin{aligned} -\Delta A(u,v) &= f_1(u,v), \quad x \in \Omega, \\ -\Delta B(u,v) &= f_2(u,v), \quad x \in \Omega, \\ u &= v = 0, \quad x \in \partial\Omega. \end{aligned} \tag{5}$$

Here let A, B, f_1, f_2 satisfy the following hypotheses conditions.

(H1) U, V are domain in $\mathbf{R}^2$, $(0,0) \in U$. (A,B) is a C^2 function about (u,v) from U to V, $A(0,0) = B(0,0) = 0$, and have a continuous inverse $(A^*, B^*) \in C^2(V,U)$. Then for all $(u,v) \in U$, let

$$w = A(u,v), \qquad z = B(u,v). \tag{6}$$

There exists only one $(w,z) \in V$, satisfying

$$u = A^*(w,z), \qquad v = B^*(w,z). \tag{7}$$

(H2) The function A^* is increasing in w and decreasing in z; B^* is decreasing in w and increasing in z.

(H3) The functions $f_1(u,v)$, $f_2(u,v)$ are Lipschitz continuous in U, and there exist positive constants M_1, M_2 such that for all $(u,v) \in U$, the function $f_1(u,v) + M_1A(u,v)$ is increasing in u; the function $f_2(u,v) + M_2B(u,v)$ is increasing in v.

According to the hypothesis (H1), (5) can be rewritten as the following equal PDE equations:

$$\begin{aligned} -\Delta w + M_1 w &= f_1(u,v) + M_1 A(u,v), \quad x \in \Omega, \\ -\Delta z + M_2 z &= f_2(u,v) + M_2 B(u,v), \quad x \in \Omega, \\ u &= A^*(w,z), \quad v = B^*(w,z), \quad x \in \Omega, \\ w &= z = 0, \quad x \in \partial\Omega. \end{aligned} \tag{8}$$

Remark 2. According to the hypothesis (H1), (5) can also be equal to the following weakly coupled elliptic equations:

$$\begin{aligned} -\Delta w &= f_1\left(A^*(w,z), B^*(w,z)\right) := g_1(w,z), \quad x \in \Omega, \\ -\Delta z &= f_2\left(A^*(w,z), B^*(w,z)\right) := g_2(w,z), \quad x \in \Omega, \\ w &= z = 0, \quad x \in \partial\Omega. \end{aligned} \tag{9}$$

In its pure form, (9) is simpler than (8). However, due to the complicity of mixed functions $g_1(w(x), z(x))$ and $g_2(w(x), z(x))$, it is difficult to find the solutions of (9) directly. Therefore, we discuss (8).

Assume functions $\overline{u}$, $\overline{v}$, $\underline{u}$, $\underline{v} \in C(\overline{\Omega})$, $\overline{w}$, $\overline{z}$, $\underline{w}$, $\underline{z} \in C^\alpha(\overline{\Omega}) \bigcap C^2(\Omega)$, the values of functions $(\overline{u}, \overline{v})$ and $(\underline{u}, \underline{v})$ are in V and the values of functions $(\overline{w}, \overline{z})$ and $(\underline{w}, \underline{z})$ are in U. To describe easily, let

$$\begin{aligned} U &= \left\{u \in C\left(\overline{\Omega}\right) : \underline{u}(x) \le u(x) \le \overline{u}(x)\right\}, \\ V &= \left\{u \in C\left(\overline{\Omega}\right) : \underline{v}(x) \le v(x) \le \overline{v}(x)\right\}. \end{aligned} \tag{10}$$

According the definition of upper and lower solutions in [4] and conditions (H1)–(H3), we give the definition of upper and lower solutions of (5).

