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Research Article

Multistability in a Multidirectional Associative Memory Neural Network with Delays

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This paper focuses on the multidirectional associative memory (MAM) neural networks with m fields which is more advanced to realize associative memory. Based on the Brouwer fixed point theorem and Dini upper right derivative, it is confirmed that the multidirectional associative memory neural network can have 3^l equilibria and 2^l equilibria of them are stable, where l is a parameter associated with the number of neurons. Furthermore, an example is given to illustrate the effectiveness of the results.

1. Introduction

The Hopfield neural network was introduced by Hopfield in 1982, which introduced the climax of the research on the neural networks. This network was extended to bidirectional associative memory (BAM) neural network by Kosko in 1987 [1] and to multidirectional associative memory (MAM) neural network by Hagiwara in 1990 [2]. They all can realize associative memory. But by using the MAM neural networks, one can achieve the many-to-many association which is a very advanced function of human brain. The many-to-many association has found wide applications in image denoising, speech recognition, pattern recognition, and intelligent information processing [2–6]. For example, it was shown that today most Indians are derived from the two-ancestor group gene by DNA analyzing [7]. If we need to distinguish which category an Indian belongs to, then this is a many-to-many associative problem.

In the designing of the associative memory neural networks to achieve associative memory, it is necessary to ensure their stability. The stability of Hopfield neural networks and BAM neural networks is discussed in a lot of recently published literature works [8–12], but the researchers about MAM neural networks are mainly focused on learning algorithms, fault tolerance, and retrieval efficiency of MAM neural networks [3–6]. To the best of our knowledge, the research on the theory of MAM neural networks was reported only in a few papers [13–18]. Chen et al. proved the stability of some specific types of MAM neural networks in [13, 14]. We studied the existence and global exponential stability of equilibrium for MAM neural networks with constant delays or time-varying delays in [15, 16]. We also obtained a sufficient condition for the global exponential stability of the discrete-time multidirectional associative memory neural network with variable delays in [17].

The multistability of a neural network describes coexistence of multiple stable patterns such as equilibria or periodic orbits. In recent years, the multistability issue of neural networks is discussed in some papers [18–32]. In [18], we discussed the existence and the exponential stability of multiple periodic solutions for an MAM neural network. The neural networks with a class of nondecreasing piecewise linear activation functions with $2r$ corner points were investigated in [19]. It was proposed that the n-neuron dynamical systems can have and only have $(2r+1)^n$ equilibria under some conditions, of which $(r+1)^n$ are locally exponentially stable and others are unstable. A class of neural networks with Mexican-hat-type activation functions was discussed in [20]. A set of new sufficient conditions were presented to ensure the multistability of the neural networks. The cellular neural

networks with time-varying unbounded delays (DCNNs) were discussed in [21]. Under some conditions, it showed that the DCNNs can exhibit 3^n equilibrium points. In [22], Huang and Cao found that the $2n$-dimensional networks can have 3^n equilibria and 2^n of them are locally exponentially stable. In [24], with perturbation techniques and the Floquet theory, Campbell et al. discussed the multistability and stable asynchronous periodic oscillations for a network of three identical neurons with multiple discrete signal transmission delays. The possible codimension one bifurcations which occur in the system were determined. In [25], Cheng et al. presented the existence of 2^n stationary solutions for a general n-dimensional delayed neural network with several classes of activation functions. It was shown that a two-dimensional neural network has n^2 isolated equilibria points which are locally stable, where the activation function has n segments. Furthermore, evoked by periodic external input, n^2 periodic orbits which are locally exponentially attractive were found [26]. Some similar results were found on n-neuron Cohen-Grossberg neural networks (CGNNs) with time-varying delays and a general class of activation functions [27]. In [29, 30], the multistability and multiperiodicity issues were discussed for competitive neural networks and high-order competitive neural networks. In [31], the stability of multiple equilibria of neural networks with time-varying delays and concave-convex characteristics was studied. Some sufficient conditions were obtained to ensure that an n-neuron neural network with concave-convex characteristics can have a fixed point located in the appointed region. By partitioning the state space, sufficient conditions were established which ensure that n-dimensional Cohen-Grossberg neural networks with k-level discontinuous activation functions can have k^n equilibrium points or k^n periodic orbits [32].

Generally, the existence of a globally stable equilibrium point or periodic solutions is necessary in solving optimization problems, but to achieve many-to-many associative memory by using MAM neural network, the system which has a globally stable equilibrium point or a globally stable periodic solution can only associate less information. So we should study the multistability of MAM neural network in order to make it achieve many-to-many associative memory. It is necessary to explore the existence, stability, and convergence speed of multiple equilibria or periodic solutions of MAM neural network. Motivated by the above, in this paper, we study the multistability issue for a delayed MAM neural network with m fields and n_k neurous in the field k as follows:

$$\frac{dx_{ki}}{dt} = -a_{ki}x_{ki}(t) + \sum_{p=1,p\neq k}^{m}\sum_{j=1}^{n_p} w_{pjki} f_{pj}\left(x_{pj}\left(t-\tau_{pjki}\right)\right) + I_{ki}, \tag{1}$$

where $k = 1, 2, \ldots, m$, $i = 1, 2, \ldots, n_k$, $x_{ki}(t)$ denotes the membrane voltage of the ith neuron in the field k at time t, $a_{ki} > 0$ denotes the decay rate of the ith neuron in the field k, $f_{pj}(\cdot)$ is the neuronal activation function of the jth neuron in the field p, w_{pjki} is the connection weight from the jth neuron in the field p to the ith neuron in the field k, I_{ki} is the external input of the ith neuron in the field k, and τ_{pjki} is the time delay of the synapse from the j neuron in the field p to the ith neuron in the field k.

Set $\tau = \max\{\tau_{pjki} \mid \text{where } 1 \le k \le m, 1 \le p \le m, p \neq k, 1 \le i \le n_k, \text{and } 1 \le j \le n_p\}$. The initial conditions with (1) are of the forms

$$x_{ki}(t) = \phi_{ki}(t), \tag{2}$$

where $k = 1, 2, \ldots, m$, $i = 1, 2, \ldots, n_k$, and $\phi_{ki} : [-\tau, 0] \to \mathbb{R}$ are continuous functions.

This paper is organized as follows. In the next section, we discuss the existence of multiple equilibria by using the Brouwer fixed point theorem. In Section 3, we analyze the exponential stability of every equilibrium. An example is given to illustrate the effectiveness of our results in Section 4. Finally, conclusions are given in Section 5.

2. Existence of Multiple Equilibria

In this section, we consider the existence of multiple equilibria by using the Brouwer fixed point theorem. Since the existence and stability of stationary patterns for neural networks certainly depend on the characteristics of activation functions, we assume the activation functions $f_{ki}(\cdot)$ $(1 \le i \le n_k, 1 \le k \le m)$ satisfy the following condition:

(H1) $f_{ki}(\cdot)$ are continuous, increasing and there exist $M_{ki} > 0$ such that $|f_{ki}(x)| \le M_{ki}$ for $1 \le i \le n_k, 1 \le k \le m$.

