

Research Article

Global Exponential Stability Criteria for Bidirectional Associative Memory Neural Networks with Time-Varying Delays

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The global exponential stability for bidirectional associative memory neural networks with time-varying delays is studied. In our study, the lower and upper bounds of the activation functions are allowed to be either positive, negative, or zero. By constructing new and improved Lyapunov-Krasovskii functional and introducing free-weighting matrices, a new and improved delay-dependent exponential stability for BAM neural networks with time-varying delays is derived in the form of linear matrix inequality (LMI). Numerical examples are given to demonstrate that the derived condition is less conservative than some existing results given in the literature.

1. Introduction

A class of neural networks related to bidirectional associative memory (BAM) has been introduced by Kosko [1]. This model generalized the single-layer autoassociative Hebbian correlator to a two-layer pattern-matched heteroassociative circuit. It is an important model with the ability of information memory and information association, which is crucial for various applications such as pattern recognition, solving optimization problems, and automatic control engineering [2–10]. In [1, 11], Kosko investigates the global stability of BAM models and obtains a severe constraint of having a symmetric connection weight matrix. Since it is impossible to maintain an absolutely symmetric connection weight matrix, asymmetric connection has been a focus of this field. Some of these applications require that there should be a well-defined computable solution for all possible initial states. From a mathematical point of view, this means that the equilibrium point of the designed cellular neural networks (CNNs) is globally asymptotically stable (GAS) or globally exponentially stable (GES). Moreover, in biological and artificial neural networks, time delays arise in the process of information transmission; for example, in the electronic implementation of analogue neural networks, time delays

occur in the communication and response of neurons owing to the finite switching speed of amplifiers. It is known that they can create an oscillatory or an unstable phenomenon. Therefore, the study of the stability and convergent dynamics of BAM neural networks with delays has raised considerable interest in recent years; see for examples [5, 7, 9, 10, 12–23] and the references cited therein. In [14, 15, 18, 20–22, 24–27], several sufficient conditions on the global exponential stability of BAM neural networks with time-varying delays have been derived. It is worth pointing out that the given criteria in [14, 15, 18, 20–22, 24–27] required the following hypothesis: the time-varying delays are continuously differentiable, the derivative of time-varying delays is smaller than one, and activation functions are bounded and monotonically nondecreasing. The common approach for studying stability of BAM neural networks is Lyapunov stability theory. With a properly designed Lyapunov-Krasovskii functional as well as introducing free-weighting matrices, one may derive stability criteria in term of linear matrix inequality (LMI) which is easily solved by several available algorithms.

Based on the above discussion, we propose to study the problem of global exponential stability of BAM neural networks with time-varying delays and generalized activation functions. The main contributions of our works are

that the system consists of both memoryless and delayed activation functions, and the lower and upper bounds of the activation functions are allowed to be either positive, negative, or zero which is more general than systems considered in [14, 15, 18, 21, 22, 24–27]. By constructing a new and improved Lyapunov-Krasovskii functional which contains some integral terms of the activation functions, less conservative results are obtained by introducing appropriate free-weighting matrices and by using some improved integral inequality. Finally, two numerical examples are presented to show that our result is less conservative than some existing ones.

Notations. Throughout the paper, \mathbb{R} denotes the set of all real numbers. $*$ denotes the elements below the main diagonal of a symmetric block matrix. $\text{diag}\{\dots\}$ denotes the diagonal matrix. For symmetric matrices X and Y , the notation $X > Y$ (resp., $X \geq Y$) means that the matrix $X - Y$ is positive definite (resp., nonnegative). $\lambda_m(\cdot)$ and $\lambda_M(\cdot)$ denote the smallest and largest eigenvalue of given square matrix, respectively.

