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Research Article

A Note on k-Potence Preservers on Matrix Spaces over Complex Field

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Let $\mathbb{C}$ be the field of all complex numbers, M_n the space of all $n \times n$ matrices over $\mathbb{C}$, and S_n the subspace of M_n consisting of all symmetric matrices. The map $\phi : S_n \to M_n$ satisfies that $A - \lambda B$ is k-potent in S_n implying that $\phi(A) - \lambda\phi(B)$ is k-potent in M_n, where $\lambda \in \mathbb{C}$, then there exist an invertible matrix $P \in M_n$ and $\epsilon \in \mathbb{C}$ with $\epsilon^k = \epsilon$ such that $\phi(X) = \epsilon P^{-1}(X)P$ for every $X \in S_n$. Moreover, the inductive method used in this paper can be used to characterise similar maps from M_n to M_n.

1. Introduction

Let $\mathbb{C}$ be the field of all complex numbers, M_n the space of all $n \times n$ matrices over $\mathbb{C}$, T_n the subspace of M_n consisting of all triangular matrices, and S_n the subspace of M_n consisting of all symmetric matrices. For fixed integer $k \geq 2$, $A \in M_n$ is called a k-potent matrix if $A^k = A$; especially, A is an idempotent matrix when $k = 2$. The map $\phi : S_n \to M_n$ satisfies that $A - \lambda B$ is a k-potent matrix in S_n implying that $\phi(A) - \lambda\phi(B)$ is a k-potent matrix in M_n, where $\lambda \in \mathbb{C}$, is a kind of the so-called weak preservers. While replacing "implying that" with "if and only if," ϕ is called strong preserver. Obviously, a strong preserver must be a weak preserver, while a weak preserver may not be a strong preserver.

The preserver problem in this paper is from LPPs but without linear assumption (more details about LPP in [1–3]). You and Wang characterized the strong k-potence preservers from M_n to M_n in [4]; then Song and Cao extended the result to weak preservers from M_n to M_n in [5]. In [6], Wang and You characterized the strong k-potence preservers from T_n to M_n. In this paper, the authors characterized the weak k-potence preservers from S_n to M_n and proved the following theorem.

Theorem 1. *Suppose $\phi : S_n \to M_n$ satisfy that $A - \lambda B$ is a k-potent matrix in S_n implying that $\phi(A) - \lambda\phi(B)$ is a k-potent matrix in M_n, where $\lambda \in \mathbb{C}$. Then there exist invertible $P \in M_n$ and $\epsilon \in \mathbb{C}$ with $\epsilon^k = \epsilon$ such that $\phi(X) = \epsilon P^{-1}XP$ for every $X \in S_n$.*

Furthermore, we can derive the following corollary from Theorem 1.

Corollary 2. *Suppose $\phi : S_n \to S_n$ satisfy that $A - \lambda B$ is a k-potent matrix in S_n implying that $\phi(A) - \lambda\phi(B)$ is a k-potent matrix in S_n, where $\lambda \in \mathbb{C}$. Then there exist invertible $P \in M_n$ and $\epsilon \in \mathbb{C}$ with $\epsilon^k = \epsilon$ such that $\phi(X) = \epsilon P^{-1}XP$ for every $X \in S_n$, where $PP^t = aI_n$ for some nonzero $a \in \mathbb{C}$.*

In fact, the proof of Theorem 1 through some adjustments is suitable for the weak k-potence preserver from M_n to M_n, and more details can be seen in remarks.

2. Notations and Lemmas

Γ_n denotes the set of all k-potent matrices in M_n, while $S\Gamma_n = \Gamma_n \cap S_n$. Λ denotes the set of all complex number ϵ satisfying $\epsilon^{k-1} = 1$, $\Delta = \Lambda \cup \{0\}$. E_{ij} denotes matrices in M_n with 1 in

(i, j) and 0 elsewhere, and I_n denotes the unit matrix in M_n. $\langle n \rangle$ denotes the set of integer s satisfy $1 \le s \le n$. GL_n denotes the general linear group consisting of all invertible matrices in M_n. D_n denotes an arbitrary diagonal matrix in M_n. For $A, B \in M_n$, A and B are orthogonal if $AB = BA = 0$. $\mathbb{C}^{n\times 1}$ denotes the space of all $n \times 1$ matrices over $\mathbb{C}$. Φ_n denotes the set of all maps $\phi : S_n \to M_n$ satisfying that $A - \lambda B$ is a k-potent matrix in S_n implying that $\phi(A) - \lambda\phi(B)$ is a k-potent matrix in M_n, where $\lambda \in \mathbb{C}$.

For an arbitrary matrix $X \in M_n$, we denote by $X[i, j]$ the term in (i, j) position of X, by $X_{[i_1,\ldots,i_s;j_1,\ldots,j_t]}$ the $s \times t$ matrix with the term in its (p, q) position equal to $X[i_p, j_q]$, where $i_1 < \cdots < i_s$ and $j_1 < \cdots < j_t$. Moreover, we denote by $X_{\{i_1,\ldots,i_s;j_1,\ldots,j_t\}}$ the $n \times n$ matrix with the term in its (i_p, j_q) position equal to $X[i_p, j_q]$ and terms elsewhere equal to 0. We especially simplify it with $X_{\{i_1,\ldots,i_s\}}$ when $s = t$, and $i_l = j_l$ for every $l \in \langle s \rangle$. Naturally, $X_{\{i\}} = X[i, i]E_{ii}$ for every $i \in \langle n \rangle$.

Without fixing X, $X_{\{i_1,\ldots,i_s;j_1,\ldots,j_t\}}$ also denotes a matrix in M_n with 0 in its (p, q) position, where $p \notin \{i_1, \ldots, i_s\}$, $q \notin \{j_1, \ldots, j_t\}$, and $1 \le i_1 < \cdots < i_s \le n$, $1 \le j_1 < \cdots < j_t \le n$.

At first, we need the following Lemmas 3, 4, 5, and 7, which are about k-potent matrices and orthogonal matrices.

Lemma 3 (see [2]). *Suppose $X, Y \in \Gamma_n$, and $X + \epsilon Y \in \Gamma_n$ for every $\epsilon \in \Lambda$; then X and Y are orthogonal.*

Lemma 4 ([7, Lemma 1]). *Suppose $A_1, A_2, \ldots, A_n$ are $n \times n$ mutually orthogonal nonzero k-potent matrices; then there exists $P \in GL_n$ such that $P^{-1}A_iP = c_iE_{ii}$ with $c_i^{k-1} = 1$ for every $i \in \langle n \rangle$.*

Lemma 5. *Suppose $Z \in M_{n-1}$, $p, q, g, h \in \mathbb{C}^{(n-1)\times 1}$ with $gh^t \ne 0$, $\delta \in \mathbb{C}$, for arbitrary nonzero $\alpha \in \mathbb{C}$ with $h^tg + \alpha^2 \ne 0$ and $\tau = (\alpha^{-1}h^tg + \alpha)^{-1}$, $\tau \left[\begin{smallmatrix} Z & p \\ q^t & \delta \end{smallmatrix}\right] + \tau \begin{bmatrix} \alpha^{-1}gh^t & g \\ h^t & \alpha \end{bmatrix} \in \Gamma_n$. Then $Z = 0$, $\delta = 0$, and there exist $\lambda_1, \lambda_2 \in \mathbb{C}$ with $(\lambda_1 + 1)(\lambda_2 + 1) = 1$ such that $p = \lambda_1 g$ and $q = \lambda_2 h$.*

Proof. By the assumption of α and τ, $\tau \begin{bmatrix} \alpha^{-1}gh^t & g \\ h^t & \alpha \end{bmatrix}$ is idempotent. Denote this matrix by X, and then we can get the following equation:

$$\begin{bmatrix} I_{n-1} & -\alpha^{-1}g \\ \tau h^t & 1 - \tau\alpha^{-1}h^tg \end{bmatrix} X \begin{bmatrix} I_{n-1} - \tau\alpha^{-1}gh^t & \alpha^{-1}g \\ -\tau h^t & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \quad (1)$$