For convenience, we give some notations as follows:

$$M = \begin{cases} m, & \text{if } m \text{ is an even number,} \\ m-1, & \text{if } m \text{ is an odd number,} \end{cases}$$

$$I_M = \{1, 3, 5, \ldots, M-1\},$$

$$l_i = l_{i+1} = \min\{n_i, n_{i+1}\}, \quad \text{for any } i \in I_M, l_m = 0, \text{ if } M \neq m,$$

$$l = \sum_{i\in I_M} l_i, \qquad n = \sum_{i=1}^{m} n_i,$$

$$\operatorname{col}(b_{ki}) = \left(b_{11}, \ldots, b_{1n_1}, b_{21}, \ldots, b_{2n_2}, \ldots, b_{m1}, \ldots, b_{mn_m}\right). \tag{3}$$

Set

$$\delta_{ki} = \frac{1}{a_{ki}}\left[\sum_{p=1,p\neq k}^{m}\sum_{j=1}^{n_p}\left|w_{pjki}\right| M_{pj} + \left|I_{ki}\right|\right],$$

$$\Omega = \{x = \operatorname{col}(x_{ki}) \in \mathbb{R}^n \mid x_{ki} \in [-\delta_{ki}, \delta_{ki}], \text{ for } 1 \le k \le m, 1 \le i \le n_k\}. \tag{4}$$

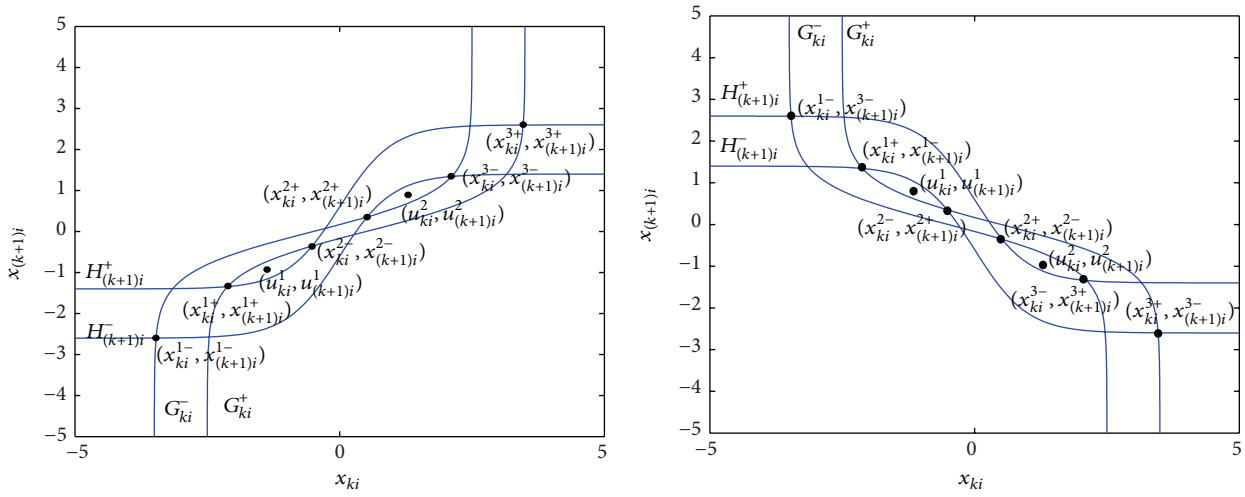

FIGURE 1: The configurations of $G^+_{ki}(y)$, $G^-_{ki}(y)$ and $H^-_{(k+1)i}(x)$, $H^+_{(k+1)i}(x)$. (a) $w_{ki(k+1)i}, w_{(k+1)iki} > 0$. (b) $w_{ki(k+1)i}, w_{(k+1)iki} < 0$.

For any $k \in I_M$, $1 \le i \le l_k$, define

$$\begin{aligned}
H^+_{(k+1)i}(x_{ki}) &= \frac{1}{a_{(k+1)i}}\left(w_{ki(k+1)i} f_{ki}(x_{ki}) - \left|w_{ki(k+1)i}\right| M_{ki}\right) + \delta_{(k+1)i},\\
H^-_{(k+1)i}(x_{ki}) &= \frac{1}{a_{(k+1)i}}\left(w_{ki(k+1)i} f_{ki}(x_{ki}) + \left|w_{ki(k+1)i}\right| M_{ki}\right) - \delta_{(k+1)i},\\
G^+_{ki}(x_{(k+1)i}) &= \frac{1}{a_{ki}}\left(w_{(k+1)iki} f_{(k+1)i}(x_{(k+1)i}) - \left|w_{(k+1)iki}\right| M_{(k+1)i}\right) + \delta_{ki},\\
G^-_{ki}(x_{(k+1)i}) &= \frac{1}{a_{ki}}\left(w_{(k+1)iki} f_{(k+1)i}(x_{(k+1)i}) + \left|w_{(k+1)iki}\right| M_{(k+1)i}\right) - \delta_{ki}.
\end{aligned} \tag{5}$$

We consider the second parameter condition which is used to establish existence of multiple equilibria for systems (1).

(H2) For any $k \in I_M$, the weights $w_{(k+1)iki} \cdot w_{ki(k+1)i} > 0$, and there exist two points $(u^1_{ki}, u^1_{(k+1)i})$, $(u^2_{ki}, u^2_{(k+1)i})$, where $u^1_{ki} < u^2_{ki}$, such that

(1) if $w_{ki(k+1)i}, w_{(k+1)iki} > 0$, then $H^+_{(k+1)i}(u^1_{ki}) < u^1_{(k+1)i}$, $G^+_{ki}(u^1_{(k+1)i}) < u^1_{ki}$, $H^-_{(k+1)i}(u^2_{ki}) > u^2_{(k+1)i}$, $G^-_{ki}(u^2_{(k+1)i}) > u^2_{ki}$;

(2) if $w_{ki(k+1)i}, w_{(k+1)iki} < 0$, then $H^-_{(k+1)i}(u^1_{ki}) > u^1_{(k+1)i}$, $G^+_{ki}(u^1_{(k+1)i}) < u^1_{ki}$, $H^+_{(k+1)i}(u^2_{ki}) < u^2_{(k+1)i}$, $G^-_{ki}(u^2_{(k+1)i}) > u^2_{ki}$.