2. Model Description and Preliminaries

Consider the following BAM neural network with time-varying delays of the form

$$\begin{aligned} \dot{u}_i(t) &= -c_i u_i(t) + \sum_{j=1}^m a_{ij}^{(1)} \tilde{g}_j(v_j(t)) \\ &\quad + \sum_{j=1}^m a_{ij}^{(2)} \tilde{g}_j(v_j(t-h(t))) + I_i, \\ \dot{v}_j(t) &= -d_j v_j(t) + \sum_{i=1}^n b_{ji}^{(1)} \tilde{f}_i(u_i(t)) \\ &\quad + \sum_{i=1}^n b_{ji}^{(2)} \tilde{f}_i(u_i(t-d(t))) + I_j, \end{aligned} \quad (1)$$

$$i = 1, 2, \dots, n, \quad j = 1, 2, \dots, m,$$

where $u_i(t)$ and $v_j(t)$ are the state of the i th neurons from the neural field F_u and the j th neurons from the neural field F_v , at time t , respectively; c_i and d_j denote the neuron charging time constants and passive delay rates, respectively; $a_{ij}^{(1)}$ and $b_{ji}^{(1)}$ are the synaptic connection strengths; $a_{ij}^{(2)}$ and $b_{ji}^{(2)}$ are delayed synaptic connection strengths; $\tilde{f}_i(\cdot)$ and $\tilde{g}_j(\cdot)$ denote the activation functions of the i th neurons from the neural field F_u and the j th neurons from the neural field F_v , respectively; I_i and I_j denote the external inputs; and $d(t)$ and $h(t)$ represent the time-varying differentiable functions which satisfy

$$\begin{aligned} \text{(i)} \quad &0 \leq d(t) \leq d, \quad \dot{d}(t) \leq \tau < 1, \\ \text{(ii)} \quad &0 \leq h(t) \leq h, \quad \dot{h}(t) \leq \mu < 1, \end{aligned} \quad (2)$$

where d , h , μ , and τ are positive scalars.

The initial conditions associated with (1) are assumed to be

$$\begin{aligned} u_i(s) &= \tilde{\phi}_i(s), \quad v_j(s) = \tilde{\varphi}_j(s), \quad s \in [-\max\{d, h\}, 0], \\ & i = 1, 2, \dots, n, \quad j = 1, 2, \dots, m. \end{aligned} \quad (3)$$

Throughout this paper, we make the following assumption on the activation function $\tilde{f}_i(\cdot)$, $\tilde{g}_j(\cdot)$.

(A1) $\tilde{f}_i(\cdot)$ and $\tilde{g}_j(\cdot)$ are bounded on \mathbb{R} .

(A2) For any $\alpha, \beta \in \mathbb{R}$, $\alpha \neq \beta$, there exist four constant matrices $E = \text{diag}(E_1, E_2, \dots, E_n)$, $F = \text{diag}(F_1, F_2, \dots, F_n)$, $M = \text{diag}(M_1, M_2, \dots, M_m)$, and $N = \text{diag}(N_1, N_2, \dots, N_m)$ satisfying

$$\begin{aligned} E_i &\leq \frac{\tilde{f}_i(\alpha) - \tilde{f}_i(\beta)}{\alpha - \beta} \leq F_i, \quad i = 1, 2, \dots, n, \\ N_j &\leq \frac{\tilde{g}_j(\alpha) - \tilde{g}_j(\beta)}{\alpha - \beta} \leq M_j, \quad j = 1, 2, \dots, m. \end{aligned} \quad (4)$$

It is clear that under (A1) and (A2), the system (1) has at least one equilibrium; see [20]. In order to simplify our proof, we shift the equilibrium point $u^* = (u_1^*, u_2^*, \dots, u_n^*)^T$, $v^* = (v_1^*, v_2^*, \dots, v_m^*)^T$ of system (1) to the origin. Let $x_i(t) = u_i(t) - u_i^*$, $y_j(t) = v_j(t) - v_j^*$, $f_i(x_i(t)) = \tilde{f}_i(x_i(t) + u_i^*) - \tilde{f}_i(u_i^*)$, $g_j(y_j(t)) = \tilde{g}_j(y_j(t) + v_j^*) - \tilde{g}_j(v_j^*)$, $f_i(x_i(t-d(t))) = \tilde{f}_i(x_i(t-d(t)) + u_i^*) - \tilde{f}_i(u_i^*)$, $g_j(y_j(t-h(t))) = \tilde{g}_j(y_j(t-h(t)) + v_j^*) - \tilde{g}_j(v_j^*)$, $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$. Then the system (1) can be transformed to

$$\begin{aligned} \dot{x}(t) &= -c_i x_i(t) + \sum_{j=1}^m a_{ij}^{(1)} g_j(y_j(t)) \\ &\quad + \sum_{j=1}^m a_{ij}^{(2)} g_j(y_j(t-h(t))), \\ \dot{y}(t) &= -d_j y_j(t) + \sum_{i=1}^n b_{ji}^{(1)} f_i(x_i(t)) \\ &\quad + \sum_{i=1}^n b_{ji}^{(2)} f_i(x_i(t-d(t))). \end{aligned} \quad (5)$$