Definition 3. A pair of functions $((\overline{u}, \overline{v}, \overline{w}, \overline{z}), (\underline{u}, \underline{v}, \underline{w}, \underline{z}))$ are called upper and lower solutions of (9) provided that they satisfy the relation $(\overline{u},\overline{v},\overline{w},\overline{z}) \geq (\underline{u},\underline{v},\underline{w},\underline{z})$, and for all $(u,v) \in U \times V$, satisfy the following inequalities:

$$\begin{aligned}
&-\Delta\overline{w} + M_1\overline{w} \geq f_1(\overline{u},v) + M_1A(\overline{u},v), \quad x \in \Omega,\\
&-\Delta\overline{z} + M_2\overline{z} \geq f_2(u,\overline{v}) + M_2B(u,\overline{v}), \quad x \in \Omega,\\
&-\Delta\underline{w} + M_1\underline{w} \leq f_1(\underline{u},v) + M_1A(\underline{u},v), \quad x \in \Omega,\\
&-\Delta\underline{z} + M_2\underline{z} \leq f_2(u,\underline{v}) + M_2B(u,\underline{v}), \quad x \in \Omega,\\
&\overline{u} \geq A^*(\overline{w},\underline{z}), \quad \overline{v} \geq B^*(\underline{w},\overline{z}), \quad x \in \Omega,\\
&\underline{u} \leq A^*(\underline{w},\overline{z}), \quad \underline{v} \leq B^*(\overline{w},\underline{z}), \quad x \in \Omega,\\
&\overline{w} \geq 0 \geq \underline{w}, \quad \overline{z} \geq 0 \geq \underline{z}, \quad x \in \partial\Omega.
\end{aligned} \tag{11}$$

We can have the following conclusion from [4, Theorem 2.1].

Proposition 4. *Assume that* (8) *has coupled upper and lower solutions* $((\overline{u},\overline{v},\overline{w},\overline{z}),(\underline{u},\underline{v},\underline{w},\underline{z}))$, *then there exists at least one solution* (u,v,w,z), *satisfying the relation*

$$(\underline{u},\underline{v},\underline{w},\underline{z}) \leq (u,v,w,z) \leq (\overline{u},\overline{v},\overline{w},\overline{z}). \tag{12}$$

Furthermore, (u,v) *is the solution of* (5).

Next, if $\overline{u},\overline{v},\underline{u},\underline{v}$ satisfy

$$\begin{aligned}
&\overline{u} = A^*(\overline{w},\underline{z}), \qquad \overline{v} = B^*(\underline{w},\overline{z}),\\
&\underline{u} = A^*(\underline{w},\overline{z}), \qquad \underline{v} = B^*(\overline{w},\underline{z}),
\end{aligned} \tag{13}$$

then

$$\begin{aligned}
&\overline{w} = A(\overline{u},\underline{v}), \qquad \overline{z} = B(\underline{u},\overline{v}),\\
&\underline{w} = A(\underline{u},\overline{v}), \qquad \underline{z} = B(\overline{u},\underline{v}),
\end{aligned} \tag{14}$$

(11) can be rewritten as

$$\begin{aligned}
&-\Delta A(\overline{u},\underline{v}) + M_1A(\overline{u},\underline{v}) \geq f_1(\overline{u},v) + M_1A(\overline{u},v), \quad x \in \Omega,\\
&-\Delta B(\underline{u},\overline{v}) + M_2B(\underline{u},\overline{v}) \geq f_2(u,\overline{v}) + M_2B(u,\overline{v}), \quad x \in \Omega,\\
&-\Delta A(\underline{u},\overline{v}) + M_1A(\underline{u},\overline{v}) \leq f_1(\underline{u},v) + M_1A(\underline{u},v), \quad x \in \Omega,\\
&-\Delta B(\overline{u},\underline{v}) + M_2B(\overline{u},\underline{v}) \leq f_2(u,\underline{v}) + M_2B(u,\underline{v}), \quad x \in \Omega,\\
&A(\overline{u},\underline{v}) \geq 0 \geq A(\underline{u},\overline{v}), \quad B(\underline{u},\overline{v}) \geq 0 \geq B(\overline{u},\underline{v}), \quad x \in \partial\Omega.
\end{aligned} \tag{15}$$

Synthetically, we have the following result.