The configuration that motivates (H2) is depicted in Figure 1. Under the above assumptions, if $w_{ki(k+1)i}$, $w_{(k+1)iki} > 0$, then there are three crossing points $(x^{1+}_{ki}, x^{1+}_{(k+1)i})$, $(x^{2-}_{ki}, x^{2-}_{(k+1)i})$, and $(x^{3+}_{ki}, x^{3+}_{(k+1)i})$ of the curves $x_{(k+1)i} = H^+_{(k+1)i}(x_{ki})$, $x_{ki} = G^+_{ki}(x_{(k+1)i})$, and there are three crossing points $(x^{1-}_{ki}, x^{1-}_{(k+1)i})$, $(x^{2+}_{ki}, x^{2+}_{(k+1)i})$, and $(x^{3-}_{ki}, x^{3-}_{(k+1)i})$ of the curves $x_{(k+1)i} = H^-_{(k+1)i}(x_{ki})$, $x_{ki} = G^-_{ki}(x_{(k+1)i})$ (see Figure 1(a)), where

$$\begin{aligned}
&x^{1-}_{ki} < x^{1+}_{ki} < u^1_{ki} < x^{2-}_{ki} < x^{2+}_{ki} < u^2_{ki} < x^{3-}_{ki} < x^{3+}_{ki},\\
&x^{1-}_{(k+1)i} < x^{1+}_{(k+1)i} < u^1_{(k+1)i} < x^{2-}_{(k+1)i}\\
&\quad < x^{2+}_{(k+1)i} < u^2_{(k+1)i} < x^{3-}_{(k+1)i} < x^{3+}_{(k+1)i}.
\end{aligned} \tag{6}$$

If $w_{ki(k+1)i}, w_{(k+1)iki} < 0$, then there are three crossing points $(x^{1-}_{ki}, x^{1+}_{(k+1)i})$, $(x^{2+}_{ki}, x^{2-}_{(k+1)i})$, and $(x^{3-}_{ki}, x^{3+}_{(k+1)i})$ of the curves $x_{(k+1)i} = H^+_{(k+1)i}(x_{ki})$, $x_{ki} = G^-_{ki}(x_{(k+1)i})$, and there are three crossing points $(x^{1+}_{ki}, x^{1-}_{(k+1)i})$, $(x^{2-}_{ki}, x^{2+}_{(k+1)i})$, and $(x^{3+}_{ki}, x^{3-}_{(k+1)i})$ of the curves $x_{(k+1)i} = H^-_{(k+1)i}(x_{ki})$, $x_{ki} = G^+_{ki}(x_{(k+1)i})$ (see Figure 1(b)), where

$$\begin{aligned}
&x^{1-}_{ki} < x^{1+}_{ki} < u^1_{ki} < x^{2-}_{ki} < x^{2+}_{ki} < u^2_{ki} < x^{3-}_{ki} < x^{3+}_{ki},\\
&x^{3-}_{(k+1)i} < x^{3+}_{(k+1)i} < u^2_{(k+1)i} < x^{2-}_{(k+1)i}\\
&\quad < x^{2+}_{(k+1)i} < u^1_{(k+1)i} < x^{1-}_{(k+1)i} < x^{1+}_{(k+1)i}.
\end{aligned} \tag{7}$$

Furthermore, it is easy to confirm that $x_{ki}^{j+}, x_{ki}^{j-} \in [-\delta_{ki}, \delta_{ki}]$, for any $1 \le k \le m$, $1 \le i \le n_k$, and $j = 1, 2, 3$.

Theorem 1. *Under conditions (H1) and (H2), there are at least 3^l equilibria of the multidirectional associative memory neural network* (1).

Proof. Set

$$\Omega^{\alpha} = \{x = \mathrm{col}\,(x_{ki}) \in \Omega \mid (x_{ki}, x_{(k+1)i}) \in \Omega_{ki}^{\alpha_{ki}}, \text{for } k \in I_M,\ 1 \le i \le l_k\}, \tag{8}$$

where $\alpha = (\alpha_{11}, \ldots, \alpha_{1l_1}, \alpha_{31}, \ldots, \alpha_{3l_3}, \ldots, \alpha_{(M-1)1}, \ldots, \alpha_{(M-1)l_{M-1}})$, $\alpha_{ki} = 1, 2$, or 3, and

$$\begin{aligned}
\Omega_{ki}^{1} &= \left\{(x_{ki}, x_{(k+1)i}) \in \mathbb{R}^2 \mid (x_{ki}, x_{(k+1)i}) \in \left[x_{ki}^{1-}, x_{ki}^{1+}\right] \times \left[x_{(k+1)i}^{1-}, x_{(k+1)i}^{1+}\right]\right\},\\
\Omega_{ki}^{2} &= \left\{(x_{ki}, x_{(k+1)i}) \in \mathbb{R}^2 \mid (x_{ki}, x_{(k+1)i}) \in \left[x_{ki}^{2-}, x_{ki}^{2+}\right] \times \left[x_{(k+1)i}^{2-}, x_{(k+1)i}^{2+}\right]\right\},\\
\Omega_{ki}^{3} &= \left\{(x_{ki}, x_{(k+1)i}) \in \mathbb{R}^2 \mid (x_{ki}, x_{(k+1)i}) \in \left[x_{ki}^{3-}, x_{ki}^{3+}\right] \times \left[x_{(k+1)i}^{3-}, x_{(k+1)i}^{3+}\right]\right\}.
\end{aligned} \tag{9}$$

Obviously, they are 3^l disjoint closed regions, and $\Omega_{ki}^{j} \subseteq [-\delta_{ki}, \delta_{ki}] \times [-\delta_{(k+1)i}, \delta_{(k+1)i}]$ for any $k \in I_M$, $1 \le i \le l_k$, $j = 1, 2, 3$.

For any fixed α and a given $\bar{x} = \mathrm{col}(\bar{x}_{ki}) \in \Omega^{\alpha}$, we study this problem on the following two cases.

Case 1 ($1 \le i \le l_k$). Consider the following equations:

$$-a_{ki}x_{ki} + w_{(k+1)iki} f_{(k+1)i}\left(x_{(k+1)i}\right) - w_{(k+1)iki} f_{(k+1)i}\left(\bar{x}_{(k+1)i}\right) + \sum_{p=1, p\neq k}^{m} \sum_{j=1}^{n_p} w_{pjki} f_{pj}\left(\bar{x}_{pj}\right) + I_{ki} = 0, \tag{10}$$

$$-a_{(k+1)i}x_{(k+1)i} + w_{ki(k+1)i} f_{ki}\left(x_{ki}\right) - w_{ki(k+1)i} f_{ki}\left(\bar{x}_{ki}\right) + \sum_{p=1, p\neq k}^{m} \sum_{j=1}^{n_p} w_{pj(k+1)i} f_{pj}\left(\bar{x}_{pj}\right) + I_{(k+1)i} = 0. \tag{11}$$