The activation functions $f_i(\cdot)$ and $g_j(\cdot)$ satisfy the following properties.

(H1) f_i and g_j are bounded on \mathbb{R} .

(H2) For any $\alpha, \beta \in \mathbb{R}, \alpha \neq \beta$, there exist constant matrices $E = \text{diag}(E_1, E_2, \dots, E_n), F = \text{diag}(F_1, F_2, \dots, F_n), M = \text{diag}(M_1, M_2, \dots, M_m)$, and $N = \text{diag}(N_1, N_2, \dots, N_m)$ satisfying

$$E_i \leq \frac{f_i(\alpha) - f_i(\beta)}{\alpha - \beta} \leq F_i, \quad i = 1, 2, \dots, n, \tag{6}$$

$$N_j \leq \frac{g_j(\alpha) - g_j(\beta)}{\alpha - \beta} \leq M_j, \quad j = 1, 2, \dots, m.$$

(H3) $f_i(0) = 0$ and $g_j(0) = 0, i = 1, 2, \dots, n, j = 1, 2, \dots, m$.

Rewrite the system (7) into the vector form

$$\begin{aligned} \dot{x}(t) &= -Cx(t) + A_1g(y(t)) + A_2g(y(t-h(t))), \\ \dot{y}(t) &= -Dy(t) + B_1f(x(t)) + B_2f(x(t-d(t))). \end{aligned} \tag{7}$$

The initial conditions associated with (7) are assumed to be

$$\begin{aligned} x_i(s) &= \phi_i(s), \quad y_j(s) = \varphi_j(s), \quad s \in [-\max\{d, h\}, 0], \\ i &= 1, 2, \dots, n, \quad j = 1, 2, \dots, m, \end{aligned} \tag{8}$$

where $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T, y(t) = (y_1(t), y_2(t), \dots, y_m(t))^T, C = \text{diag}(c_1, c_2, \dots, c_n), D = \text{diag}(d_1, d_2, \dots, d_m), A_1 = (a_{ij}^{(1)})_{m \times n}, A_2 = (a_{ij}^{(2)})_{m \times n}, B_1 = (b_{ij}^{(1)})_{n \times m}, B_2 = (b_{ij}^{(2)})_{n \times m}, f(\cdot) = (f_1(\cdot), f_2(\cdot), \dots, f_n(\cdot))^T, g(\cdot) = (g_1(\cdot), g_2(\cdot), \dots, g_m(\cdot))^T$.

Definition 1 (see [14]). The trivial solution of system (7) is said to be globally exponentially stable if there exist constants $k > 0$ and $\rho \geq 1$ such that

$$\|x(t)\|^2 + \|y(t)\|^2 \leq \rho e^{-2kt} (\|\phi\|^2 + \|\varphi\|^2), \quad \forall t \geq 0, \tag{9}$$

where one denotes

$$\begin{aligned} \|\phi\|^2 + \|\varphi\|^2 &= \sup_{-\max\{d, h\} \leq s \leq 0} \|\phi(s)\|^2 \\ &+ \sup_{-\max\{d, h\} \leq s \leq 0} \|\varphi(s)\|^2. \end{aligned} \tag{10}$$

