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Research Article

The Fractional Quadratic-Form Identity and Hamiltonian Structure of an Integrable Coupling of the Fractional Broer-Kaup Hierarchy

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A fractional quadratic-form identity is derived from a general isospectral problem of fractional order, which is devoted to constructing the Hamiltonian structure of an integrable coupling of the fractional BK hierarchy. The method can be generalized to other fractional integrable couplings.

1. Introduction

The theory of derivatives of noninteger order can go back to Leibniz, Liouville, Grunwald, Letnikov, and Riemann. And the fractional analysis has attracted increasing interest of many researchers, because fractional analysis has numerous applications: kinetic theories [1–3], such as statistical mechanics [4–6], dynamics in complex media [7, 8], and many others [9–16]. In recent studies in physics, the researchers have found many applications of the derivatives and integrals of fractional order [16, 17]. They also pointed out that fractional-order models are more appropriate than integer-order models for various real materials. The main advantage of fractional derivative in comparison with classical integer-order models is that it provides an effective instrument for the description of memory and hereditary properties of various materials and progress. Also, the advantages of the fractional derivatives become apparent in modeling mechanical and electrical properties of real materials and in the description of rheological properties of rocks, as well as in many other fields.

The fractional calculus is a generalization of ordinary differentiation and integration to arbitrary order [17–20]. Since Riewe [4, 21] presented a concept of nonconservation mechanics, fractional conservation laws [22], Lie symmetries [9], and fractional Hamiltonian systems [23–33] have been receiving more and more attention.

It is an important and interesting topic to search for new Hamiltonian hierarchies of soliton equations and their integrable couplings in soliton theory. Tu once proposed a simple and efficient method to construct the integrable systems and Hamiltonian structures [34], which was called the Tu scheme by Ma [35]. Later, many integrable systems and their Hamiltonian structures were worked out [36–39]. Recently, Wu and Zhang proposed the generalized Tu formula and searched for the Hamiltonian structure of fractional AKNS hierarchy [40]. In [41], a generalized Hamiltonian structure of the fractional soliton equation hierarchy was presented. Very recently, Wang and Xia obtained the fractional supersoliton hierarchies and their super-Hamiltonian structures by using fractional supertrace identity [42, 43]. Then, how to generate integrable coupling system and Hamiltonian structure of fractional soliton equation?

In this paper, begining with a general isospectral problem of fractional order, we propose a fractional quadratic-form identity, from which the Hamiltonian structure of an integrable coupling of the fractional BK hierarchy is derived.

2. Brief Overview of Fractional Differentiable Functions

Several local versions have been presented [44–52], among which Jumarie's derivative is defined as follows [52]:

$$D_x^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-\xi)^{-\alpha} (f(\xi) - f(0))\, d\xi, \quad (0 < \alpha < 1); \tag{1}$$

some properties of the fractional differentiable functions are given as follows.

(a) The Leibniz product law.

Assuming that $f(x)$ is an α order differentiable function in the area of point x, from the Jumarie-Kolwankar's Taylor series [52–54], we can have

$$D_x^{\alpha} f(x) = \lim_{y \to x^+} \frac{\Gamma(1+\alpha)(f(y) - f(x))}{(y-x)^{\alpha}}, \quad (0 < \alpha \le 1). \tag{2}$$

If $g(x)$ is a differentiable function of α order, the Leibniz product law can hold for the nondifferentiable functions [39, 44, 45]

$$D_x^{\alpha}(f(x) g(x)) = g(x) D_x^{\alpha} f(x) + f(x) D_x^{\alpha} g(x). \tag{3}$$

(b) Denoting ${}_0I_x^{\alpha}$ as the Riemann-Liouville integration in the following form:

$${}_0I_x^{\alpha} f(x) = D_x^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha+1)} \int_0^x f(\xi)(d\xi)^{\alpha}, \quad (0 < \alpha \le 1), \tag{4}$$

we can have a generalized Newton-Leibniz formulation

$$\begin{gathered}
\frac{1}{\Gamma(1+\alpha)} \int_0^1 D_x^{\alpha} f(x)(dx)^{\alpha} = f(1) - f(0), \\
\frac{1}{\Gamma(1+\alpha)} \int_0^x D_x^{\xi} f(\xi)(d\xi)^{\alpha} = f(x) - f(0), \\
\frac{D_x^{\alpha}}{\Gamma(1+\alpha)} \int_0^x f(\xi)(d\xi)^{\alpha} = f(x).
\end{gathered} \tag{5}$$