It follows from (10) that

$$\begin{aligned}
x_{ki} &\le \frac{1}{a_{ki}}\left[w_{(k+1)iki} f_{(k+1)i}\left(x_{(k+1)i}\right) + \sum_{j=1, j\neq i}^{n_{k+1}} \left|w_{(k+1)jki}\right| M_{(k+1)j} + \left|I_{ki}\right|\right] + \frac{1}{a_{ki}} \sum_{\substack{p=1, p\neq k,\\ p\neq k+1}}^{m} \sum_{j=1}^{n_p} \left|w_{pjki}\right| M_{pj}\\
&= \frac{1}{a_{ki}}\left[w_{(k+1)iki} f_{(k+1)i}\left(x_{(k+1)i}\right) - \left|w_{(k+1)iki}\right| M_{(k+1)i} + \left|I_{ki}\right|\right] + \frac{1}{a_{ki}} \sum_{p=1, p\neq k}^{m} \sum_{j=1}^{n_p} \left|w_{pjki}\right| M_{pj}\\
&= \frac{1}{a_{ki}}\left[w_{(k+1)iki} f_{(k+1)i}\left(x_{(k+1)i}\right) - \left|w_{(k+1)iki}\right| M_{(k+1)i}\right] + \delta_{ki}\\
&= G_{ki}^{+}\left(x_{(k+1)i}\right).
\end{aligned} \tag{12}$$

Similarly, from (10), we also can obtain that

$$\begin{aligned}
x_{ki} &\ge \frac{1}{a_{ki}}\left[w_{(k+1)iki} f_{(k+1)i}\left(x_{(k+1)i}\right) + \left|w_{(k+1)iki}\right| M_{(k+1)i}\right] - \delta_{ki} = G_{ki}^{-}\left(x_{(k+1)i}\right).
\end{aligned} \tag{13}$$

So the curve of (10) is between the curves of $x_{ki} = G_{ki}^{-}(x_{(k+1)i})$ and $x_{ki} = G_{ki}^{+}(x_{(k+1)i})$.

Under similar analysis we can obtain that the curve of (11) is between the curves of $x_{(k+1)i} = H_{(k+1)i}^{-}(x_{ki})$ and $x_{(k+1)i} = H_{(k+1)i}^{+}(x_{ki})$. So we can affirm that there are at least three solutions of (10) and (11); each solution lies separately in Ω_{ki}^{1}, Ω_{ki}^{2}, and Ω_{ki}^{3}. Get one that is in $\Omega_{ki}^{\alpha_{ki}}$, and denote it by $(\bar{x}_{ki}, \bar{x}_{(k+1)i})$.

Case 2 ($l_k + 1 \le i \le n_k$). Let

$$\bar{x}_{ki} = \frac{1}{a_{ki}}\left[\sum_{p=1, p\neq k}^{m} \sum_{j=1}^{n_p} w_{pjki} f_{pj}\left(\bar{x}_{pj}\right) + I_{ki}\right]. \tag{14}$$

Then

$$\begin{aligned}
\left|\bar{x}_{ki}\right| &\le \frac{1}{a_{ki}}\left[\sum_{p=1, p\neq k}^{m} \sum_{j=1}^{n_p} \left|w_{pjki} f_{pj}\left(x_{pj}\right)\right| + I_{ki}\right]\\
&\le \frac{1}{a_{ki}}\left[\sum_{p=1, p\neq k}^{m} \sum_{j=1}^{n_p} \left|w_{pjki}\right| M_{pj} + \left|I_{ki}\right|\right] = \delta_{ki}.
\end{aligned} \tag{15}$$

From the two cases above, we can obtain $\bar{x}_{ki} \in \Omega^{\alpha}$ for any given $\bar{x} \in \Omega^{\alpha}$. Set $\bar{x} = \mathrm{col}(\bar{x}_{ki})$. It is obvious that $\bar{x} \in \Omega^{\alpha}$. Set mapping $F^{\alpha} : \Omega^{\alpha} \to \Omega^{\alpha}$ as follows:

$$F^{\alpha}(\bar{x}) = \bar{x}. \tag{16}$$

Because $f_{ki}(\cdot)$ are continuous mappings, so F^{α} is continuous mapping. By the Brouwer fixed point theorem, there is at least one fixed point of F^{α}, that is, zero point of (1) in Ω^{α}. Therefore, there are at least 3^l equilibria of the multidirectional associative memory neural network (1). □

3. Stability Analysis

In this section, the stability of the equilibria is considered. Set

$$\Psi = \{\phi = \mathrm{col}\,(\phi_{ki}) \in C\,([-\tau, 0], \mathbb{R}^n) \mid -\delta_{ki} \le \phi_{ki}\,(s) \le \delta_{ki}, \text{ for } 1 \le k \le m, 1 \le i \le n_k\},$$
$$\eta = \left(\eta_{11}, \ldots, \eta_{1l_1}, \eta_{31}, \ldots, \eta_{3l_3}, \ldots, \eta_{(M-1)1}, \ldots, \eta_{(M-1)l_{M-1}}\right), \tag{17}$$

with $\eta_{ki} = 1$ or $\eta_{ki} = 3$.

We consider the following 2^l subsets of Ψ:

$$\Psi^\eta = \{\phi = \mathrm{col}\,(\phi_{ki}) \in \Psi \mid (\phi_{ki}, \phi_{(k+1)i}) \in \Psi_{ki}^{\eta_{ki}}, \text{ for } k \in I_M, 1 \le i \le l_k\}, \tag{18}$$

where

$$\Psi_{ki}^1 = \left\{(\phi_{ki}, \phi_{(k+1)i}) \in C\left([-\tau, 0], \mathbb{R}^2\right) \mid \phi_{pi}\,(s) \in \left[x_{pi}^{1-}, x_{pi}^{1+}\right], \text{ for } p = k, k+1\right\},$$
$$\Psi_{ki}^3 = \left\{(\phi_{ki}, \phi_{(k+1)i}) \in C\left([-\tau, 0], \mathbb{R}^2\right) \mid \phi_{pi}\,(s) \in \left[x_{pi}^{3-}, x_{pi}^{3+}\right], \text{ for } p = k, k+1\right\}. \tag{19}$$

Lemma 2. *Under assumptions (H1) and (H2), each Ψ^η is a positive invariant set with respect to the solution flow generated by system* (1).

Proof. For any given η and any initial condition $\phi = \mathrm{col}(\phi_{ki}) \in \Psi^\eta$, let $x(t;\phi) = \mathrm{col}(x_{ki}(t;\phi))$ be the solution of (1) with initial condition $\phi \in \Psi^\eta$.

For any $1 \le k \le m$, $1 \le i \le n_k$, and any $t > 0$, from (1), we obtain that

$$-a_{ki}\delta_{ki} \le \frac{dx_{ki}\,(t;\phi)}{dt} + a_{ki}x_{ki}\,(t;\phi) \le a_{ki}\delta_{ki}. \tag{20}$$

Multiplying both sides of (20) by $e^{a_{ki}t}$, we have

$$-a_{ki}\delta_{ki}e^{a_{ki}t} \le \frac{dx_{ki}\,(t;\phi)\,e^{a_{ki}t}}{dt} \le a_{ki}\delta_{ki}e^{a_{ki}t}. \tag{21}$$

Integrating (21) over $[0, t]$, we obtain

$$-\delta_{ki}\left(e^{a_{ki}t} - 1\right) \le x_{ki}\,(t;\phi)\,e^{a_{ki}t} - \phi_{ki}\,(0) \le \delta_{ki}\left(e^{a_{ki}t} - 1\right). \tag{22}$$

Hence it follows that, for $t > 0$,

$$\left|x_{ki}\,(t;\phi)\right| \le e^{-a_{ki}t}\left(\left|\phi_{ki}\,(0)\right| - \delta_{ki}\right) + \delta_{ki}. \tag{23}$$

Therefore, for any given initial condition $\phi \in \Psi^\eta$, we have, for $t > 0$,

$$\left|x_{ki}\,(t;\phi)\right| \le \delta_{ki}. \tag{24}$$

That is, $x(t,\phi) \in \Psi$ for $t > 0$.