Since the matrices on both sides of X satisfy the following equation:

$$\begin{bmatrix} I_{n-1} & -\alpha^{-1}g \\ \tau h^t & 1 - \tau\alpha^{-1}h^tg \end{bmatrix} \begin{bmatrix} I_{n-1} - \tau\alpha^{-1}gh^t & \alpha^{-1}g \\ -\tau h^t & 1 \end{bmatrix} = I_n \quad (2)$$

then the following matrix is k-potent by the assumption of lemma:

$$\begin{aligned} &\begin{bmatrix} I_{n-1} & -\alpha^{-1}g \\ \tau h^t & 1 - \tau\alpha^{-1}h^tg \end{bmatrix} \left(\tau \begin{bmatrix} Z & p \\ q^t & \delta \end{bmatrix} + X \right) \\ &\times \begin{bmatrix} I_{n-1} - \tau\alpha^{-1}gh^t & \alpha^{-1}g \\ -\tau h^t & 1 \end{bmatrix}. \end{aligned} \quad (3)$$

We denote by A the following matrix:

$$\begin{bmatrix} I_{n-1} & -\alpha^{-1}g \\ \tau h^t & 1 - \tau\alpha^{-1}h^tg \end{bmatrix} \begin{bmatrix} Z & p \\ q^t & \delta \end{bmatrix} \begin{bmatrix} I_{n-1} - \tau\alpha^{-1}gh^t & \alpha^{-1}g \\ -\tau h^t & 1 \end{bmatrix}; \quad (4)$$

then the following equation is obvious:

$$\left(\tau A + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)^k = \tau A + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \quad (5)$$

Unfolding it, we get $\tau^kA^k + \tau^{k-1}(\cdots)_{k-1} + \cdots + \tau(\cdots)_1 + \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right] = \tau A + \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$; that is, $\tau^kA^k + \tau^{k-1}(\cdots)_{k-1} + \cdots + \tau(\cdots)_1 - \tau A = 0$, where $(\cdots)_i$ is the coefficient matrix of τ^i for every $i \in \langle k-1 \rangle$.

Let $A = \left[\begin{smallmatrix} Z_1 & p_1 \\ q_1^t & \delta_1 \end{smallmatrix}\right]$, then we calculate it and get the following equations:

$$\begin{aligned} Z_1 &= \left(Z - \alpha^{-1}gq^t\right)\left(I_{n-1} - \tau\alpha^{-1}gh^t\right) - \left(p - \delta\alpha^{-1}g\right)\tau h^t, \\ p_1 &= \left(Z - \alpha^{-1}gq^t\right)\alpha^{-1}g + p - \delta\alpha^{-1}g, \\ q_1^t &= \left(\tau h^tZ + q^t - \tau\alpha^{-1}q^th^tg\right)\left(I_{n-1} - \tau\alpha^{-1}gh^t\right) \\ &\quad - \left(\tau h^tp + \delta - \delta\tau\alpha^{-1}h^tg\right)\tau h^t, \\ \delta_1 &= \left(\tau h^tZ + q^t - \tau\alpha^{-1}q^th^tg\right)\alpha^{-1}g \\ &\quad + \tau h^tp + \delta - \delta\tau\alpha^{-1}h^tg. \end{aligned} \quad (6)$$

It is easy to get $\tau(\cdots)_1 = \tau \left[\begin{smallmatrix} 0 & p_1 \\ q_1^t & k\delta_1 \end{smallmatrix}\right]$ and the following equation:

$$\begin{aligned} &\tau^kA^k + \tau^{k-1}(\cdots)_{k-1} + \cdots + \tau^2(\cdots)_2 \\ &+ \tau \begin{bmatrix} -Z_1 & 0 \\ 0 & (k-1)\delta_1 \end{bmatrix} = 0. \end{aligned} \quad (7)$$

Note that the highest degree of α in $\tau^{-2}A$ is 2; then the highest degree of α in $\tau^{-3k+i}(\cdots)_{k-i}$ is less or equal to $3k - i$ for every i with $2 \le i \le k - 1$, and the highest degree of α in $\tau^{-3k+1}\left[\begin{smallmatrix} -Z_1 & 0 \\ 0 & (k-1)\delta_1 \end{smallmatrix}\right]$ is $3k - 1$, where Z is the coefficient matrix of α^{3k-1} in Z_1 and δ is the coefficient of α^{3k-1} in δ_1.

By the assumption of α, we have $Z = 0$ and $\delta = 0$. Then the following equations are true:

$$\begin{aligned} Z_1 &= -\alpha^{-1}gq^t\left(I_{n-1} - \tau\alpha^{-1}gh^t\right) - p\tau h^t, \\ p_1 &= -\alpha^{-1}gq^t\alpha^{-1}g + p, \\ q_1^t &= \left(q^t - \tau\alpha^{-1}q^th^tg\right)\left(I_{n-1} - \tau\alpha^{-1}gh^t\right) - \tau h^tp\tau h^t, \\ \delta_1 &= \left(q^t - \tau\alpha^{-1}q^th^tg\right)\alpha^{-1}g + \tau h^tp \end{aligned} \quad (8)$$

and $\tau^{-3k+1}Z_1 = \tau^{-3k+1}[-\alpha^{-1}gq^t + \alpha^{-1}gq^t\tau\alpha^{-1}gh^t - p\tau h^t] = \tau^{-3k+2}[-\tau^{-1}\alpha^{-1}gq^t + \alpha^{-2}gq^tgh^t - ph^t]$, where the highest degree of α is $3k - 2$ and $-gq^t - ph^t$ is the coefficient matrix of α^{3k-2}.

Now, we calculate the upper left part of $\tau^{-3k+2}(\cdots)_2$.

When $k = 2$, $\tau^{-3k+2}(\cdots)_2 = \tau^{-4}A^2$, of which the upper left part is $\tau^{-4}[pq^t(I_{n-1} - \tau\alpha^{-1}gh^t) - q^t p\alpha^{-1}g\tau h^t] = \tau^{-4}[pq^t - \tau\alpha^{-1}pq^t gh^t - \tau\alpha^{-1}q^t pgh^t]$. Then in the upper left part of $\tau^{-4}A^2 + \tau^{-5}\left[\begin{smallmatrix} -Z_1 & 0 \\ 0 & (k-1)\delta_1 \end{smallmatrix}\right]$, the highest degree of α is 4, and the coefficient matrix is $pq^t + gq^t + ph^t$.

When $k > 2$, if $\left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ appears in the left (or right) end of an additive item of $\tau^{-3k+2}(\cdots)_2$, then the upper left part of this item is 0. So, the upper left part of $\tau^{-3k+2}(\cdots)_2$ is equal to the upper left part of $\tau^{-3k+2}A\left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]^{k-2}A$; that is, the upper left part is $\tau^{-3k+2}p_1q_1^t = \tau^{-3k+4}[\tau^{-1}\alpha pq^t - (q^t g + h^t p)ph^t - \tau^{-1}\alpha^{-1}gq^t gq^t + \alpha^{-2}q^t g(q^t g + h^t p)gh^t]$, and the highest degree of α is $3k-2$ with pq^t as the coefficient matrix of α^{3k-2}.

By the assumption of α, we have $pq^t + gq^t + ph^t = 0$.

By $gh^t \neq 0$, we have $g \neq 0$, $h \neq 0$, and $p = 0$ if and only if $q = 0$. When $p \neq 0$, we can get $p = \lambda_1 g$ by $p(q^t + h^t) + gq^t = 0$, and $q = \lambda_2 h$ by $(p+g)q^t + ph^t = 0$, where λ_1 and λ_2 satisfy $\lambda_1\lambda_2 gh^t + \lambda_2 gh^t + \lambda_1 gh^t = 0$; that is, $\lambda_1\lambda_2 + \lambda_2 + \lambda_1 = 0$ by $gh^t \neq 0$, which is equivalent to $(\lambda_1 + 1)(\lambda_2 + 1) = 1$. When $p = q = 0$, $\lambda_1 = \lambda_2 = 0$. □

Remark 6. Replacing $gh^t \neq 0$ with $gh^t = 0$ in Lemma 5, we have $g = 0$ implies $p = 0$ or $q + h = 0$, and $h = 0$ implies $q = 0$ or $p + g = 0$. These cases will not appear in the proof of Theorem 1, but are necessary for the weak preservers from M_n to M_n.