(c) With the properties (a) and (b), integration by parts for α order differentiable functions $f(x)$ and $g(x)$ can be generated as

$$\begin{aligned}
&\frac{1}{\Gamma(1+\alpha)} \int_a^b g(x) D_x^{\alpha} f(x)(dx)^{\alpha} \\
&\quad = g(x) f(x)\Big|_a^b - \frac{1}{\Gamma(1+\alpha)} \int_a^b f(x) D_x^{\alpha} g(x)(dx)^{\alpha}.
\end{aligned} \tag{6}$$

(d) From [31, 32, 55], the fractional variational derivative is written as

$$\frac{\delta L}{\delta y} = \frac{\partial L}{\partial y} + \sum_{k=1} (-1)^k (D_x^{\alpha})^k \left(\frac{\partial L}{\partial (D_x^{\alpha})^k y} \right), \tag{7}$$

where k is a positive integer. In this paper, we propose a generalized quadratic-form identity for fractional soliton hierarchy from (7).

3. Fractional Exterior Differential and Hamiltonian Equations

Since Adda proposed the fractional generalization of differential forms [56, 57], several versions of fractional exterior differential approaches and applications related to different forms of fractional derivatives appeared in some parts of the open literature [58, 59]. The properties of fractional derivatives are discussed in [60].

The exterior derivative is defined as

$$d = \sum_{m=1}^{n} dx_m \frac{\partial}{\partial x_m}. \tag{8}$$

The exterior derivative map k forms into $k+1$ forms and has the following algebraic results. Let γ and λ be k forms, and let μ be an m form; we have

$$\begin{gathered}
d(\gamma + \lambda) = d\gamma + d\lambda, \\
d(\gamma \wedge \mu) = (d\gamma) \wedge \mu + (-1)^k \gamma \wedge d\mu, \\
d(d\gamma) = 0.
\end{gathered} \tag{9}$$

The last identity is called the Poincaré lemma. A form γ is called closed if $d\gamma = 0$. A form γ is called exact if there exists a form μ such that $d\mu = \gamma$. The order of μ is one less than the order of γ. Exact forms are always closed, closed forms are not always exact.

Next, we introduce the fractional exterior derivative

$$d^{\alpha} = (dx_i)^{\alpha} D_{x_i}^{\alpha}. \tag{10}$$

A differential 1-form is defined by

$$\omega_{\alpha} = F^i(x)(dx_i)^{\alpha}, \tag{11}$$

with the vector field $F^i(x)$ that can be represented as $F^i(x) = -D_{x_i}^{\alpha} V$ and $V(x)$ is a continuously differentiable function. Using (10), the exact fractional form can be expressed as

$$\omega_{\alpha} = -d^{\alpha} V = -(dx_i)^{\alpha} D_{x_i}^{\alpha} V. \tag{12}$$

Note that (11) is a fractional generalization of the differential form (8). It is easy to find that fractional 1-form ω_{α} can be closed when the differential 1-form $\omega = \omega_1$ is not closed.

Then, we define the fractional functional

$$J[p,q] = \frac{1}{\Gamma(1+\alpha)} \int [p D_t^{\alpha} q - H(t,p,q)] (dt)^{\alpha}; \tag{13}$$

hence, we can readily derive the generalized Poincare-Cartan 1-form, which reads

$$\omega = p d^{\alpha} q - H (dt)^{\alpha}. \tag{14}$$

From (14), one has

$$\begin{aligned} d^{\alpha}\omega &= p_t^{\alpha}(dt)^{\alpha} \wedge d^{\alpha}q + d^{\alpha}p \wedge d^{\alpha}q \\ &\quad - \frac{\partial H}{\partial p} d^{\alpha}p \wedge (dt)^{\alpha} - \frac{\partial H}{\partial q} d^{\alpha}q \wedge (dt)^{\alpha} \\ &= \left[p_t^{\alpha} + \frac{\partial H}{\partial q} \right] (dt)^{\alpha} \wedge d^{\alpha}q \\ &\quad + \left[\frac{\partial H}{\partial p} (dt)^{\alpha} - d^{\alpha}q \right] \wedge d^{\alpha}p. \end{aligned} \tag{15}$$

In the previous derivation, p and q are fractional differentiable functions with respect to t.