We claim that $x(t,\phi)$ remains in Ψ^η for any $t > 0$. If it is not true, there exist $k \in I_M$ and $1 \le i \le l_k$ that $(x_{ki}(t), x_{(k+1)i}(t))$ firstly (or one of the first) escapes from $\Psi_{ki}^{\eta_{ki}}$.

Case 1 ($w_{ki(k+1)i}, w_{(k+1)iki} > 0$). If $\eta_{ki} = 1$, then exists $t_0 > 0$, such that $(x_{ki}(t_0), x_{(k+1)i}(t_0))$ is on the edges of $\Psi_{ki}^{\eta_{ki}}$, and for any $t < t_0$, $x_{ki}^{1-} < x_{ki}(t) < x_{ki}^{1+}$, $x_{(k+1)i}^{1-} < x_{(k+1)i}(t) < x_{(k+1)i}^{1+}$. If $x_{ki}(t_0) = x_{ki}^{1+}$, $x_{(k+1)i}(t_0) \in [x_{(k+1)i}^{1-}, x_{(k+1)i}^{1+}]$, then

$$\begin{aligned}
\left.\frac{dx_{ki}\,(t)}{dt}\right|_{t=t_0} &= -a_{ki}x_{ki}\,(t_0) + \sum_{p=1,p\ne k}^{m}\sum_{j=1}^{n_p} w_{pjki}f_{pj} \\
&\quad \times \left(x_{pj}\left(t_0 - \tau_{pjki}\right)\right) + I_{ki} \\
&\le -a_{ki}x_{ki}^{1+} + w_{(k+1)iki}f_{(k+1)i}\left(x_{(k+1)i}^{1+}\right) \\
&\quad - \left|w_{(k+1)iki}\right| M_{(k+1)i} \\
&\quad + \sum_{p=1,p\ne k}^{m}\sum_{j=1}^{n_p}\left|w_{pjki}\right| M_{pj} + \left|I_{ki}\right| \\
&= -a_{ki}\left\{x_{ki}^{1+} - \frac{1}{a_{ki}}\left[w_{(k+1)iki}f_{(k+1)i}\left(x_{(k+1)i}^{1+}\right)\right.\right. \\
&\quad \left.\left. - \left|w_{(k+1)iki}\right| M_{(k+1)i}\right] - \delta_{ki}\right\} \\
&= -a_{ki}\left[x_{ki}^{1+} - G_{ki}^{+}\left(x_{(k+1)i}^{1+}\right)\right] = 0.
\end{aligned} \tag{25}$$

Therefore, $(x_{ki}(t), x_{(k+1)i}(t))$ cannot escape from $\Psi_{ki}^{\eta_{ki}}$ through the edge of $x_{ki}(t) = x_{ki}^{1+}$, $x_{(k+1)i}(t) \in [x_{(k+1)i}^{1-}, x_{(k+1)i}^{1+}]$. With similar proof, we can obtain that $(x_{ki}(t), x_{(k+1)i}(t))$ cannot escape from $\Psi_{ki}^{\eta_{ki}}$ through the other three edges. Hence $(x_{ki}(t), x_{(k+1)i}(t))$ cannot escape from Ψ_{ki}^1. We can also prove that $(x_{ki}(t), x_{(k+1)i}(t))$ cannot escape from Ψ_{ki}^3.

Case 2 ($w_{ki(k+1)i}, w_{(k+1)iki} < 0$). The proof is similar to that of Case 1.

This completes the proof. □

We give the criterions concerning the stability for the multiple equilibria of system (1).

(H3) There exist constants $L_{ki} > 0$ ($1 \le k \le m$ and $1 \le i \le n_k$) such that

$$\left|f_{ki}\,(x) - f_{ki}\,(y)\right| \le L_{ki}\left|x - y\right|, \tag{26}$$

for each x, y in a subset $R_{ki} \subset \mathbb{R}$, where R_{ki} is defined as follows:

$$R_{ki} = \begin{cases} \left[x_{ki}^{1-}, x_{ki}^{1+}\right] \cup \left[x_{ki}^{3-}, x_{ki}^{3+}\right], & \text{for } 1 \le k \le M, 1 \le i \le l_k, \\ \left[\delta_{ki}^{-}, \delta_{ki}^{+}\right], & \text{for } 1 \le k \le m, \quad l_k + 1 \le i \le n_k. \end{cases} \tag{27}$$

(H4) For any $1 \le k \le m, 1 \le i \le n_k$, it is satisfied that

$$a_{ki} - \sum_{p=1,p\ne k}^{m}\sum_{j=1}^{n_p}\left|w_{pjki}\right| L_{ki} > 0. \tag{28}$$

Theorem 3. *There are 2^l exponential stability equilibria of the multidirectional associative memory neural network* (1)*, if the conditions (H1)–(H4) hold.*

Proof. According to Theorem 1, for any α, the multidirectional associative memory neural network (1) has an equilibrium in Ω^α; let it be $\overline{x}$. According to Lemma 2, for any η, the solution $x(t,\phi)$ of the multidirectional associative memory neural network (1) is in Ψ^η under initial condition $\phi \in \Psi^\eta$. Let

$$y(t) = \operatorname{col}\{y_{ki}(t)\} = x(t,\phi) - \overline{x}. \tag{29}$$

System (1) becomes

$$\begin{aligned}\frac{dy_{ki}(t)}{dt} &= -a_{ki}y_{ki}(t) \\ &\quad + \sum_{p=1,p\neq k}^{m}\sum_{j=1}^{n_p} w_{pjki}\left\{f_{pj}\left[y_{pj}\left(t-\tau_{pjki}\right)+\overline{x}_{pj}\right]\right. \\ &\qquad \left. -f_{pj}\left(\overline{x}_{pj}\right)\right\},\end{aligned} \tag{30}$$

for $1 \le k \le m$, $1 \le i \le n_k$.