Lemma 7. *Suppose* $A = \left[\begin{smallmatrix} 0 & (\lambda(a)-\lambda(b))/(a-b) \\ (\lambda(a)^{-1}-\lambda(b)^{-1})/(a-b) & 1 \end{smallmatrix}\right] \in \Gamma_2$ *for arbitrary* $a, b \in \mathbb{C}$ *with* $a \neq b$, *where* $\lambda : \mathbb{C} \to \mathbb{C}$ *is a map satisfying* $\lambda(x) \neq 0$ *for every* $x \in \mathbb{C}$. *Then there exists nonzero* $\lambda_0 \in \mathbb{C}$ *such that* $\lambda(x) = \lambda_0$ *for every* $x \in \mathbb{C}$.

Proof. Since the trace of A is equal to 1, then $(\lambda(a) - \lambda(b))(\lambda^{-1}(a) - \lambda^{-1}(b))/(a-b)^2 = 0$, or -1, especially, when equal to -1, $k - 1 = 6p$ with $p \in Z^+$. Denote $\lambda(a)/\lambda(b)$ by y, and $a - b$ by c; then we have $(2 - y - y^{-1})/c^2 = 0$ or -1.

(1) If $(2 - y - y^{-1})/c^2 = 0$, then $y = 1$, that is, $\lambda(a) = \lambda(b)$;

(2) if $(2 - y - y^{-1})/c^2 = -1$, then $y = (2 + c^2 \pm \sqrt{4c^2 + c^4})/2$. When $c = 1$, $\lambda(b+1)/\lambda(b) = (3 \pm \sqrt{5})/2$; when $c = 2$, $\lambda(b+2)/\lambda(b) = (6 \pm \sqrt{32})/2 = 3 \pm 2\sqrt{2}$. But $\lambda(b+1)/\lambda(b) = (3 \pm \sqrt{5})/2$ implies $\lambda(b+2)/\lambda(b) = (\lambda(b+2)/\lambda(b+1))(\lambda(b+1)/\lambda(b)) = 1$, or $(7 \pm 3\sqrt{5})/2$. It is a contradiction! So it is impossible that $(2 - y - y^{-1})/x^2 = -1$.

Hence, there exists nonzero $\lambda_0 \in \mathbb{C}$ such that $\lambda(x) = \lambda_0$ for every $x \in \mathbb{C}$. □

We can prove the following Lemmas 8 and 9 similar as Lemmas 4 and 5 in [4].

Lemma 8 (see [4], Lemma 4). *Suppose* $\phi \in \Phi_n$, *A and B are* $n \times n$ *orthogonal k-potent matrices; then* $\phi(A)$ *and* $\phi(B)$ *are orthogonal.*

Lemma 9 (see [4], Lemma 5). *Suppose* $\phi \in \Phi_n$*; then* ϕ *are homogeneous; that is,* $\phi(\lambda X) = \lambda\phi(X)$ *for every* $X \in S_n$ *and every* $\lambda \in \mathbb{C}$.

Corollary 10. *Suppose* $\phi \in \Phi_n$, $A + B$, $C \in S\Gamma_n$, *and for every* $\epsilon \in \Lambda$, $A + B + \epsilon C \in S\Gamma_n$, $\phi(B + \epsilon C) = \phi(B) + \phi(\epsilon C)$. *Then* $\phi(A) + \phi(B)$ *and* $\phi(C)$ *are orthogonal.*

Proof. By the assumption and Lemma 9, we have $\phi(A) + \phi(B) \in \Gamma_n$, $\phi(C) \in \Gamma_n$, $\phi(A) + \phi(B + \epsilon C) = \phi(A) + \phi(B) + \epsilon\phi(C) \in \Gamma_n$. By Lemma 3, $\phi(A) + \phi(B)$ and $\phi(C)$ are orthogonal. □

Corollary 11. *Suppose* $\phi \in \Phi_n$ *and* $\phi(D_n) = D_n$ *for arbitrary diagonal matrix* $D_n \in M_n$. *Then for every* $i, j \in \langle n \rangle$ *with* $i \neq j$, $\phi(E_{ij} + E_{ji} + D_n) = \lambda_{ij}E_{ij} + \lambda_{ij}^{-1}E_{ji} + D_n$, *where* $\lambda_{ij} \in \mathbb{C}$ *is only decided by i and j.*

Proof. Let $A = (1/2)(E_{ij} + E_{ji} + D_n)$, $B = (1/2)(E_{ii} + E_{jj} - D_n)$, and $C = \sum_{l \neq i,j} E_{ll}$; then A, B and C satisfy the assumption of Corollary 10, and $\phi(A) + \phi(B)$ and $\phi(C)$ are orthogonal; that is, $\phi((E_{ij} + E_{ji} + D_n)) = \alpha_{ii}E_{ii} + \beta_{ij}E_{ij} + \gamma_{ji}E_{ji} + \delta_{jj}E_{jj} + D_n$ for some α_{ii}, β_{ij}, γ_{ji}, and $\delta_{jj} \in \mathbb{C}$.

Since $(\eta^{-1} + \eta)^{-1}[(E_{ij} + E_{ji} + D_n) - (D_n - \eta^{-1}E_{ii} - \eta E_{jj})] = (\eta^{-1} + \eta)^{-1}(\eta^{-1}E_{ii} + E_{ij} + E_{ji} + \eta E_{jj}) \in S\Gamma_n$ for arbitrary nonzero $\eta \in \mathbb{C}$ with $1 + \eta^2 \neq 0$, after applying ϕ, we have $(\eta^{-1} + \eta)^{-1}[\alpha_{ii}E_{ii} + \beta_{ij}E_{ij} + \gamma_{ji}E_{ji} + \delta_{jj}E_{jj} + \eta^{-1}E_{ii} + \eta E_{jj}] = (\eta^{-1} + \eta)^{-1}[\alpha_{ii}E_{ii} + (\beta_{ij} - 1)E_{ij} + (\gamma_{ji} - 1)E_{ji} + \delta_{jj}E_{jj}] + (\eta^{-1} + \eta)^{-1}(\eta^{-1}E_{ii} + E_{ij} + E_{ji} + \eta E_{jj}) \in \Gamma_n$. By Lemma 5, $\alpha_{ii} = \delta_{jj} = 0$, $\beta_{ij}\gamma_{ji} = 1$.

Let $D_n = \sum_{l=1}^n x_l E_{ll}$, where $x_l \in \mathbb{C}$ for every $l \in \langle n \rangle$; then β_{ij} is the function of i, j, and x_l and denote by $\beta_{ij}(D_n)$ the value of β_{ij} on $x_1, \ldots, x_n$, i, and j.

Fix i, j, and D_n and add a free variable x to x_l for some $l \in \langle n \rangle$; then $\beta_{ij}(D_n + xE_{ll})$ becomes into a map of x. Since $(1/(a-b))(E_{ij} + E_{ji} + D_n + aE_{jj}) - (1/(a-b))(E_{ij} + E_{ji} + D_n + bE_{jj}) \in S\Gamma_n$ for arbitrary a and $b \in \mathbb{C}$ with $a - b \neq 0$, then by $\phi(E_{ij} + E_{ji} + D_n + aE_{jj}) = \beta_{ij}(D_n + aE_{jj})E_{ij} + \beta_{ij}^{-1}(D_n + aE_{jj})E_{ji} + D_n + aE_{jj}$ and $\phi(E_{ij} + E_{ji} + D_n + bE_{jj}) = \beta_{ij}(D_n + bE_{jj})E_{ij} + \beta_{ij}^{-1}(D_n + bE_{jj})E_{ji} + D_n + bE_{jj}$, we can derive that $((\beta_{ij}(D_n + aE_{jj}) - \beta_{ij}(D_n + bE_{jj}))/(a-b))E_{ij} + ((\beta_{ij}^{-1}(D_n + aE_{jj}) - \beta_{ij}^{-1}(D_n + bE_{jj}))/(a-b))E_{ji} + E_{jj} \in \Gamma_n$. By Lemma 7, $\beta_{ij}(D_n + aE_{jj}) = \beta_{ij}(D_n + bE_{jj})$ for fixed i, j, and D_n; that is, $\beta_{ij}(D_n + xE_{jj}) = \beta_{ij}(D_n)$ for arbitrary $x \in \mathbb{C}$. Similarly, we can prove $\beta_{ij}(D_n + xE_{ii}) = \beta_{ij}(D_n)$ for arbitrary $x \in \mathbb{C}$.