The fractional closed condition $d^{\alpha}\omega = 0$ admits the fractional Hamilton's equations [40]

$$q_t^{(\alpha)} = \frac{\partial H}{\partial p}, \qquad p_t^{(\alpha)} = -\frac{\partial H}{\partial q}, \tag{16}$$

which can be generalized to the following case [31]:

$$q_i^{(\alpha)}(t) = \frac{\partial H}{\partial p_i}, \qquad p_i^{(\alpha)}(t) = -\frac{\partial H}{\partial q_i}, \tag{17}$$

4. The Fractional Quadratic-Form Identity

Guo and Zhang once proposed quadratic-form identity [61], which is very efficient tool to systematically generate integrable couplings and their Hamiltonian structures. In the following, the fractional quadratic-form identity is presented.

Set G to be an s-dimensional Lie algebra with the basis

$$e_1, e_2, \ldots, e_s, \tag{18}$$

whose corresponding loop algebra $\widetilde{G}$ possesses the following basis:

$$\begin{gathered} e_i(m) = e_i \lambda^m, \quad i = 1, 2, \ldots, s, \\ m = 0, \pm 1, \pm 2, \ldots, \\ \left[e_i(m), e_j(m) \right] = \left[e_i, e_j \right] \lambda^{m+n}. \end{gathered} \tag{19}$$

In terms of $\widetilde{G}$, we construct the following isospectral problem:

$$\psi_x^{(\alpha)} = [U, \psi], \qquad \psi_t^{(\beta)} = [V, \psi]. \tag{20}$$

The compatibility condition of (20) gives rise to the generalized zero curvature equation:

$$U_t^{(\beta)} - V_x^{(\alpha)} + [U, V] = 0. \tag{21}$$

Taking $\alpha = \beta = 1$ (21) reduces to the classical zero curvature equation. For λ and u_i $(i = 1, 2, \ldots, p)$ in $U = U(\lambda, u) = \Sigma_{i=1}^{s} U_i e_i$, defining $\operatorname{rank}(\lambda) = \deg(\lambda)$, then $\operatorname{rank}(e_i(\lambda)) = \alpha_i$, $0 \le i \le s$ can be presented. If the ranks of u_i are taken as $\zeta - \alpha_i$, $1 \le i \le s$, then each term in U has the homogeneous rank α which is denoted by

$$\operatorname{rank}(U) = \operatorname{rank}\left(\frac{\partial^{\alpha}}{\partial x^{\alpha}} \right) = \zeta. \tag{22}$$

Set $V = \sum_{m \ge 0} V_m \lambda^{-m}$, $V_m = \Sigma_{i=1}^{s} V_{mi} e_i \in G$, as a solution of the stationary zero curvature equation

$$-V_x^{(\alpha)} + [U, V] = 0, \tag{23}$$

and $\operatorname{rank}(V_m)_{\lambda}$ is assumed to be given so that $\operatorname{rank}(V_m)_{\lambda} = \xi$, $m \ge 0$; each team in V has the same rank as follows:

$$\operatorname{rank}(V) = \operatorname{rank}\left(\frac{\partial^{\beta}}{\partial t^{\beta}} \right) = \xi. \tag{24}$$

Let the two arbitrary solutions V_1 and V_2 of (23) with the same rank be linearly related by

$$\overline{V} = \gamma V, \qquad \gamma = \text{const.} \tag{25}$$

In the following, relation (25) will be used when deducing the fractional quadratic-form identity. For $a, b \in \widetilde{G}$, the s-order matrix $R(b)$ is determined by

$$[a, b]^T = a^T R(b), \tag{26}$$

and constant matrix $F = (f_{ij})_{s \times s}$ is determined by

$$F = F^T, \qquad R(b) F = -(R(b) F)^T. \tag{27}$$

Defining functional $\{a, b\} = a^T F b$ satisfies the symmetry

$$\{a, b\} = \{b, a\}, \tag{28}$$

and the bilinear relation

$$\{c_1 a_1 + c_2 a_2, b\} = c_1 \{a_1, b\} + c_2 \{a_2, b\}. \tag{29}$$

In the sense of the local fractional derivative, the gradient $\nabla_b \{a, b\}$ of the functional $\{a, b\}$ is defined by

$$\frac{\partial}{\partial \epsilon} \{a, b + \epsilon V\} = \left(\delta_b \{a, b\}, V \right), \quad a, b, V \in \widetilde{G}, \tag{30}$$