According to condition (H4), there exists $\mu > 0$ such that

$$a_{ki} - \mu - \sum_{p=1,p\neq k}^{m}\sum_{j=1}^{n_p}\left|w_{pjki}\right| L_{ki}e^{\mu\tau_{pjki}} > 0. \tag{31}$$

Define that $u_{ki}(t) = e^{\mu t}|y_{ki}(t)|$, for $1 \le k \le m$ and $1 \le i \le n_k$. It is obvious that $u_{ki}(t) > 0$. Denote

$$Q = \max_{1\le k\le m, 1\le i\le n_k}\left\{\sup_{\theta\in(-\tau,0]}\left|x_{ki}(\theta) - \overline{x}_{ki}\right|\right\}. \tag{32}$$

Let $\delta > 1$ be an arbitrary real number. For any $\theta \in (-\tau, 0]$, $1 \le k \le m$, $1 \le i \le n_k$, it is easy to obtain that $u_{ki}(t) < Q\delta$. We shall prove that

$$u_{ki}(t) < Q\delta, \tag{33}$$

for $1 \le k \le m$, $1 \le i \le n_k$, and any $t > 0$. Suppose this is not the case; then there exist $k = q$, $i = r$, and a time t_0 such that $u_{ki}(t) \le Q\delta$ for $t \in (-\tau, t_0]$, $1 \le k \le m$ and $1 \le i \le n_k$, $u_{qr}(t_0) = Q\delta$, and $D^+u_{qr}(t_0) \ge 0$. From (30), we derive that

$$\begin{aligned}D^+\left|y_{qr}(t_0)\right| &\le -a_{qr}\left|y_{qr}(t_0)\right| \\ &\quad + \sum_{p=1,p\neq q}^{m}\sum_{j=1}^{n_p}\left|w_{pjqr}\left|L_{pj}\right| y_{pj}\left(t_0-\tau_{pjqr}\right)\right|.\end{aligned} \tag{34}$$

Hence, from (31) and (34),

$$\begin{aligned}&D^+\left|u_{qr}(t_0)\right| \\ &\le \mu e^{\mu t_0}\left|y_{qr}(t_0)\right| + e^{\mu t_0} \\ &\quad \times\left[-a_{qr}\left|y_{qr}(t_0)\right| + \right. \\ &\qquad \left. \sum_{p=1,p\neq q}^{m}\sum_{j=1}^{n_p}\left|w_{pjqr}\right| L_{pj}\left|y_{pj}\left(t_0-\tau_{pjqr}\right)\right|\right] \\ &= -\left(a_{qr}-\mu\right)e^{\mu t_0}\left|y_{qr}(t_0)\right| \\ &\quad + \sum_{p=1,p\neq q}^{m}\sum_{j=1}^{n_p}\left|w_{pjqr}\right| L_{pj}e^{\mu\tau_{pjqr}}\left|u_{pj}\left(t_0-\tau_{pjqr}\right)\right| \\ &\le -\left(a_{qr}-\mu\right)u_{qr}(t_0) \\ &\quad + \sum_{p=1,p\neq q}^{m}\sum_{j=1}^{n_p}\left|w_{pjqr}\right| L_{pj}e^{\mu\tau_{pjqr}}\left[\sup_{\theta\in\left[t_0-\tau,t_0\right]}u_{pj}(\theta)\right] \\ &\le -\left(a_{qr}-\mu-\sum_{p=1,p\neq q}^{m}\sum_{j=1}^{n_p}\left|w_{pjqr}\right| L_{pj}e^{\mu\tau_{pjqr}}\right)Q\delta \\ &\le 0,\end{aligned} \tag{35}$$

which is contradicting $D^+u_{qr}(t_0) \ge 0$. Hence the inequality (33) holds. Since $\delta > 1$ is arbitrary, by allowing $\delta \to 1+$, we have $u_{ki}(t) \le Q$. Therefore, $x(t,\phi)$ is exponentially convergent to $\overline{x}$. □

Remark 4. The dynamic system (1) studied in this paper is different from the system in [18]. First, the coefficients of system (1) are constants, which are T-periodic functions in [18]. Second, the delays of system (1) are constant delays while they are distributed delays in [18]. Above all, our conclusions in this paper are different from those in [18], and the proof methods are different. In [18], we obtained the existence of $2^{n_0[m/2]}$ exponentially stable T-periodic solutions, where $n_0 = \min\{n_1, n_2, \ldots, n_m\}$ if m is an even number or $n_0 = \min\{n_1, n_2, \ldots, n_{m-1}\}$ if m is an odd number. But we obtain the existence of 3^l equilibria by Theorem 1 and the exponential stability of 2^l equilibria of them by Theorem 3. Because $l_k = \min\{n_k, n_{k+1}\}$, so $n_0 = \min\{l_k \mid k \in I_M\}$. It follows that $l = \sum_{k\in I_M} l_k \ge n_0[m/2]$. Therefore, the number of equilibria obtained in this paper is more than that of [18] if the conclusions of [18] are used to the constant coefficient system (1).