In fact, we have proved that $\beta_{ij}(D_n + xE_{ii}) = \beta_{ij}(D_n)$ and $\beta_{ij}(D_n + yE_{jj}) = \beta_{ij}(D_n)$ for arbitrary $x, y \in \mathbb{C}$ and arbitrary D_n; then $\beta_{ij}(D_n + xE_{ii} + yE_{jj}) = \beta_{ij}(D_n + xE_{ii})(= \beta_{ij}(D_n + yE_{jj})) = \beta_{ij}(D_n)$ follows.

Since $\beta_{ij}(D_n + xE_{jj} + yE_{ll}) = \beta_{ij}(D_n + yE_{ll})$ for fixed i, j, and l with $l \neq i, j$, and arbitrary $x, y \in \mathbb{C}$, then $(1/(a-b))(E_{ij} + E_{ji} + D_n + (a-b)E_{jj} + aE_{ll}) - (1/(a-b))(E_{ij} + E_{ji} + D_n + bE_{ll}) \in S\Gamma_n$ implies $((\beta_{ij}(D_n + aE_{ll}) - \beta_{ij}(D_n + bE_{ll}))/(a-b))E_{ij} + ((\beta_{ij}^{-1}(D_n + aE_{ll}) - \beta_{ij}^{-1}(D_n + bE_{ll}))/(a-b))E_{ji} + E_{jj} + E_{ll} \in \Gamma_n$. By Lemma 7, we can get $\beta_{ij}(D_n + aE_{ll}) = \beta_{ij}(D_n + bE_{ll})$ for arbitrary a and

$b \in \mathbb{C}$ with $a - b \neq 0$; that is, $\beta_{ij}(D_n + xE_{ll}) = \beta_{ij}(D_n)$ for arbitrary $x \in \mathbb{C}$.

Until now, we have proved that $\beta_{ij}(D_n) = \beta_{ij}(\sum_{l=1}^{n} x_l E_{ll}) = \beta_{ij}(\sum_{l=1}^{n-1} x_l E_{ll}) = \cdots = \beta_{ij}(x_1 E_{11}) = \beta_{ij}(0)$ for arbitrary D_n; that is, β_{ij} is only decided by i and j. □

Remark 12. The proof of Corollary 11 presents the basic procedure of proof of Theorem 1. In order to decide the image of matrix A, we use Corollary 10 and the images of B and C, which usually are diagonal matrices or some matrices with images already decided.

If ϕ is a weak preserver from M_n to M_n, then Corollary 11 is also true. Let $A = E_{ij} + D_n$, $B = -(E_{ij} + E_{ji} + D_n) + E_{ii}$, and $C = \sum_{l \neq i,j} E_{ll}$; then we can prove $\phi(A) = a_{ii}E_{ii} + a_{ij}E_{ij} + a_{ji}E_{ji} + a_{jj}E_{jj} + D_n$ similarly as proving $\phi((E_{ij} + E_{ji} + D_n)) = \alpha_{ii}E_{ii} + \beta_{ij}E_{ij} + \gamma_{ji}E_{ji} + \delta_{jj}E_{jj} + D_n$, and $(a_{ii} + 1)E_{ii} + (a_{ij} - \lambda_{ij})E_{ij} + (a_{ji} - \lambda_{ij}^{-1})E_{ji} + a_{jj}E_{jj} \in \Gamma_n$. Since $\alpha^{-1}A + \alpha^{-1}(-(E_{ij} + E_{ji} + D_n) + \alpha E_{ii}) = -\alpha^{-1}E_{ji} + E_{ii} \in \Gamma_n$ for arbitrary nonzero α, then the following matrix is k-potent:

$$\alpha^{-1}\begin{bmatrix} a_{ii} & a_{ij} - \lambda_{ij} \\ a_{ji} - \lambda_{ji}^{-1} & a_{jj} \end{bmatrix} + \alpha^{-1}\begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix}. \quad (9)$$

Remark 6 tells us that $a_{ii} = a_{jj} = 0$, $a_{ij} - \lambda_{ij} = 0$, or $a_{ji} - \lambda_{ij}^{-1} = 0$; that is, $\phi(A) = \lambda_{ij}E_{ij} + D_n$, or $\phi(A) = \lambda_{ij}^{-1}E_{ji} + D_n$, $\phi(A) = \lambda_{ij}E_{ij} + \lambda_{ij}^{-1}E_{ji} + D_n$. Similarly, we can prove $\phi(E_{ji} + D_n) = \lambda_{ij}^{-1}E_{ji} + D_n$, $\phi(E_{ji} + D_n) = \lambda_{ij}E_{ij} + D_n$, or $\phi(E_{ji} + D_n) = \lambda_{ij}E_{ij} + \lambda_{ij}^{-1}E_{ji} + D_n$. Since D_n is arbitrary, we set $D_n = 0$ for convenience.

If $\phi(E_{ij}) = \lambda_{ij}E_{ij} + \lambda_{ij}^{-1}E_{ji}$; then $(1/3)E_{ij} + (1/3)(E_{ij} + E_{ji} + 2E_{ii} + E_{jj}) = (1/3)(2E_{ij} + E_{ji} + 2E_{ii} + E_{jj}) \in \Gamma_n$ implies $(1/3)(\lambda_{ij}E_{ij} + \lambda_{ij}^{-1}E_{ji}) + (1/3)(\lambda_{ij}E_{ij} + \lambda_{ij}^{-1}E_{ji} + 2E_{ii} + E_{jj}) = (1/3)(2\lambda_{ij}E_{ij} + 2\lambda_{ij}^{-1}E_{ji} + 2E_{ii} + E_{jj}) \in \Gamma_n$; that is, $-2/9 \in \Delta$, which is a contradiction. Hence, we proved that it is impossible $\phi(E_{ij}) = \lambda_{ij}E_{ij} + \lambda_{ij}^{-1}E_{ji}$ or $\phi(E_{ji}) = \lambda_{ij}E_{ij} + \lambda_{ij}^{-1}E_{ji}$.

If $\phi(E_{ij}) = \lambda_{ij}E_{ij}$ and $\phi(E_{ji}) = \lambda_{ij}E_{ij}$, then $(1/2)(E_{ii} + E_{ij} + E_{ji} + E_{jj}) \in \Gamma_n$ implies $(1/2)(\phi(E_{ij}) + \phi(E_{ii} + E_{ji} + E_{jj})) \in \Gamma_n$; that is, $(1/2)(E_{ii} + 2\lambda_{ij}E_{ij} + E_{jj}) \in \Gamma_n$, which is a contradiction. Hence, we proved that $\phi(E_{ij}) = \lambda_{ij}E_{ij}$ and $\phi(E_{ji}) = \lambda_{ij}^{-1}E_{ji}$, or $\phi(E_{ij}) = \lambda_{ij}^{-1}E_{ji}$ and $\phi(E_{ji}) = \lambda_{ij}E_{ij}$.

3. Proof of Theorem 1

Suppose $\phi \in \Phi_n$, then we can derive Theorem 1 from Propositions 13, 14, and 16.

Proposition 13. *Suppose $i, j \in \langle n \rangle$ with $i \neq j$; then $\phi(E_{ii}) = 0$ if and only if $\phi(E_{jj}) = 0$.*

Proof. Suppose $\phi(E_{ii}) = 0$ and $\phi(E_{jj}) \neq 0$ for some $i, j \in \langle n \rangle$ with $i \neq j$. At first, we prove that $\phi(aE_{ii} + E_{jj}) = \phi(E_{jj})$ for arbitrary $a \in \mathbb{C}$. Since the equation is already true when $a = 0$, then we assume $a \neq 0$ in the following proof.

Let $A = a^{-1}(aE_{ii} + E_{jj})$, $B = -a^{-1}E_{jj}$, and $C = E_{jj}$; then it is easy to verify A, B, and C satisfying the assumption of Corollary 10. So $\phi(a^{-1}(aE_{ii} + E_{jj})) + \phi(-a^{-1}E_{jj})$ and $\phi(E_{jj})$ are orthogonal. Moreover, we can derive $\phi(aE_{ii} + E_{jj}) \in \Gamma_n$ from $(aE_{ii} + E_{jj}) - aE_{ii} \in S\Gamma_n$ and $\phi(E_{ii}) = 0$. Let $a^{-1}(\phi(aE_{ii} + E_{jj}) - \phi(E_{jj})) = D$, then D and $\phi(E_{jj})$ are orthogonal k-potent matrices. While $\phi(aE_{ii} + E_{jj}) \in \Gamma_n$ implies $aD + \phi(E_{jj}) \in \Gamma_n$; then $aD \in \Gamma_n$. There are two cases on a.