Theorem 5. *If there is a pair of functions* $((\overline{u},\overline{v}),(\underline{u},\underline{v}))$, *satisfying*

$$(\overline{u},\overline{v},A(\overline{u},\underline{v}),B(\underline{u},\overline{v})) \geq (\underline{u},\underline{v},A(\underline{u},\overline{v}),B(\overline{u},\underline{v})), \tag{16}$$

and for all $(u,v) \in U \times V$, (15) *is satisfied, then* (5) *has at least one solution* (u,v), *satisfying the relation* $(\underline{u},\underline{v}) \leq (u,v) \leq (\overline{u},\overline{v})$.

To make sure the upper and lower solutions reasonable, we give the following two lemmas; more details can be found in [8, 9].

Lemma 6. *If the functions* $u,v \in C^1(\overline{\Omega})$ *satisfy* $u|_{\partial\Omega} = v|_{\partial\Omega} = 0$, $u|_{\Omega} > 0$, $(\partial u/\partial \nu)|_{\partial\Omega} < 0$, ν *is the outer unit normal vector of* $\partial\Omega$, *then there exists positive constant* ε, *such that* $u(x) > \varepsilon v(x)$, *for all* $x \in \Omega$.

For the equation:

$$\begin{aligned}
-\Delta u &= u(a-u), \quad x \in \Omega,\\
u &= 0, \quad x \in \partial\Omega.
\end{aligned} \tag{17}$$

Lemma 7. *If* $a > \lambda_1$, *then* (17) *has a unique positive solution* θ_a *satisfying* $\theta_a \leq a$. *In addition,* θ_a *is increasing with respect to* a.

3. A Lotka-Volterra Competition Model with Two Cross-Diffusions

In this section, the existence of positive solutions of (1) corresponding to $\alpha \geq 0$, $\beta \geq 0$, is investigated by applying Theorem 5 to prove Theorem 1.

Proof. We seek some positive constants $R, K, \delta, R, K > \lambda_1$ sufficiently large and δ sufficiently small, Lemma 6 may guarantee the existence of θ_R and θ_K. It can be easily known from Hopf boundary lemma:

$$\frac{\partial\phi}{\partial\nu}(x) < 0, \quad \frac{\partial\theta_R}{\partial\nu}(x) < 0, \quad \frac{\partial\theta_K}{\partial\nu}(x) < 0, \quad \forall x \in \partial\Omega. \tag{18}$$

Observe that $\min\{a-cb, b-da\} > \lambda_1$, using Lemma 7, we can have R, K, δ, $a < R < (b-\lambda_1)/d$, $b < K < (a-\lambda_1)/c$, satisfying the following three conditions:

(i) $\delta\phi(x) < \theta_R(x)$, $\delta\phi(x) < \theta_K(x)$, for all $x \in \Omega$;
(ii) $(\partial(\theta_R - \delta\phi)/\partial\nu)(x) < 0$, $(\partial(\theta_K - \delta\phi)/\partial\nu)(x) < 0$;
(iii) $\delta < \min\{a - \lambda_1 - cK, b - \lambda_1 - dR\}$.

Let $M_1 = 2R + cK$, $M_2 = 2K + dR$. Using Lemma 7 again, there exist $\overline{\alpha} = \overline{\alpha}(a,b,c,d,\Omega) < 1$, $\overline{\beta} = \overline{\beta}(a,b,c,d,\Omega) < 1$, for all $(\rho,\tau) \in [0,\overline{\alpha}) \times [0,\overline{\beta})$, for all $x \in \Omega$, satisfying

(iv) $\theta_R - \delta\phi > \rho(\theta_K - \delta\phi)$, $\theta_K - \delta\phi > \tau(\theta_R - \delta\phi)$;
(v) $(R-a)\theta_R > \rho[M_1\theta_K - (M_1 + \lambda_1)\delta\phi]$, $(K-b)\theta_K > \tau[M_2\theta_R - (M_2 + \lambda_1)\delta\phi]$;
(vi) $(a - \lambda_1 - \delta - cK)\delta\phi > \rho[(K + M_1 - \theta_K)\theta_K - M_1\delta\phi]$;
(vii) $(b - \lambda_1 - \delta - dR)\delta\phi > \tau[(R + M_2 - \theta_R)\theta_R - M_2\delta\phi]$.