Lemma 2 (see [28]). *If there exist symmetric positive-definite matrix $X_{33} > 0$ and arbitrary matrices $X_{11}, X_{12}, X_{13}, X_{22}$, and X_{23} such that*

$$X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^T & X_{22} & X_{23} \\ X_{13}^T & X_{23}^T & X_{33} \end{bmatrix} \geq 0, \tag{11}$$

then,

$$\begin{aligned} & - \int_{t-h(t)}^t \dot{x}(s) X_{33} \dot{x}(s) ds \\ & \leq \int_{t-h(t)}^t \begin{bmatrix} x^T(t) & x^T(t-h(t)) & \dot{x}^T(s) \end{bmatrix} \\ & \quad \times \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^T & X_{22} & X_{23} \\ X_{13}^T & X_{23}^T & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h(t)) \\ \dot{x}(s) \end{bmatrix} ds. \end{aligned} \tag{12}$$

Lemma 3 (see [25]). *For any real vectors a, b and any matrix $Q > 0$ with appropriate dimensions, it follows that*

$$2a^T b \leq a^T Q a + b^T Q^{-1} b. \tag{13}$$

Lemma 4 (see [25]). *Suppose that (H2) holds; then*

$$\begin{aligned} \int_v^u [f_i(s) - f_i(v)] ds &\leq [u - v] [f_i(u) - f_i(v)], \\ i &= 1, 2, \dots, n, \\ \int_v^u [g_j(s) - g_j(v)] ds &\leq [u - v] [g_j(u) - g_j(v)], \\ j &= 1, 2, \dots, m. \end{aligned} \tag{14}$$

3. Main Result

In this section, we present a theorem which states the conditions that guarantee the global exponential stability of the system (7) employing the Lyapunov stability theory and linear matrix inequality approach.

Theorem 5. *Under the assumptions (H1)–(H3), for given four diagonal matrices $E = \text{diag}(E_1, E_2, \dots, E_n), F = \text{diag}(F_1, F_2, \dots, F_n), M = \text{diag}(M_1, M_2, \dots, M_m)$, and $N = \text{diag}(N_1, N_2, \dots, N_m)$ and positive constants d, h, τ, μ , and k , the system (7) is globally exponentially stable with the convergent rate k , if there exist positive matrices $P_i, W_i, Z_i, i = 1, 2, Q_j, j = 1, 2, 3, 4$, positive diagonal matrices $K = \text{diag}(k_1, k_2, \dots, k_n), R = \text{diag}(r_1, r_2, \dots, r_m)$, and positive-definite matrices*

$$\begin{aligned} S &= \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{12}^T & S_{22} & S_{23} \\ S_{13}^T & S_{23}^T & Z_1 \end{bmatrix}, \\ T &= \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{12}^T & T_{22} & T_{23} \\ T_{13}^T & T_{23}^T & Z_2 \end{bmatrix}, \end{aligned} \tag{15}$$

such that the following LMI holds:

$$\Xi = \begin{bmatrix} \Sigma_1^1 & \Sigma_2^1 & \Sigma_3^1 & 0 & 0 & 0 & \Sigma_7^1 & \Sigma_8^1 \\ * & \Sigma_2^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & \Sigma_3^3 & \Sigma_4^3 & \Sigma_5^3 & 0 & \Sigma_7^3 & \Sigma_8^3 \\ * & * & * & \Sigma_4^4 & \Sigma_5^4 & 0 & \Sigma_7^4 & 0 \\ * & * & * & * & \Sigma_5^5 & \Sigma_6^5 & \Sigma_7^5 & 0 \\ * & * & * & * & * & \Sigma_6^6 & 0 & 0 \\ * & * & * & * & * & * & \Sigma_7^7 & \Sigma_8^7 \\ * & * & * & * & * & * & * & \Sigma_8^8 \end{bmatrix} < 0, \tag{16}$$