(1) If $a \notin \Lambda$, then $D = 0$; that is, $\phi(aE_{ii} + E_{jj}) = \phi(E_{jj})$;

(2) if $a \in \Lambda$, we can derive that $(1/3)\phi(aE_{ii} + E_{jj}) - (1/3)\phi[(a-3)E_{ii} + E_{jj}] \in \Gamma_n$ from $(1/3)(aE_{ii} + E_{jj}) - (1/3)[(a-3)E_{ii} + E_{jj}] \in S\Gamma_n$. Note that $a - 3 \notin \Lambda$, so it is true that $\phi[(a-3)E_{ii} + E_{jj}] = \phi(E_{jj})$; that is, $(1/3)\phi(aE_{ii} + E_{jj}) - (1/3)\phi(E_{jj}) = (a/3)D \in \Gamma_n$. Finally, we can derive $D = 0$ from $a/3 \notin \Lambda$ and $D \in \Gamma_n$. At the same time, $\phi(aE_{ii} + E_{jj}) = \phi(E_{jj})$.

Anyway, $\phi(aE_{ii} + E_{jj}) = \phi(E_{jj})$ for arbitrary $a \in \mathbb{C}$.

Since $(b^{-1} + b)^{-1}(b^{-1}E_{ii} + E_{ij} + E_{ji} + bE_{jj}) \in S\Gamma_n$ for every nonzero $b \in \mathbb{C}$ with $1 + b^2 \neq 0$, then $(b^{-1} + b)^{-1}[\phi(E_{ij} + E_{ji}) + \phi(b^{-1}E_{ii} + bE_{jj})] \in \Gamma_n$, and $(b^{-1} + b)^{-1}[\phi(E_{ij} + E_{ji}) + b\phi(E_{jj})] \in \Gamma_n$ by $\phi(b^{-1}E_{ii} + bE_{jj}) = b\phi(E_{jj})$. While the equation $(b^{-1} + b)^{-k}[\phi(E_{ij} + E_{ji}) + b\phi(E_{jj})]^k = (b^{-1} + b)^{-1}[\phi(E_{ij} + E_{ji}) + b\phi(E_{jj})]$ is equivalent to $b^{k-1}[\phi(E_{ij} + E_{ji}) + b\phi(E_{jj})]^k = (1 + b^2)^{k-1}[\phi(E_{ij} + E_{ji}) + b\phi(E_{jj})]$. Note that $\phi(E_{ij} + E_{ji})$ is the constant term of the equation; then $\phi(E_{ij} + E_{ji}) = 0$ by the infinite property of b, and $(b^{-1} + b)^{-1}b\phi(E_{jj}) \in \Gamma_n$ follows. Then we can derive $\phi(E_{jj}) = 0$ which is a contradiction to the assumption. □

Proposition 14. *Suppose $\phi(E_{ii}) = 0$ for every $i \in \langle n \rangle$; then $\phi(X) = 0$ for arbitrary $X \in S_n$.*

Proof. The proof will be completed by induction on the following equation for arbitrary $X \in S_n$ with $X[i,i] = x_i$ for every $i \in \langle n \rangle$:

$$\phi\left(X_{\{1,\ldots,m\}} + \sum_{i=m+1}^{n} x_i E_{ii}\right) = 0, \quad (10)$$

where $1 \leq m \leq n-1$.

When $m = 1$, (10) is equivalent to $\phi(\sum_{i=1}^{n} a_i E_{ii}) = 0$ for arbitrary $D_n = \sum_{i=1}^{n} a_i E_{ii} \in S_n$.

At first, by the assumption, it is already true that $\phi(E_{ii}) = 0$ for every $i \in \langle n \rangle$.

Suppose $\phi(\sum_{j=1}^{s} a_{i_j} E_{i_j i_j}) = 0$ for every $s \in \langle n-1 \rangle$ with $1 \leq i_1 < \cdots < i_s \leq n$; then by the homogeneity of ϕ, we just need to prove the following equation for i_{s+1} with $i_s < i_{s+1} \leq n$:

$$\phi\left(\sum_{j=1}^{s} a_{i_j} E_{i_j i_j} + E_{i_{s+1} i_{s+1}}\right) = 0. \quad (11)$$

There are two cases on $B_s = \sum_{j=1}^{s} a_{i_j} E_{i_j i_j}$.

(1) If $B_s \notin S\Gamma_n$, then there exists $l \in \langle s \rangle$ such that $a_{i_l} \notin \Delta$, and the following statements are true:

$$\left(B_s + E_{i_{s+1} i_{s+1}}\right) - B_s = E_{i_{s+1} i_{s+1}} \in S\Gamma_n,$$
$$a_{i_l}^{-1}\left(B_s + E_{i_{s+1} i_{s+1}}\right) - a_{i_l}^{-1}\left(B_s + E_{i_{s+1} i_{s+1}} - a_{i_l} E_{i_l i_l}\right) = E_{i_l i_l} \in S\Gamma_n. \quad (12)$$

Note that $\phi(B_s) = 0$ and $\phi(B_s + E_{i_{s+1}i_{s+1}} - a_{i_l}E_{i_l i_l}) = 0$ by the assumption; then the following statements are true:

$$\phi\left(B_s + E_{i_{s+1}i_{s+1}}\right) \in \Gamma_n, \qquad a_{i_l}^{-1}\phi\left(B_s + E_{i_{s+1}i_{s+1}}\right) \in \Gamma_n. \tag{13}$$

Since $a_{i_l} \notin \Delta$, then $a_{i_l}^{-1} \notin \Delta$, and $\phi(B_s + E_{i_{s+1}i_{s+1}}) = 0$ follows.

(2) If $B_s \in S\Gamma_n$, then we have the following statements:

$$B_s + E_{i_{s+1}i_{s+1}} \in S\Gamma_n,$$
$$\frac{1}{3}\left(B_s + E_{i_{s+1}i_{s+1}}\right) - \frac{1}{3}\left(-3E_{i_1 i_1} + B_s + E_{i_{s+1}i_{s+1}}\right) = E_{i_1 i_1} \in S\Gamma_n. \tag{14}$$

Since $a_{i_1} - 3 \notin \Delta$; then $\phi(-3E_{i_1 i_1} + B_s + E_{i_{s+1}i_{s+1}}) = 0$ by case 1, and $(1/3)\phi(B_s + E_{i_{s+1}i_{s+1}}) \in \Gamma_n$ follows. While $\phi(B_s + E_{i_{s+1}i_{s+1}}) \in \Gamma_n$, hence we get $\phi(B_s + E_{i_{s+1}i_{s+1}}) = 0$.

Anyway, we prove $\phi(\sum_{j=1}^{s} a_{i_j}E_{i_j i_j} + E_{i_{s+1}i_{s+1}}) = 0$; then by the induction, (10) is true for $m = 1$.

Suppose (10) is true for $m \in \langle n-1\rangle$, then we prove the case on $m+1$.

Let $X_m = X_{[1,\ldots,m;\,1,\ldots,m]}$, $g = X_{[1,\ldots,m;m+1]}$, $A_{n-m} = \sum_{i=1}^{n-m} x_{i+m}E_{ii} \in M_{n-m}$; then we have $g^t = X_{[m+1;\,1,\ldots,m]}$ and the following equation:

$$\phi\left(\begin{bmatrix} X_m & 0 \\ 0 & A_{n-m} \end{bmatrix}\right) = 0. \tag{15}$$

We will prove the following equation which is equivalent to (10) on $m+1$:

$$\phi\left(\begin{bmatrix} X_m & g & 0 \\ g^t & x_{m+1} & 0 \\ 0 & 0 & A_{n-m-1} \end{bmatrix}\right) = 0 \tag{16}$$

For arbitrary nonzero $\alpha \in \mathbb{C}$ with $g^t g + \alpha^2 \neq 0$, the following $n \times n$ matrix B is idempotent:

$$B = \tau \begin{bmatrix} \alpha^{-1}gg^t & g & 0 \\ g^t & \alpha & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{17}$$

where $\tau = (\alpha^{-1}g^t g + \alpha)^{-1}$.