We will verify $\overline{\alpha}$, $\overline{\beta}$ satisfying Theorem 5. Suppose that $(\alpha,\beta) \in [0,\overline{\alpha}) \times [0,\overline{\beta})$. Then we construct a pair of upper and lower solutions of the form

$$(\overline{u},\overline{v}) = (\theta_R,\theta_K), \qquad (\underline{u},\underline{v}) = (\delta\phi,\delta\phi), \tag{19}$$

where δ satisfies conditions (i)–(iii). Let

$$A(u,v) = u + \alpha v, \qquad B(u,v) = \beta u + v. \tag{20}$$

Then

$$A^*(w,z) = \frac{w-\alpha z}{1-\alpha\beta}, \qquad B^*(w,z) = \frac{z-\beta w}{1-\alpha\beta}. \tag{21}$$

By simply computing, (H1) and (H2) are satisfied, where $U = [0,R]\times[0,K]$, $V = [0, R+\alpha K]\times[0, K+\beta R]$.

Note

$$f_1(u,v) = u(a-u-cv), \qquad f_2(u,v) = v(b-v-du). \tag{22}$$

And for all $(u,v)\in U$, we have

$$\begin{aligned}
[f_1(u,v) + M_1 A(u,v)]_u &= a - 2u - cv + M_1 \\
&\geq -2R - cK + M_1 = 0, \\
[f_2(u,v) + M_2 B(u,v)]_v &= b - 2v - du + M_2 \\
&\geq -2K - dR + M_2 = 0.
\end{aligned} \tag{23}$$

So (H3) is satisfied; observer that $\overline{u}|_{\partial\Omega} = \overline{v}|_{\partial\Omega} = \underline{u}|_{\partial\Omega} = \underline{v}|_{\partial\Omega} = 0$, $(\overline{u},\overline{v}) \geq (\underline{u},\underline{v})$ and (iv) and (15) and the boundary conditions of (16) can be checked. Therefore, if we want to obtain the existence of solutions through [4, Theorem 2.1], we should only verify for all $(u,v)\in U\times V$,

$$\begin{aligned}
&-\Delta A(\overline{u},\underline{v}) + M_1 A(\overline{u},\underline{v}) \geq f_1(\overline{u},v) + M_1 A(\overline{u},v), \quad x\in\Omega, \\
&-\Delta B(\underline{u},\overline{v}) + M_2 B(\underline{u},\overline{v}) \geq f_2(u,\overline{v}) + M_2 B(u,\overline{v}), \quad x\in\Omega, \\
&-\Delta A(\underline{u},\overline{v}) + M_1 A(\underline{u},\overline{v}) \leq f_1(\underline{u},v) + M_1 A(\underline{u},v), \quad x\in\Omega, \\
&-\Delta B(\overline{u},\underline{v}) + M_2 B(\overline{u},\underline{v}) \leq f_2(u,\underline{v}) + M_2 B(u,\underline{v}), \quad x\in\Omega.
\end{aligned} \tag{24}$$

Because f_1 is decreasing in v, f_2 is decreasing in u, and $A(u,v)$ is increasing in v, $B(u,v)$ is increasing in u, only to verify the following inequations:

$$\begin{aligned}
&-\Delta A(\overline{u},\underline{v}) + M_1 A(\overline{u},\underline{v}) \geq f_1(\overline{u},\underline{v}) + M_1 A(\overline{u},\overline{v}), \quad x\in\Omega, \\
&-\Delta B(\underline{u},\overline{v}) + M_2 B(\underline{u},\overline{v}) \geq f_2(\underline{u},\overline{v}) + M_2 B(\overline{u},\overline{v}), \quad x\in\Omega, \\
&-\Delta A(\underline{u},\overline{v}) + M_1 A(\underline{u},\overline{v}) \leq f_1(\underline{u},\overline{v}) + M_1 A(\underline{u},\underline{v}), \quad x\in\Omega, \\
&-\Delta B(\overline{u},\underline{v}) + M_2 B(\overline{u},\underline{v}) \leq f_2(\overline{u},\underline{v}) + M_2 B(\underline{u},\underline{v}), \quad x\in\Omega.
\end{aligned} \tag{25}$$

It is easy to check (25) by (v), (vi), and (vii). So from [4, Theorem 2.1], (1) has a solution (u,v), in addition $(\overline{u},\overline{v}) \geq (u,v) \geq (\underline{u},\underline{v}) > (0,0)$. □

In the end, before investigating the nonexistence of positive solutions of (1), we give its priori bound of positive solutions.