where δ_b is variational derivative with respect to b. With the fractional variational derivative (7), one can have

$$\delta_b \left\{ a, b_x^{(k\alpha)} \right\} = (-1)^k a_x^{(k\alpha)}, \tag{31}$$

where k is a positive integer and $D_x^{k\alpha} = \underbrace{D_x^{\alpha} \cdots D_x^{\alpha}}_{k}$. The communication relationship of $\{a, b\}$ can be given as

$$\{[a, b], c\} = \{a, [b, c]\}, \quad a, b, c \in \widetilde{G}. \tag{32}$$

Introduce a functional

$$W = \left\{ V, U_{\lambda} \right\} + \left\{ \Lambda, V_x^{(\alpha)} - [U, V] \right\}, \tag{33}$$

where U, V meet (23), while $\Lambda(\in \widetilde{G})$ is to be determined; using (7), we can obtain the following fractional variation constraint conditions:

$$\frac{\delta W}{\delta \Lambda} = V_x^{(\alpha)} - [U, V], \qquad \frac{\delta W}{\delta V} = U_\lambda - \Lambda_x^{(\alpha)} + [U, \Lambda]; \quad (34)$$

according to the Jacobi identity and the previous equations, we can have

$$[\Lambda, V]_x^\alpha = [U_\lambda, V] + [U, [\Lambda, V]], \quad (35)$$

$Z = [\Lambda, V] - V_\lambda$ and V/λ are solutions of (23); using (25) and $\text{rank}(Z) = \text{rank}(V_\lambda) = \text{rank}(V/\lambda)$, due to V/λ satisfying (34), we can have $Z = (\gamma/\lambda)V$. From (23) and (33), a fractional quadratic-form identity is firstly presented as follows:

$$\begin{aligned}\frac{\delta}{\delta u_i}\{V, U_\lambda\} &= \left\{V, \frac{\partial U_\lambda}{\partial u_i}\right\} + \left\{[\Lambda, V], \frac{\partial U}{\partial u_i}\right\} \\ &= \left\{V, \frac{\partial U_\lambda}{\partial u_i}\right\} + \left\{V_\lambda, \frac{\partial U}{\partial u_i}\right\} + \frac{\gamma}{\lambda}\left\{V, \frac{\partial U}{\partial u_i}\right\} \\ &= \frac{\partial}{\partial \lambda}\left\{V, \frac{\partial U}{\partial u_i}\right\} + \left(\lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\right)\left\{V, \frac{\partial U}{\partial u_i}\right\} \\ &= \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\left(\lambda^{\gamma}\left\{V, \frac{\partial U}{\partial u_i}\right\}\right), \quad 1 \le i \le p.\end{aligned} \quad (36)$$

5. Application of the Fractional Quadratic-Form Identity

Introduce a loop algebra $\widetilde{G}_6 = \{a = (a_1, a_2, \ldots, a_6)^T, a_k = \sum_m \lambda^m\}$, with the commuting relations

$$\begin{aligned}[a, b] = (&a_2 b_3 - a_3 b_2, 2a_1 b_2 - 2a_2 b_1, 2a_3 b_1 \\ &- 2a_1 b_3, a_2 b_6 - a_6 b_2 + a_5 b_3 - a_3 b_5, 2a_1 b_5 \\ &- 2a_5 b_1 + 2a_4 b_2 - 2a_2 b_4, 2a_6 b_1 \\ &-2a_1 b_6 + 2a_3 b_4 - 2a_4 b_3)^T.\end{aligned} \quad (37)$$

Consider the following spectral problem:

$$\begin{gathered}\psi_x^{(\alpha)} = [U, \psi], \\ U = \left(-\lambda + \frac{v}{2}, 1, -w, u_1, 0, u_2\right)^T \\ V = (a, b, c, d, e, f)^T.\end{gathered} \quad (38)$$