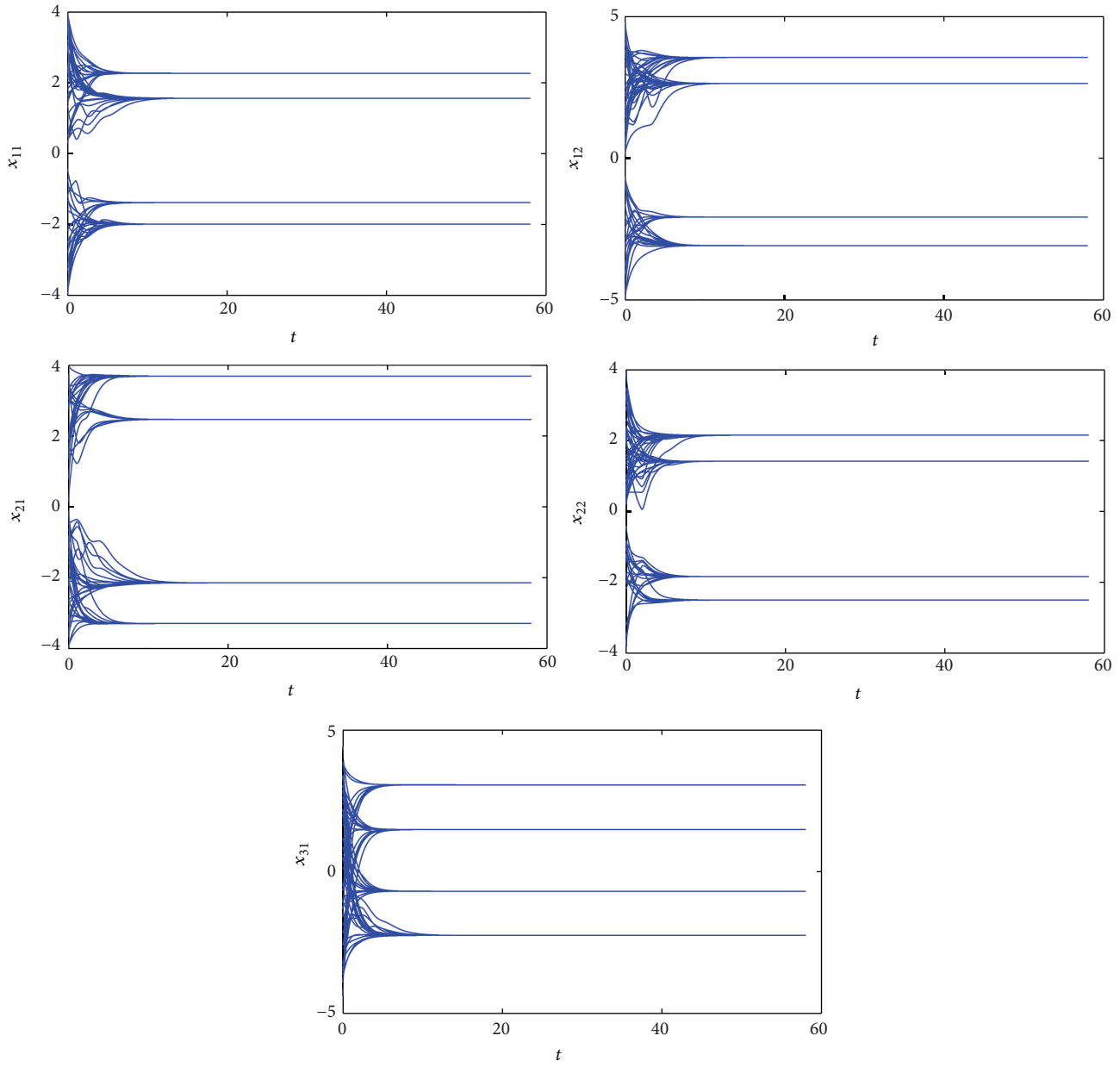

FIGURE 2: The exponential stability of 4 equilibria of the MAM neural network (36).

4. Numerical Example

In this section, a numerical example is given to illustrate the validity of results. Consider an MAM neural network with three fields as follows:

$$\frac{dx_{ki}}{dt} = -a_{ki}x_{ki}(t) + \sum_{p=1,p\neq k}^{3}\sum_{j=1}^{n_p} w_{pjki} f_{pj} \times \left(x_{pj}\left(t-\tau_{pjki}\right)\right) + I_{ki}, \tag{36}$$

where $k = 1, 2, 3$, $n_1 = n_2 = 2, n_3 = 1$, the neuronal signal decay rates $a_{11} = a_{12} = a_{21} = a_{22} = a_{31} = 1$, the external input $I_{11} = 0.1$, $I_{12} = -0.4$, $I_{21} = 0.2$, $I_{22} = -0.2$, $I_{31} = 0.4$, the connection weights

$$\begin{pmatrix} 0 & 0 & w_{2111} & w_{2211} & w_{3111} \\ 0 & 0 & w_{2112} & w_{2212} & w_{3112} \\ w_{1121} & w_{1221} & 0 & 0 & w_{3121} \\ w_{1122} & w_{1222} & 0 & 0 & w_{3122} \\ w_{1131} & w_{1231} & w_{2131} & w_{2231} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -2 & 0.3 & 0.2 \\ 0 & 0 & 0.4 & 3 & 0.1 \\ -3 & -0.5 & 0 & 0 & 0.1 \\ 0.3 & 2 & 0 & 0 & -0.1 \\ 1 & 2 & 1 & -1 & 0 \end{pmatrix}, \tag{37}$$

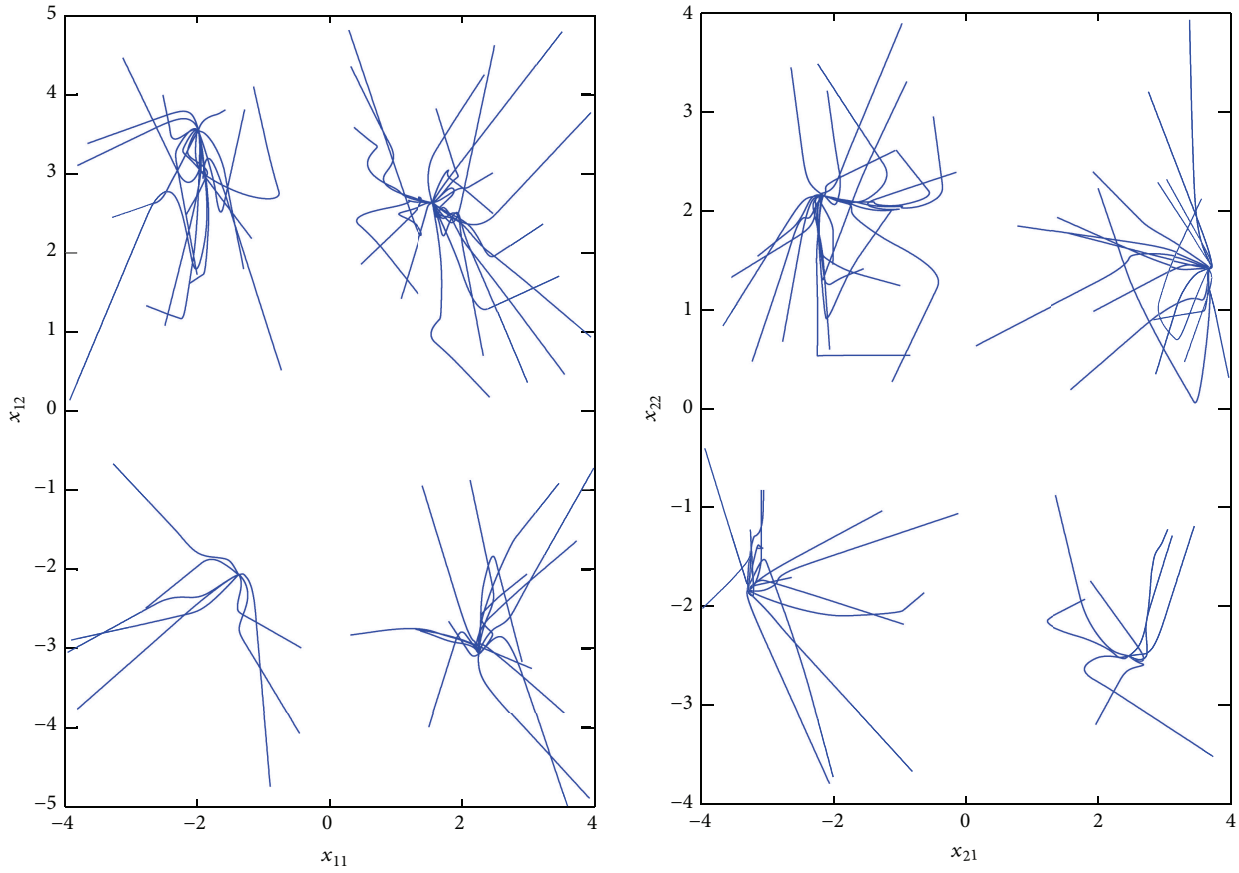

FIGURE 3: The phase of the MAM neural network (36).

and the delays

$$\begin{pmatrix} 0 & 0 & \tau_{2111} & \tau_{2211} & \tau_{3111} \\ 0 & 0 & \tau_{2112} & \tau_{2212} & \tau_{3112} \\ \tau_{1121} & \tau_{1221} & 0 & 0 & \tau_{3121} \\ \tau_{1122} & \tau_{1222} & 0 & 0 & \tau_{3122} \\ \tau_{1131} & \tau_{1231} & \tau_{2131} & \tau_{2231} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 2 & 1.2 \\ 0 & 0 & 0.4 & 1 & 1.5 \\ 1 & 0.4 & 0 & 0 & 0.8 \\ 2 & 2 & 0 & 0 & 0.7 \\ 1.2 & 1 & 1 & 2 & 0 \end{pmatrix}. \tag{38}$$

The neuronal activation functions $f_{pj}(x) = \tanh(x)$ $(1 \le p \le 3,\ 1 \le j \le n_p)$ are continuous, increasing, and bounded functions, and there are $M_{pj} = 1 > 0$ such that $|f_{pj}(x)| \le M_{pj}$. Hence condition (H1) is satisfied.