Theorem 8. *Any positive solutions (u,v) of (1) have a priori bound; that is*

$$u(x) \leq \frac{b}{d}, \qquad v(x) \leq \frac{a}{c}. \tag{26}$$

Proof. Let $w = u + \alpha v$, $z = \beta u + v$; then

$$u = \frac{w-\alpha z}{1-\alpha\beta}, \qquad v = \frac{z-\beta w}{1-\alpha\beta}. \tag{27}$$

Equation (1) can be rewritten as

$$\begin{gathered}
-\Delta w = \frac{w-\alpha z}{1-\alpha\beta}\left(a - \frac{w-\alpha z}{1-\alpha\beta} - c\frac{z-\beta w}{1-\alpha\beta}\right), \quad x\in\Omega, \\
-\Delta z = \frac{z-\beta w}{1-\alpha\beta}\left(b - \frac{z-\beta w}{1-\alpha\beta} - d\frac{w-\alpha z}{1-\alpha\beta}\right), \quad x\in\Omega, \\
(w,z) = (0,0), \quad x\in\partial\Omega.
\end{gathered} \tag{28}$$

Since $(u,v) > (0,0)$, it easily follows that $w-\alpha z > 0$, $z-\beta w > 0$. Assume that $z(x)$ attains its positive maximum at $x_0 \in \Omega$, then

$$\begin{gathered}
a(1-\alpha\beta) - w(x_0) + \alpha z(x_0) - cz(x_0) + c\beta w(x_0) > 0 \\
a(1-\alpha\beta) - cz(x_0) + c\beta\alpha z(x_0) > 0, \\
z(x) \leq z(x_0) \leq \frac{a}{c}
\end{gathered} \tag{29}$$

so that

$$v = z - \beta u \leq z(x_0) \leq \frac{a}{c}. \tag{30}$$

Similarly, we can obtain the desired result

$$u \leq \frac{b}{d}. \tag{31}$$

□

Theorem 9. *If one of the following conditions:*

(i) $b \leq ad$, $\lambda_1 \geq (b + c\beta(b/d))/(1-\alpha\beta)$;

(ii) $(1-(\alpha+\beta)/2)\lambda_1 \geq \max\{a,b\}$;

is satisfied, then (1) with $\alpha < \overline{\alpha}$, $\beta < \overline{\beta}$ has no positive solution.

Proof. Multiplying u and v to the first and second equations in (1), and integrating these equations on Ω, we have

$$\begin{aligned}
\int_\Omega |\nabla u|^2 dx + \alpha\int_\Omega \nabla u\nabla v dx &= \int_\Omega u^2(a-u-cv)\,dx, \\
\alpha\int_\Omega |\nabla v|^2 dx + \int_\Omega \nabla u\nabla v dx &= \int_\Omega uv(a-u-cv)\,dx, \\
\beta\int_\Omega |\nabla u|^2 dx + \int_\Omega \nabla u\nabla v dx &= \int_\Omega uv(b-v-du)\,dx, \\
\int_\Omega |\nabla v|^2 dx + \beta\int_\Omega \nabla u\nabla v dx &= \int_\Omega v^2(b-v-du)\,dx.
\end{aligned} \tag{32}$$

(i) Suppose, by contradiction that (1) has a positive solution (u, v), then the second and fourth equations in (32) yield

$$\begin{aligned}\int_\Omega v^3 dx + \beta \int_\Omega uv(a-u)\,dx &= -(1-\alpha\beta)\int_\Omega |\nabla v|^2 dx + \beta\int_\Omega cuv^2 dx \\ &\quad + \int_\Omega v^2(b-du)\,dx.\end{aligned} \tag{33}$$