where

$$\begin{aligned}
\Sigma_1^1 &= 2kP_1 - P_1C - C^T P_1 - 2kE^T K - 2kK^T E \\
&\quad + E^T KC + C^T KE + Q_1 + dC^T Z_1 C \\
&\quad + e^{-2kd} (dS_{11} + S_{13}^T + S_{13}) + W_1, \\
\Sigma_2^1 &= e^{-2kd} (dS_{12} + S_{23}^T - S_{13}), \\
\Sigma_3^1 &= 2kK, \\
\Sigma_7^1 &= P_1 A_1 - E^T K A_1 - dC^T Z_1 A_1, \\
\Sigma_8^1 &= P_2 A_2 - E^T K A_2 - dC^T Z_1 A_2, \\
\Sigma_2^2 &= e^{-2kd} (dS_{22} - S_{23}^T - S_{23}) - e^{-2kd} Q_1, \\
\Sigma_3^3 &= hB_1^T Z_2 B_1 + Q_3 - 2KCF^{-1}, \\
\Sigma_4^3 &= hB_1^T Z_2 B_2, \\
\Sigma_5^3 &= P_2 B_1 - N^T R B_1 - hB_1^T Z_2 D, \\
\Sigma_7^3 &= K A_1 + R B_1, \\
\Sigma_8^3 &= K A_2, \\
\Sigma_4^4 &= hB_2^T Z_2 B_2 - (1 - \tau) e^{-2kd} Q_3 \\
&\quad - (1 - \tau) e^{-2kd} F^{-1} W_1 F^{-1}, \\
\Sigma_5^4 &= P_2 B_2 - N^T R B_2 - hB_2^T Z_2 D, \\
\Sigma_7^4 &= R B_2, \\
\Sigma_5^5 &= 2kP_2 - P_2 D - D^T P_2 - 2kN^T R - 2kR^T N \\
&\quad + N^T R D + D^T R N + Q_2 + hD^T Z_2 D \\
&\quad + e^{-2kh} (hT_{11} + T_{13}^T + T_{13}) + W_2, \\
\Sigma_6^5 &= e^{-2kh} (hT_{12} + T_{23}^T - T_{13}), \\
\Sigma_7^5 &= 2kR, \\
\Sigma_6^6 &= e^{-2kh} (hT_{22} - T_{23}^T - T_{23}) - e^{-2kh} Q_2, \\
\Sigma_7^7 &= dA_1^T Z_1 A_1 + Q_4 - 2RDM^{-1}, \\
\Sigma_8^7 &= dA_1^T Z_1 A_2, \\
\Sigma_8^8 &= dA_2^T Z_1 A_2 - (1 - \mu) e^{-2kh} Q_4 \\
&\quad - (1 - \mu) e^{-2kh} M^{-1} W_2 M^{-1}.
\end{aligned} \tag{17}$$

Proof. Choose the Lyapunov-Krasovskii function candidate for the system (7) to be

$$V(t) = \sum_{l=1}^5 V_l(t), \tag{18}$$

where

$$\begin{aligned}
V_1(t) &= e^{2kt} x^T(t) P_1 x(t) \\
&\quad + 2 \sum_{i=1}^n k_i e^{2kt} \int_0^{x_i(t)} [f_i(s) - E_i s] ds \\
&\quad + e^{2kt} y^T(t) P_2 y(t) \\
&\quad + 2 \sum_{j=1}^m r_j e^{2kt} \int_0^{y_j(t)} [g_j(s) - N_j s] ds, \\
V_2(t) &= \int_{t-d}^t e^{2ks} x^T(s) Q_1 x(s) ds \\
&\quad + \int_{t-h}^t e^{2ks} y^T(s) Q_2 y(s) ds, \\
V_3(t) &= \int_{t-d(t)}^t e^{2ks} f^T(x(s)) Q_3 f(x(s)) ds \\
&\quad + \int_{t-h(t)}^t e^{2ks} g^T(y(s)) Q_4 g(y(s)) ds, \\
V_4(t) &= \int_{t-d(t)}^t e^{2ks} x^T(s) W_1 x(s) ds \\
&\quad + \int_{t-h(t)}^t e^{2ks} y^T(s) W_2 y(s) ds, \\
V_5(t) &= \int_{t-d}^t \int_s^t e^{2k\theta} \dot{x}^T(\theta) Z_1 \dot{x}(\theta) d\theta ds \\
&\quad + \int_{t-h}^t \int_s^t e^{2k\theta} \dot{y}^T(\theta) Z_2 \dot{y}(\theta) d\theta ds.
\end{aligned} \tag{19}$$