Solving equation

$$-V_x^{(\alpha)} + [U, V] = 0 \quad (39)$$

leads to

$$\begin{gathered}a_{mx}^{(\alpha)} = c_m + w b_m, \\ b_{mx}^{(\alpha)} = -2b_{m+1} + v b_m - 2a_m, \\ c_{mx}^{(\alpha)} = 2c_{m+1} - v c_m - 2w a_m, \\ d_{mx}^{(\alpha)} = f_m + w e_m - u_2 b_m, \\ e_{mx}^{(\alpha)} = -2e_{m+1} + v e_m - 2d_m + 2u_1 b_m, \\ f_{mx}^{(\alpha)} = 2f_{m+1} - v f_m - 2w d_m - 2u_1 c_m + 2u_2 a_m, \\ a_1 = d_1 = e_1 = c_2 = 0, \qquad b_1 = 1, \qquad c_1 = -w, \\ f_1 = u_2, \qquad a_2 = -\frac{1}{2}w, \\ b_2 = \frac{1}{2}v, \qquad d_2 = \frac{1}{2}u_2, \\ e_2 = u_1, \qquad f_2 = \frac{1}{2}u_{2x}^{(\alpha)} + \frac{1}{2}u_2 v - w u_1, \ldots.\end{gathered} \quad (40)$$

Set

$$\begin{aligned}V^{(n)} = &\sum_{m=0}^{n}(a_m, b_m, c_m, d_m, e_m, f_m)^T \lambda^{n-m} \\ &+ (b_{n+1}, 0, 0, 0, 0, 0)^T;\end{aligned} \quad (41)$$

then the generalized zero curvature equation, $D_t^\beta U - D_x^\alpha V^{(n)} + [U, V^{(n)}] = 0$, gives rise to a system

$$\begin{aligned}u_{t_n}^{(\beta)} &= \begin{pmatrix} v_{t_n}^{(\beta)} \\ w_{t_n}^{(\beta)} \\ u_{1t_n}^{(\beta)} \\ u_{2t_n}^{(\beta)} \end{pmatrix} = \begin{pmatrix} 2b_{n+1,x}^{(\alpha)} \\ -2a_{n+1,x}^{(\alpha)} \\ e_{n+1,x}^{(\alpha)} \\ 2d_{n+1,x}^{(\alpha)} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 2D_x^\alpha \\ 0 & 0 & -D_x^\alpha & 0 \\ 0 & -D_x^\alpha & 0 & -D_x^\alpha \\ 2D_x^\alpha & 0 & -D_x^\alpha & 0 \end{pmatrix} \\ &\quad \times \begin{pmatrix} a_{n+1} + d_{n+1} \\ -b_{n+1} - e_{n+1} \\ 2a_{n+1} \\ b_{n+1} \end{pmatrix} = J P_{n+1},\end{aligned} \quad (42)$$

where J is a Hamiltonian operator. From (40), we have a recurrence operator

$$L=\begin{pmatrix}\frac{1}{2}D_x^{\alpha}+\frac{1}{2}D_x^{-\alpha}vD_x^{\alpha} & \frac{1}{2}\left(w+D_x^{-\alpha}wD_x^{\alpha}\right) & \frac{1}{2}D_x^{-\alpha}u_1D_x^{\alpha} & \frac{1}{2}\left(u_2+D_x^{-\alpha}u_2D_x^{\alpha}\right)\\ 1 & \frac{1}{2}\left(v-D_x^{\alpha}\right) & 0 & -u_1\\ 0 & 0 & \frac{1}{2}\left(D_x^{\alpha}+D_x^{-\alpha}vD_x^{\alpha}\right) & -w-D_x^{-\alpha}wD_x^{\alpha}\\ 0 & 0 & -\frac{1}{2} & \frac{1}{2}\left(v-D_x^{\alpha}\right)\end{pmatrix} \tag{43}$$

which meets $P_{n+1}=LP_n$. Hence, expression (42) can be written as

$$u_{t_n}^{(\beta)}=\begin{pmatrix}v_{t_n}^{(\beta)}\\ w_{t_n}^{(\beta)}\\ u_{1t_n}^{(\beta)}\\ u_{2t_n}^{(\beta)}\end{pmatrix}=JL^n\begin{pmatrix}0\\-1\\0\\1\end{pmatrix}. \tag{44}$$

From expression (37), we have

$$[a,b]^T=a^T\begin{pmatrix}0 & 2b_2 & -2b_3 & 0 & 2b_5 & -2b_6\\ b_3 & -2b_1 & 0 & b_6 & -2b_4 & 0\\ -b_2 & 0 & 2b_1 & -b_5 & 0 & 2b_4\\ 0 & 0 & 0 & 0 & 2b_2 & -2b_3\\ 0 & 0 & 0 & b_3 & -2b_1 & 0\\ 0 & 0 & 0 & -b_2 & 0 & 2b_1\end{pmatrix} = a^TR(b). \tag{45}$$