Note that $X_{m+1} = \begin{bmatrix} X_m & g \\ g^t & x_{m+1} \end{bmatrix}$ and A_{n-m-1} satisfy the following equation:

$$\tau \begin{bmatrix} X_m & g & 0 \\ g^t & x_{m+1} & 0 \\ 0 & 0 & A_{n-m-1} \end{bmatrix} - \tau \begin{bmatrix} X_m - \alpha^{-1}gg^t & 0 & 0 \\ 0 & x_{m+1} - \alpha & 0 \\ 0 & 0 & A_{n-m-1} \end{bmatrix} = B. \tag{18}$$

After applying ϕ on the above matrices, we have $\tau\phi(X_{m+1} \oplus A_{n-m-1}) \in \Gamma_n$ by the inductive assumption. Then $\phi(X_{m+1} \oplus A_{n-m-1}) = 0$ because of the assumption of α; that is, (10) holds for $m+1$.

Finally, we prove that $\phi(X) = 0$ for every $X \in S_n$ by the induction. □

Remark 15. If ϕ is a weak k-potence preserver from M_n to M_n; then Propositions 13 and 14 (replacing g^t with h^t for arbitrary $X \in M_n$ in the proof of Proposition 14) hold since Corollary 10 is true under this assumption.

Proposition 16. *Suppose $\phi(E_{ii}) \neq 0$ for every $i \in \langle n\rangle$, then there exist $P \in GL_n$ and $c \in \Lambda$ such that $\phi(X) = cP^{-1}XP$ for every $X \in S_n$.*

Proof. The proof will be completed in the following 4 steps.

Step 1. $\phi(E_{ii}) = c_i E_{ii}$, where $c_i \in \Lambda$ for every $i \in \langle n\rangle$.

Since $\phi(E_{ii})$ is nonzero k-potent, then we can derive from Lemma 4 that there exists $P_1 \in GL_n$ such that $P_1^{-1}\phi(E_{ii})P_1 = c_i E_{ii}$ for every $i \in \langle n\rangle$, where $c_i \in \Lambda$. It is obvious that the following map $\varphi \in \Phi_n$ and $\varphi(E_{ii}) = c_i E_{ii}$ for every $i \in \langle n\rangle$.

$$\varphi(X) = P_1^{-1}\phi(X)P_1. \tag{19}$$

Without loss of generality, we can assume $\phi(E_{ii}) = c_i E_{ii}$.

Step 2. $\phi(\sum_{i=1}^{n} a_i E_{ii}) = \sum_{i=1}^{n} a_i\phi(E_{ii})$, for arbitrary diagonal matrix $\sum_{i=1}^{n} a_i E_{ii}$.

The proof of this step can be seen in Step 3, Section 3 in [5].

Step 3. $c_i = c \in \Lambda$ for every $i \in \langle n\rangle$.

Let $A = (1/2)(E_{ij} + E_{ji})$, $B = (1/2)(E_{ii} + E_{jj})$, and $C = \sum_{l \in \langle n\rangle \setminus \{i,j\}} E_{ll}$, we can derive the following equation from Step 2 and Corollary 10:

$$\phi\left(E_{ij} + E_{ji}\right) = \alpha_0 E_{ii} + \beta_0 E_{ij} + \gamma_0 E_{ji} + \delta_0 E_{jj}, \tag{20}$$

where $\alpha_0, \beta_0, \gamma_0, \delta_0 \in \mathbb{C}$, $i, j \in \langle n\rangle$ with $i \neq j$.

Note that $pE_{ii} + q(E_{ij} + E_{ji}) + (1-p)E_{jj} \in S\Gamma_n$ for $p, q \in \mathbb{C}$ with $q^2 = p(1-p)$. In fact, 0 and 1 are all the eigenvalues of this matrix. Applying ϕ on the matrix $q(E_{ij} + E_{ji}) + [pE_{ii} + (1-p)E_{jj}]$, we have $H(p) = q(\alpha_0 E_{ii} + \beta_0 E_{ij} + \gamma_0 E_{ji} + \delta_0 E_{jj}) + pc_i E_{ii} + (1-p)c_j E_{jj} = (pc_i + q\alpha_0)E_{ii} + q\beta_0 E_{ij} + q\gamma_0 E_{ji} + ((1-p)c_j + q\delta_0)E_{jj} \in \Gamma_n$.

Since k is fixed, then Δ is the finite set which contains all of eigenvalues of $H(p)$, and there exists $w \in \{c + d \mid c, d \in \Delta\}$ such that the trace of $H(p)$ is w for infinite choices of p; that is, there exist (p_1, p_2) with $p_1 \neq p_2$ such that the traces of $H(p_1)$ and $H(p_2)$ are all equal to w; then we have the following equation:

$$\begin{aligned}(p_1 c_i + q_1\alpha_0) + \left((1-p_1)c_j + q_1\delta_0\right) \\ = (p_2 c_i + q_2\alpha_0) + \left((1-p_2)c_j + q_2\delta_0\right)\end{aligned} \tag{21}$$

which is equivalent to

$$(q_1 - q_2)(\alpha_0 + \delta_0) = (p_2 - p_1)(c_i - c_j), \quad (22)$$

where $q_s^2 = p_s(1 - p_s)$, for $s = 1, 2$.

Naturally, there are infinite choices of p_2 for fixed p_1 such that the above equation is true. If $(q_1 - q_2)/(p_2 - p_1)$ is equal to some $a \in \mathbb{C}$, where $p_2 \neq p_1$, p_1 and q_1 are fixed, then we can derive from the following equation:

$$(a^2 + 1) p_2^2 - (2aq_1 + 2a^2 p_1 + 1) p_2 + (q_1 + ap_1)^2 = 0 \quad (23)$$

that there are infinite choices of p_2 for constant $(q_1 - q_2)/(p_2 - p_1)$ if and only if $a^2 + 1 = 2aq_1 + 2a^2 p_1 + 1 = (q_1 + ap_1)^2 = 0$. While $a^2 + 1 = (q_1 + ap_1)^2 = 0$ and $q_1^2 = p_1(1 - p_1)$ imply $p_1 = q_1 = 0$, which is a contradiction to $2aq_1 + 2a^2 p_1 + 1 = 0$, hence $(q_1 - q_2)/(p_2 - p_1)$ varies with p_2.

Since $\alpha_0 + \delta_0$ and $c_i - c_j$ are all fixed numbers for fixed ϕ, then $\alpha_0 + \delta_0 \neq 0$ implies that there are at least two different values of $c_i - c_j = (q_1 - q_2)/(p_2 - p_1)(\alpha_0 + \delta_0)$ for fixed p_1 and infinite choices of p_2; it is a contradiction. So $\alpha_0 + \delta_0 = 0$ and $c_i = c_j$ follows. Hence $c_i = c \in \Lambda$ for every $i \in \langle n \rangle$.

Step 4. $\phi(X) = X$ for every $X \in S_n$.

After the discussion in Steps 1, 2, and 3, we already have the following equation:

$$\phi\left(\sum_{i=1}^{n} a_i E_{ii}\right) = c \sum_{i=1}^{n} a_i E_{ii}, \quad (24)$$

where $c \in \Lambda$, $a_i \in \mathbb{C}$ for every $i \in \langle n \rangle$. Since the map $c^{-1}\phi \in \Phi_n$, then we can assume $\phi(\sum_{i=1}^{n} a_i E_{ii}) = \sum_{i=1}^{n} a_i E_{ii}$ without loss of generality.

The proof in this step will be completed by induction on the following equation for arbitrary $X \in S_n$ with $X[i,i] = x_i$ for every $i \in \langle n \rangle$:

$$\phi\left(X_{\{i_1,\ldots,i_m\}} + \sum_{j \in \langle n \rangle \setminus \{i_1,\ldots,i_m\}} x_j E_{jj}\right) = X_{\{i_1,\ldots,i_m\}} + \sum_{j \in \langle n \rangle \setminus \{i_1,\ldots,i_m\}} x_j E_{jj}, \quad (25)$$

where $1 \le i_1 < \cdots < i_m \le n$ with $2 \le m \le n - 1$.