The derivative of $V(t)$ along the trajectories of system (7) is given by

$$\begin{aligned}
\dot{V}_1(t) &= 2ke^{2kt} x^T(t) P_1 x(t) + 2e^{2kt} x^T(t) P_1 \dot{x}(t) \\
&\quad + 4k \sum_{i=1}^n k_i e^{2kt} \int_0^{x_i(t)} [f_i(s) - E_i s] ds \\
&\quad + 2 \sum_{i=1}^n k_i e^{2kt} \dot{x}_i(t) [f_i(x_i(t)) - E_i x_i(t)] \\
&\quad + 2ke^{2kt} y^T(t) P_2 y(t) + 2e^{2kt} y^T(t) P_2 \dot{y}(t) \\
&\quad + 4k \sum_{j=1}^m r_j e^{2kt} \int_0^{y_j(t)} [g_j(s) - N_j s] ds \\
&\quad + 2 \sum_{j=1}^m r_j e^{2kt} \dot{y}_j(t) [g_j(y_j(t)) - N_j y_j(t)]
\end{aligned}$$

$$\begin{aligned}
 &\leq e^{2kt} \left\{ 2kx^T(t) P_1 x(t) + 2x^T(t) \right. \\
 &\quad \times P_1 [-Cx(t) + A_1 g(y(t)) \\
 &\quad \quad \left. + A_2 g(y(t-h(t)))] \right. \\
 &\quad + 4k[f(x(t)) - Ex(t)]^T Kx(t) \\
 &\quad + 2[f(x(t)) - Ex(t)]^T \\
 &\quad \times K [-Cx(t) + A_1 g(y(t)) \\
 &\quad \quad \left. + A_2 g(y(t-h(t)))] \right. \\
 &\quad + 2ky^T(t) P_2 y(t) \\
 &\quad + 2y^T(t) P_2 [-Dy(t) + B_1 f(x(t)) \\
 &\quad \quad \left. + B_2 f(x(t-d(t)))] \right. \\
 &\quad + 4k[g(y(t)) - Ny(t)]^T Ry(t) \\
 &\quad + 2[g(y(t)) - Ny(t)]^T R \\
 &\quad \times [-Dy(t) + B_1 f(x(t)) \\
 &\quad \quad \left. + B_2 f(x(t-d(t)))] \right\}. \tag{20}
 \end{aligned}$$

By (H2), we have

$$\begin{aligned}
 -2f^T(x(t)) K C x(t) &\leq -2f^T(x(t)) K C F^{-1} f(x(t)), \\
 -2g^T(y(t)) R D y(t) &\leq -2g^T(y(t)) R D M^{-1} g(y(t)).
 \end{aligned} \tag{21}$$

Substituting (21) into (20), we obtain

$$\begin{aligned}
 \dot{V}_1(t) &\leq e^{2kt} \\
 &\times \left\{ 2kx^T(t) P_1 x(t) - 2x^T(t) P_1 C x(t) \right. \\
 &\quad + 2x^T(t) P_1 A_1 g(y(t)) + 2x^T(t) P_1 A_2 \\
 &\quad \times g(y(t-h(t))) + 4kf^T(x(t)) Kx(t) \\
 &\quad - 4kx^T(t) E^T Kx(t) - 2f^T(x(t)) K C \\
 &\quad \times F^{-1} f(x(t)) + 2f^T(x(t)) K A_1 g(y(t)) \\
 &\quad + 2f^T(x(t)) K A_2 g(y(t-h(t))) \\
 &\quad + 2x^T(t) E^T K C x(t) \\
 &\quad - 2x^T(t) E^T K A_1 g(y(t)) \\
 &\quad - 2x^T(t) E^T K A_2 g(y(t-h(t))) \\
 &\quad + 2ky^T(t) P_2 y(t) - 2y^T(t) P_2 D y(t) \\
 &\quad \left. + 2y^T(t) P_2 B_1 f(x(t)) \right\}.
 \end{aligned}$$