Obviously, $n = 5$, $M = 2$, $l_1 = l_2 = 2$, and $l = 2$. Through calculations, we have

$$\begin{aligned} H_{21}^{\pm}(x_{11}) &= -3f_{11}(x_{11}) \pm 0.8, \\ G_{11}^{\pm}(x_{21}) &= -2f_{21}(x_{21}) \pm 0.6, \\ H_{22}^{\pm}(x_{12}) &= 2f_{21}(x_{12}) \pm 0.6, \\ G_{12}^{\pm}(x_{22}) &= 3f_{22}(x_{22}) \pm 0.9. \end{aligned} \tag{39}$$

There exist four points $(u_{11}^1, u_{21}^1) = (-0.8, 1)$, $(u_{11}^2, u_{21}^2) = (0.9, -1.2)$, $(u_{12}^1, u_{22}^1) = (-1, -0.9)$, and $(u_{12}^2, u_{22}^2) = (1.1, 0.9)$, where $u_{11}^1 < u_{11}^2$, $u_{12}^1 < u_{12}^2$, such that

$$\begin{aligned} &H_{21}^{-}(u_{11}^1) = 1.1921 > 1, \quad G_{11}^{+}(u_{21}^1) = -0.9232 < -0.8, \\ &H_{21}^{+}(u_{11}^2) = -1.3489 < -1.2, \quad G_{11}^{-}(u_{21}^2) = 0.9673 > 0.9, \\ &H_{22}^{+}(u_{12}^1) = -0.9232 < -0.9, \quad G_{12}^{+}(u_{22}^1) = -1.2489 < -1, \\ &H_{22}^{-}(u_{12}^2) = 1.0010 > 0.9, \quad G_{12}^{-}(u_{22}^2) = 1.2489 > 1.1. \end{aligned} \tag{40}$$

Hence condition (H2) holds. The curves $x_{21} = H_{21}^{+}(x_{11})$, $x_{11} = G_{11}^{-}(x_{21})$ have three crossing points $(x_{11}^{1-}, x_{21}^{1+}) = (-2.5979, 3.7669)$, $(x_{11}^{2+}, x_{21}^{2-}) = (0.5198, 0.6327)$, and $(x_{11}^{3-}, x_{21}^{3+}) = (1.2886, -1.7763)$, and the curves $x_{21} = H_{21}^{+}(x_{11})$, $x_{11} = G_{11}^{-}(x_{21})$ have three crossing points $(x_{11}^{1+}, x_{21}^{1-}) = (-1.2886, 1.7763)$, $(x_{11}^{2-}, x_{21}^{2+}) = (-0.5198, 0.6327)$, and $(x_{11}^{3+}, x_{21}^{3-}) = (2.5979, -3.7669)$. The curves $x_{22} = H_{22}^{+}(x_{12}), x_{12} = G_{12}^{+}(x_{22})$ have three crossing points $(x_{12}^{1+}, x_{22}^{1+}) = (-1.6517, -1.2582)$, $(x_{12}^{2-}, x_{22}^{2-}) = (-0.6830, -0.5869)$, and $(x_{12}^{3+}, x_{22}^{3+}) = (3.8670, 2.5982)$, and the curves $x_{22} = H_{22}^{-}(x_{12})$, $x_{12} = G_{12}^{-}(x_{22})$ have three crossing points $(x_{12}^{1-}, x_{22}^{1-}) = (-3.8670, -2.5982)$, $(x_{12}^{2+}, x_{22}^{2+}) =$

$(0.6830, 0.5869)$, and $(x_{12}^{3-}, x_{22}^{3-}) = (1.6517, 1.2582)$. From (27), we obtain the subsets

$$\begin{aligned} R_{11} &= [-2.2579, -1.2886] \cup [1.2886, 2.2579], \\ R_{21} &= [-3.7669, -1.7763] \cup [1.7763, 3.7669], \\ R_{12} &= [-3.8670, -1.6517] \cup [1.6571, 3.8670], \\ R_{22} &= [-2.5982, -1.2582] \cup [1.2582, 2.5982], \\ R_{31} &= [-5.4, 5.4]. \end{aligned} \tag{41}$$

Because the activation functions $f_{pj}(x) = \tanh(x)$ are differentiable, we can let $L_{pj} = \max\{f_{pj}'(x) \mid x \in R_{pj}\} > 0$; then $L_{11} = 0.2625$, $L_{12} = 0.1083$, $L_{21} = 0.1368$, $L_{22} = 0.2765$, and $L_{31} = 1$. By Lagrange's mean value theorem, condition (H3) holds. Through calculations we have

$$\begin{aligned} 1 - 2L_{21} - 0.3L_{22} + 0.2L_{31} &= 0.4434 > 0, \\ 1 - 0.4L_{21} - 3L_{22} + 0.1L_{31} &= 0.0158 > 0, \\ 1 - 3L_{11} - 0.5L_{12} + 0.1L_{31} &= 0.0583 > 0, \\ 1 - 0.3L_{11} + 2L_{12} + 0.1L_{31} &= 0.6047 > 0, \\ 1 - L_{11} - 2L_{12} - L_{21} - L_{22} &= 0.1076 > 0. \end{aligned} \tag{42}$$

Hence condition (H4) holds. Then by Theorems 1 and 3, the MAM neural network (36) has $3^l = 9$ equilibria, and $2^l = 4$ of these equilibria are exponentially stable.

The dynamics of the MAM neural network system (36) are illustrated in Figures 2 and 3. Evolutions of sixty initial conditions of the MAM neural network system (36) have been tracked in Figure 2, which clearly displays that there exist four stable equilibria of the dynamical system, as confirmed by our theorems. Figure 3 shows the phases of the evolutions from time 5 to time 60 with sixty initial conditions, which shows that each evolution has converged to one of the four stable equilibria at time 60.

5. Conclusions

In this paper, the multistability has been studied for MAM neural networks. Sufficient conditions are obtained which ensure the existence of 3^l equilibria. It is proved that 2^l of the equilibria are exponentially stable. In [18], we have discussed the existence and the exponential stability of multiple periodic solutions for an MAM neural network. Furthermore, the coexistence of multiple stable equilibria and periodic solutions of an MAM neural network is an interesting topic. It will be investigated in the near future.

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