When $m = 2$, (25) is equivalent to $\phi(E_{ij} + E_{ji} + D_n) = E_{ij} + E_{ji} + D_n$ for arbitrary diagonal matrix $D_n \in S_n$ and $i, j \in \langle n \rangle$ with $i < j$, since ϕ is homogeneous. The proof will be completed in the following (1) and (2).

(1) $\phi(E_{ii+1} + E_{i+1i} + D_n) = E_{ii+1} + E_{i+1i} + D_n$ for every $i \in \langle n - 1 \rangle$.

We already derive from Corollary 11 that $\phi(E_{ii+1} + E_{i+1i} + D_n) = \lambda_i E_{ii+1} + \lambda_i^{-1} E_{i+1i} + D_n$ for every $i \in \langle n - 1 \rangle$, where $\lambda_i \in \mathbb{C}$ is only decided by i.

Suppose the map $\rho : S_n \to M_n$ satisfies the following equation for every $X \in S_n$,

$$\rho(X) = \operatorname{diag}\left(1, \lambda_1, \lambda_1\lambda_2, \ldots, \prod_{i=1}^{n-1} \lambda_i\right) \phi(X) \times \operatorname{diag}\left(1, \lambda_1^{-1}, \lambda_1^{-1}\lambda_2^{-1}, \ldots, \prod_{i=1}^{n-1} \lambda_i^{-1}\right); \quad (26)$$

then $\rho \in \Phi_n$, and for arbitrary diagonal matrix D_n and every $i \in \langle n - 1 \rangle$, $\rho(D_n) = D_n$ and $\rho(E_{ii+1} + E_{i+1i} + D_n) = E_{ii+1} + E_{i+1i} + D_n$.

Without loss of generality, we can assume $\phi(E_{ii+1} + E_{i+1i} + D_n) = E_{ii+1} + E_{i+1i} + D_n$ for every $i \in \langle n - 1 \rangle$ and arbitrary D_n.

(2) Suppose $\phi(E_{ij} + E_{ji} + D_n) = E_{ij} + E_{ji} + D_n$ for every i, j with $1 \le j - i < s < n - 1$; then $\phi(E_{ij} + E_{ji} + D_n) = E_{ij} + E_{ji} + D_n$ for every i, j with $j - i = s$.

At first, we have to prove that $\phi(x_{ii+1}(E_{ii+1} + E_{i+1i}) + x_{i+1i+m}(E_{i+1i+m} + E_{i+mi+1}) + D_n) = x_{ii+1}(E_{ii+1} + E_{i+1i}) + x_{i+1i+m}(E_{i+1i+m} + E_{i+mi+1}) + D_n$ for arbitrary nonzero x_{ii+1} and $x_{i+1i+m} \in \mathbb{C}$.

By the assumption, we already have the following equations:

$$\begin{aligned} \phi(x_{ii+1}(E_{ii+1} + E_{i+1i}) + D_n) &= x_{ii+1}(E_{ii+1} + E_{i+1i}) + D_n, \\ \phi(x_{i+1i+m}(E_{i+1i+m} + E_{i+mi+1}) + D_n) &= x_{i+1i+m}(E_{i+1i+m} + E_{i+mi+1}) + D_n. \end{aligned} \quad (27)$$

Let $X_1 = x_{ii+1}(E_{ii+1} + E_{i+1i}) + x_{i+1i+m}(E_{i+1i+m} + E_{i+mi+1}) + D_n$, $X_2 = x_{ii+1}(E_{ii+1} + E_{i+1i}) + D_n$, and $X_3 = x_{i+1i+m}(E_{i+1i+m} + E_{i+mi+1}) + D_n$. Then the following statements are true

$$\begin{gathered} X_1 - (X_2 - a_{i+1}E_{i+1i+1} - a_{i+m}E_{i+mi+m}) \in S\Gamma_n, \\ X_1 - (X_2 - a_{i+1}E_{i+1i+1} - a_{i+m}E_{i+mi+m}) + \epsilon \sum_{l \neq i+1, i+m}^{n} E_{ll} \in S\Gamma_n, \\ X_1 - (X_3 - b_i E_{ii} - b_{i+1}E_{i+1i+1}) \in S\Gamma_n, \\ X_1 - (X_3 - b_i E_{ii} - b_{i+1}E_{i+1i+1}) + \epsilon \sum_{l \neq i, i+1}^{n} E_{ll} \in S\Gamma_n, \end{gathered} \quad (28)$$

where $x_{i+1i+m}(E_{i+1i+m} + E_{i+mi+1}) + a_{i+1}E_{i+1i+1} + a_{i+m}E_{i+mi+m}$ and $x_{ii+1}(E_{ii+1} + E_{i+1i}) + b_i E_{ii} + b_{i+1}E_{i+1i+1}$ are k-potent.

Let $A = X_1$, $B = -(X_2 - a_{i+1}E_{i+1i+1} - a_{i+m}E_{i+mi+m})$, and $C = \sum_{l \neq i+1, i+m}^{n} E_{ll}$, then A, B, and C satisfy the assumption of Corollary 10. Hence we get $\phi(A) + \phi(B)$ and $\phi(C)$ are orthogonal; that is,

$$\begin{aligned} \phi(X_1) = X_2 &+ y_{i+1}E_{i+1i+1} + y_{i+m}E_{i+mi+m} \\ &+ y_{i+1i+m}E_{i+1i+m} + y_{i+mi+1}E_{i+mi+1}. \end{aligned} \quad (29)$$

Similarly, we can derive the following equation from Corollary 10:

$$\begin{aligned} \phi(X_1) = X_3 &+ z_i E_{ii} + z_{i+1}E_{i+1i+1} \\ &+ z_{ii+1}E_{ii+1} + z_{i+1i}E_{i+1i}. \end{aligned} \quad (30)$$

Comparing the above two equations, we have $z_i = y_{i+m} = 0$, $z_{i+1} = y_{i+1}$, $z_{ii+1} = z_{i+1i} = x_{ii+1}$, and $y_{i+1i+m} = y_{i+mi+1} = x_{i+1i+m}$, that is, $\phi(X_1) = X_1 + y_{i+1}E_{i+1i+1}$.

We will prove $z_{i+1} = y_{i+1} = 0$. For arbitrary nonzero α with $x_{i+1i+m}^2 + \alpha^2 \neq 0$, let $\tau = (\alpha^{-1}x_{i+1i+m}^2 + \alpha)^{-1}$, and $X_4 =$

$-X_2 + \alpha^{-1}x_{i+1i+m}^2 E_{i+1i+1} + \alpha E_{i+mi+m}$; then $\tau X_1 + \tau X_4 \in S\Gamma_n$ implies $\tau\phi(X_1) + \tau\phi(X_4) \in \Gamma_n$; that is, the following matrix is k-potent since $\phi(X_4) = X_4$ by the assumption

$$\tau \begin{bmatrix} y_{i+1} & 0 \\ 0 & 0 \end{bmatrix} + \tau \begin{bmatrix} \alpha^{-1}x_{i+1i+m}^2 & x_{i+1i+m} \\ x_{i+1i+m} & \alpha \end{bmatrix} \quad (31)$$

by Lemma 5, $y_{i+1} = 0$. Hence we prove $\phi(X_1) = X_1$.

Now we prove $\phi(E_{ii+m} + E_{i+mi} + D_n) = E_{ii+m} + E_{i+mi} + D_n$.

By Corollary 11, we already have $\phi(E_{ii+m} + E_{i+mi} + D_n) = \lambda_{ii+m}E_{ii+m} + \lambda_{ii+m}^{-1}E_{i+mi} + D_n$.

For arbitrary nonzero α with $2 + \alpha^2 \neq 0$, $(2\alpha^{-1} + \alpha)^{-1}(E_{ii+m} + E_{i+mi} + D_n) - (2\alpha^{-1} + \alpha)^{-1}(-\alpha^{-1}(E_{ii} + E_{ii+1} + E_{i+1i} + E_{i+1i+1}) - E_{i+1i+m} - E_{i+mi+1} - \alpha E_{i+mi+m} + D_n) = (2\alpha^{-1} + \alpha)^{-1}(\alpha^{-1}(E_{ii} + E_{ii+1} + E_{i+1i} + E_{i+1i+1}) + (E_{ii+m} + E_{i+1i+m}) + (E_{i+mi} + E_{i+mi+1}) + \alpha E_{i+mi+m})$ is idempotent.