Solving the matrix equation (27) for F leads to

$$F=\begin{pmatrix}2&0&0&2&0&0\\0&0&1&0&0&1\\0&1&0&0&1&0\\2&0&0&0&0&0\\0&0&1&0&0&0\\0&1&0&0&0&0\end{pmatrix}. \tag{46}$$

Let

$$\{a,b\}=2\left(a_1+a_4\right)b_1+\left(a_3+a_6\right)b_2+\left(a_2+a_5\right)b_3 + 2a_1b_4+a_3b_5+a_2b_6; \tag{47}$$

we have

$$\left\{V,\frac{\partial U}{\partial v}\right\}=a+d,\qquad \left\{V,\frac{\partial U}{\partial w}\right\}=-b-e,$$
$$\left\{V,\frac{\partial U}{\partial u_1}\right\}=2a,\qquad \left\{V,\frac{\partial U}{\partial u_2}\right\}=b, \tag{48}$$
$$\left\{V,\frac{\partial U}{\partial \lambda}\right\}=-2a-2d.$$

Substituting the previous results into the fractional quadratic-form identity (36) gives

$$\frac{\delta}{\delta u}\left(-2a-2d\right)=\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\begin{pmatrix}a+d\\-b-e\\2a\\b\end{pmatrix}. \tag{49}$$

Comparing the coefficients of λ^{-n-1} on both sides of (49) yields

$$\frac{\delta}{\delta u}\left(-2a_{n+1}-2d_{n+1}\right)=\left(\gamma-n\right)\begin{pmatrix}a_n+d_n\\-b_n-e_n\\2a_n\\b_n\end{pmatrix}. \tag{50}$$

It is easy to find that $\gamma=0$; then we obtain the fractional Hamiltonian structure of (42)

$$u_{t_n}^{(\beta)}=\begin{pmatrix}v_{t_n}^{(\beta)}\\ w_{t_n}^{(\beta)}\\ u_{1t_n}^{(\beta)}\\ u_{2t_n}^{(\beta)}\end{pmatrix}=J\begin{pmatrix}a_{n+1}+d_{n+1}\\-b_{n+1}-e_{n+1}\\2a_{n+1}\\b_{n+1}\end{pmatrix}=J\frac{\delta H_n}{\delta u}, \tag{51}$$

where $H_n = (2a_{n+1} + 2d_{n+1})/n$ and $(n = 0, 1, 2, \ldots)$ is the fractional Hamiltonian function. When taking $n = 2$, we have an integrable coupling of a fractional BK hierarchy

$$\begin{aligned} D_{t_2}^{\beta} v &= -\frac{1}{2} D_x^{\alpha} D_x^{\alpha} v + v D_x^{\alpha} v + D_x^{\alpha} w, \\ D_{t_2}^{\beta} w &= \frac{1}{2} D_x^{\alpha} D_x^{\alpha} w + D_x^{\alpha} (wv), \\ D_{t_2}^{\beta} u_1 &= -\frac{1}{2} D_x^{\alpha} D_x^{\alpha} u_1 + D_x^{\alpha} (u_1 v) - \frac{1}{2} D_x^{\alpha} u_2, \\ D_{t_2}^{\beta} u_2 &= \frac{1}{2} D_x^{\alpha} D_x^{\alpha} u_2 + D_x^{\alpha} (u_2 v) - 2 D_x^{\alpha} (w u_1). \end{aligned} \tag{52}$$

Reduction Cases

Case 1. When $\alpha = \beta = 1$, $u_1 = u_2 = 0$, $t_2 = t$; (52) reduces to the BK hierarchy

$$\begin{aligned} v_t &= -\frac{1}{2} v_{xx} + v v_x + w_x, \\ w_t &= \left(vw + \frac{1}{2} w_x \right)_x. \end{aligned} \tag{53}$$

Case 2. Let $v = -q$, $w = r + 1 + (1/2) v_x$, (53) is transformed to the classical Boussinesq equation

$$\begin{aligned} q_t &= -q q_x - r_x, \\ r_t &= -\frac{1}{4} q_{xxx} - (q (r + 1))_x. \end{aligned} \tag{54}$$

6. Conclusion

A way to construct the Hamiltonian structure of integrable coupling of fractional soliton equation hierarchy is presented. As an application, the Hamiltonian structure of an integrable coupling of the fractional BK hierarchy is obtained by use of the fractional quadratic-form identity. The method can be generalized to other fractional integrable couplings.

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