Since $u \le b/d$ by Theorem 8, the left-hand side of (33) must be positive. On the other hand, the Poincare inequality, $\|\nabla v\|^2_{L^2} \ge \lambda_1 \|v\|^2_{L^2}$, for $v \in W^1_2(\Omega)$ and the given assumption shows the following contradiction:

$$\begin{aligned}&-(1-\alpha\beta)\int_\Omega |\nabla v|^2 dx + \beta\int_\Omega cuv^2 dx + \int_\Omega v^2(b-du)\,dx \\ &\quad\le -\left[(1-\alpha\beta)\lambda_1 - c\beta\frac{b}{d} - b\right]\int_\Omega v^2 dx \le 0.\end{aligned} \tag{34}$$

(ii) A contraction argument is also used assuming that (1) has a positive solution (u, v). Adding the first equation to the fourth equation, and then subtracting $a\int_\Omega u^2 dx + b\int_\Omega v^2 dx$ from the both sides, the following identity is obtained:

$$\begin{aligned}&\int_\Omega |\nabla u|^2 dx + (\alpha+\beta)\int_\Omega \nabla u \nabla v dx \\ &\quad + \int_\Omega |\nabla v|^2 dx - a\int_\Omega u^2 dx - b\int_\Omega v^2 dx \\ &\quad = -\int_\Omega u^2(u+cv)\,dx - \int_\Omega v^2(v+du)\,dx.\end{aligned} \tag{35}$$

Since $2\nabla u\nabla v = |\nabla(u+v)|^2 - |\nabla u|^2 - |\nabla v|^2$ and $(1-(\alpha+\beta)/2)\lambda_1 \ge \max\{a, b\}$, the Poincare inequality shows that the left-hand side of (35) must be nonnegative, more precisely,

$$\begin{aligned}&\int_\Omega |\nabla u|^2 dx + (\alpha+\beta)\int_\Omega \nabla u\nabla v dx \\ &\quad + \int_\Omega |\nabla v|^2 dx - a\int_\Omega u^2 dx - b\int_\Omega v^2 dx \\ &= \left(1-\frac{\alpha+\beta}{2}\right)\int_\Omega |\nabla u|^2 dx + \frac{\alpha+\beta}{2}\int_\Omega |\nabla(u+v)|^2 dx \\ &\quad + \left(1-\frac{\alpha+\beta}{2}\right)\int_\Omega |\nabla v|^2 dx - a\int_\Omega u^2 dx - b\int_\Omega v^2 dx \\ &\ge \left[\left(1-\frac{\alpha+\beta}{2}\right)\lambda_1 - a\right]\int_\Omega u^2 dx + \frac{\alpha+\beta}{2} \\ &\quad \cdot \int_\Omega |\nabla(u+v)|^2 dx + \left[\left(1-\frac{\alpha+\beta}{2}\right)\lambda_1 - b\right]\int_\Omega v^2 dx \\ &\ge \left[\left(1-\frac{\alpha+\beta}{2}\right)\lambda_1 - a\right]\int_\Omega u^2 dx \\ &\quad + \left[\left(1-\frac{\alpha+\beta}{2}\right)\lambda_1 - b\right]\int_\Omega v^2 dx \ge 0.\end{aligned} \tag{36}$$

However, this results in a contradiction since the right-hand side of (35) is clearly strictly negative by the positivity of u and v. □

Remark 10. Before closing this section, more sufficient conditions of the nonexistence of positive solutions of (1) with $\alpha+\beta > 0$, $\alpha\beta = 0$ are investigated. Take $\alpha = 0$, $\beta > 0$ for example, then (1) may be reduced as

$$\begin{aligned}-\Delta u &= u(a-u-cv), \quad x \in \Omega, \\ -\Delta(\beta u + v) &= v(b-v-du), \quad x \in \Omega, \\ (u, v) &= (0, 0), \quad x \in \partial\Omega.\end{aligned} \tag{37}$$

Using the same method, we can obtain that (37) has no positive solution, if one of the following conditions is satisfied:

(i) $\lambda_1 \ge b + \beta ca$;

(ii) $\lambda_1 \ge a$;

(iii) $(1-\beta/2)\lambda_1 \ge \max\{a, b\}$;

(iv) $c < 1 < a/b$ and $(1-d)/\beta \le \lambda_1/(b+\beta a) \le 1$.

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