After applying ϕ on the above matrices, we have $(2\alpha^{-1} + \alpha)^{-1}\phi(E_{ii+m} + E_{i+mi} + D_n) - (2\alpha^{-1} + \alpha)^{-1}\phi(-\alpha^{-1}(E_{ii} + E_{ii+1} + E_{i+1i} + E_{i+1i+1}) - E_{i+1i+m} - E_{i+mi+1} - \alpha E_{i+mi+m} + D_n) = (2\alpha^{-1} + \alpha)^{-1}(\alpha^{-1}(E_{ii} + E_{ii+1} + E_{i+1i} + E_{i+1i+1}) + (E_{ii+m} + E_{i+1i+m}) + (E_{i+mi} + E_{i+mi+1}) + \alpha E_{i+mi+m}) + (2\alpha^{-1} + \alpha)^{-1}((\lambda_{ii+m} - 1)E_{ii+m} + (\lambda_{ii+m}^{-1} - 1)E_{i+mi}) \in \Gamma_n$.

Then $\lambda_{ii+m} = 1$ by Lemma 5.

By the induction, we prove $\phi(E_{ij} + E_{ji} + D_n) = E_{ij} + E_{ji} + D_n$ for every i, j with $1 \leq i < j \leq n$.

(3) Suppose (25) is true for every s with $2 \leq s < m \leq n$; then we prove it holds on m.

For arbitrary $X \in S_n$ with $X[i,i] = x_i$ for every $i \in \langle n \rangle$, let A, B, U, V, y_{i_m}, and τ satisfy the following equations:

$$\begin{aligned} A &= X_{\{i_1,\ldots,i_m\}} + \sum_{j\in\langle n\rangle\setminus\{i_1,\ldots,i_m\}} x_j E_{jj}, \\ B &= X_{\{i_1,\ldots,i_{m-1}\}} + \sum_{j\in\langle n\rangle\setminus\{i_1,\ldots,i_{m-1}\}} x_j E_{jj}, \\ U &= X_{\{i_1,\ldots,i_{m-1};i_m\}}, \\ V &= X_{\{i_m;i_1,\ldots,i_{m-1}\}}, \\ y_{i_m} &= \left(X_{\{i_m;i_1,\ldots,i_{m-1}\}} X_{\{i_1,\ldots,i_{m-1};i_m\}}\right)[i_m, i_m], \\ \tau &= \left(\alpha^{-1} y_{i_m} + \alpha\right)^{-1}. \end{aligned} \quad (32)$$

Then $\tau A + \tau(-B + \alpha^{-1}UV + \alpha E_{i_m i_m})$ is idempotent for arbitrary nonzero α with $y_{i_m} + \alpha^2 \neq 0$. Applying ϕ on it, we have $\tau\phi(A) + \tau\phi(-B + \alpha^{-1}UV + \alpha E_{i_m i_m}) \in \Gamma_n$. Let $C = -B + \alpha^{-1}UV + \alpha E_{i_m i_m}$; then by $\tau A + \tau C + \epsilon \sum_{j\in\langle n\rangle\setminus\{i_1,\ldots,i_m\}} E_{jj} \in S\Gamma_n$ for every $\epsilon \in \Lambda$, we have $\tau\phi(A) + \phi(\tau C + \epsilon \sum_{j\in\langle n\rangle\setminus\{i_1,\ldots,i_m\}} E_{jj}) \in \Gamma_n$.

Note that $\phi(\tau C + \epsilon \sum_{j\in\langle n\rangle\setminus\{i_1,\ldots,i_m\}} E_{jj}) = \tau C + \epsilon \sum_{j\in\langle n\rangle\setminus\{i_1,\ldots,i_m\}} E_{jj}$ and $\phi(C) = C$ by the assumption; then $\tau\phi(A) + \tau\phi(C)$ and $\sum_{j\in\langle n\rangle\setminus\{i_1,\ldots,i_m\}} E_{jj}$ are orthogonal by Corollary 10; that is, $\phi(A) = Y_{\{i_1,\ldots,i_m\}} + \sum_{j\in\langle n\rangle\setminus\{i_1,\ldots,i_m\}} x_j E_{jj}$ for some $Y \in M_n$.

On the other hand, $C = -(X_{\{i_1,\ldots,i_{m-1}\}} + \sum_{j\in\langle n\rangle\setminus\{i_1,\ldots,i_{m-1}\}} x_j E_{jj}) + \alpha^{-1}UV + \alpha E_{i_m i_m} = -(X_{\{i_1,\ldots,i_m\}} + \sum_{j\in\langle n\rangle\setminus\{i_1,\ldots,i_m\}} x_j E_{jj}) + \alpha^{-1}UV + \alpha E_{i_m i_m} + U + V$ implies $\tau(Y_{\{i_1,\ldots,i_m\}} - X_{\{i_1,\ldots,i_m\}} + \alpha^{-1}UV + \alpha E_{i_m i_m} + U + V) = \tau(Y_{\{i_1,\ldots,i_m\}} - X_{\{i_1,\ldots,i_m\}}) + \tau(\alpha^{-1}UV + \alpha E_{i_m i_m} + U + V) \in \Gamma_n$ by $\tau\phi(A) + \tau\phi(C) \in \Gamma_n$. By Lemma 5, we can derive the following equations:

$$\begin{aligned} Y_{\{i_1,\ldots,i_{m-1}\}} &= X_{\{i_1,\ldots,i_{m-1}\}}, \\ Y[i_m, i_m] &= X[i_m, i_m], \\ Y_{\{i_1,\ldots,i_{m-1}\},\{i_m\}} &= \eta U, \\ Y_{\{i_m\},\{i_1,\ldots,i_{m-1}\}} &= \eta^{-1} V \end{aligned} \quad (33)$$

that is, $\phi(A) = X_{\{i_1,\ldots,i_{m-1}\}} + \eta U + \eta^{-1}V + \sum_{j\in\langle n\rangle\setminus\{i_1,\ldots,i_{m-1}\}} x_j E_{jj}$.

Let B_1 and B_2 satisfy the following equations:

$$\begin{aligned} B_1 &= X_{\{i_2,\ldots,i_m\}} + \sum_{j\in\langle n\rangle\setminus\{i_2,\ldots,i_m\}} x_j E_{jj}, \\ B_2 &= X_{\{i_1,\ldots,i_{m-2},i_m\}} + \sum_{j\in\langle n\rangle\setminus\{i_1,\ldots,i_{m-2},i_m\}} x_j E_{jj}; \end{aligned} \quad (34)$$

then we can prove

$$\begin{aligned} Y_{\{i_2,\ldots,i_m\}} &= X_{\{i_2,\ldots,i_m\}}, \\ Y[i_1, i_1] &= X[i_1, i_1], \\ Y_{\{i_2,\ldots,i_m\},\{i_1\}} &= \beta X_{\{i_2,\ldots,i_m\},\{i_1\}}, \\ Y_{\{i_1\},\{i_2,\ldots,i_m\}} &= \beta^{-1} X_{\{i_1\},\{i_2,\ldots,i_m\}}, \\ Y_{\{i_1,\ldots,i_{m-2},i_m\}} &= X_{\{i_1,\ldots,i_{m-2},i_m\}}, \\ Y[i_{m-1}, i_{m-1}] &= X[i_{m-1}, i_{m-1}], \\ Y_{\{i_1,\ldots,i_{m-2},i_m\},\{i_{m-1}\}} &= \gamma X_{\{i_1,\ldots,i_{m-2},i_m\},\{i_{m-1}\}}, \\ Y_{\{i_{m-1}\},\{i_1,\ldots,i_{m-2},i_m\}} &= \gamma^{-1} X_{\{i_{m-1}\},\{i_1,\ldots,i_{m-2},i_m\}}. \end{aligned} \quad (35)$$

Comparing the above three sets of equations, we can get $\phi(A) = A$, which is equivalent to (25) on m.

By the induction, we prove that $\phi(X) = X$ for arbitrary $X \in S_n$. □

Remark 17. If ϕ is a weak k-potence preserver from M_n to M_n, then the proof in Steps 1, 2, and 3 of Proposition 16 holds, and we prove $\phi(X) = X$ or $\phi(X) = X^t$ in Step 4. We omit the detailed proof since the case on X^t is totally the same after changing relevant notations.

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