$$\begin{aligned}
 &+ 2y^T(t) P_2 B_2 f(x(t-d(t))) \\
 &+ 4kg^T(y(t)) Ry(t) - 4ky^T(t) N^T Ry(t) \\
 &- 2g^T(y(t)) R D M^{-1} g(y(t)) \\
 &+ 2g^T(y(t)) R B_1 f(x(t)) \\
 &+ 2g^T(y(t)) R B_2 f(x(t-d(t))) \\
 &+ 2y^T(t) N^T R D y(t) \\
 &- 2y^T(t) N^T R B_1 f(x(t)) \\
 &- 2y^T(t) N^T R B_2 f(x(t-d(t))) \Big\}, \\
 \dot{V}_2(t) &= e^{2kt} \left\{ x^T(t) Q_1 x(t) \right. \\
 &\quad - e^{-2kd} x^T(t-d) Q_1 x(t-d) \\
 &\quad + y^T(t) Q_2 y(t) \\
 &\quad \left. - e^{-2kh} y^T(t-h) Q_2 y(t-h) \right\}, \\
 \dot{V}_3(t) &= e^{2kt} \\
 &\times \left\{ f^T(x(t)) Q_3 f(x(t)) \right. \\
 &\quad - (1-\dot{d}(t)) e^{-2kd(t)} f^T(x(t-d(t))) \\
 &\quad \times Q_3 f(x(t-d(t))) \\
 &\quad + g^T(y(t)) Q_4 g(y(t)) \\
 &\quad - (1-\dot{h}(t)) e^{-2kh(t)} \\
 &\quad \times g^T(y(t-h(t))) Q_4 g(y(t-h(t))) \Big\} \\
 &\leq e^{2kt} \left\{ f^T(x(t)) Q_3 f(x(t)) \right. \\
 &\quad - (1-\tau) e^{-2kd} f^T(x(t-d(t))) \\
 &\quad \times Q_3 f(x(t-d(t))) \\
 &\quad + g^T(y(t)) Q_4 g(y(t)) \\
 &\quad - (1-\mu) e^{-2kh} g^T(y(t-h(t))) \\
 &\quad \times Q_4 g(y(t-h(t))) \Big\}, \tag{22}
 \end{aligned}$$

$$\begin{aligned}
 \dot{V}_4(t) &= e^{2kt} \left\{ x^T(t) W_1 x(t) \right. \\
 &\quad - (1-\dot{d}(t)) e^{-2kd(t)} x^T(t-d(t)) \\
 &\quad \times W_1 x(t-d(t)) \\
 &\quad + y^T(t) W_2 y(t) - (1-\dot{h}(t)) e^{-2kh(t)} \\
 &\quad \times y^T(t-h(t)) (s) W_2 y^T(t-h(t)) \Big\}
 \end{aligned}$$

$$\begin{aligned}
&\leq e^{2kt} \left\{ x^T(t) W_1 x(t) - (1-\tau) e^{-2kd} \right. \\
&\quad \times x^T(t-d(t)) W_1 x(t-d(t)) \\
&\quad + y^T(t) W_2 y(t) - (1-\mu) e^{-2kh} \\
&\quad \left. \times y^T(t-h(t)) W_2 y(t-h(t)) \right\} \\
&\quad \times h y^T(t) Z_2 \dot{y}(t) \\
&\quad - e^{-2kh} \int_{t-h}^t \dot{y}^T(s) Z_2 \dot{y}(s) ds \Big\}. \tag{26}
\end{aligned}$$

By Lemma 2, we obtain

$$\begin{aligned}
& - \int_{t-d}^t \dot{x}^T(s) Z_1 \dot{x}(s) ds \\
& \leq \int_{t-d}^t \left[x^T(t) \quad x^T(t-d) \quad \dot{x}^T(s) \right] \\
& \quad \times \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{12}^T & S_{22} & S_{23} \\ S_{13}^T & S_{23}^T & 0 \end{bmatrix} \\
& \quad \times \begin{bmatrix} x(t) \\ x(t-d) \\ \dot{x}(s) \end{bmatrix} ds \\
& = \int_{t-d}^t \left(x^T(t) S_{11} + x^T(t-d) S_{12}^T + \dot{x}^T(s) S_{13}^T \right) x(t) ds \\
& \quad + \int_{t-d}^t \left(x^T(t) S_{12} + x^T(t-d) S_{22} + \dot{x}^T(s) S_{23}^T \right) \\
& \quad \times x(t-d) ds \\
& \quad + \int_{t-d}^t \left(x^T(t) S_{13} + x^T(t-d) S_{23} \right) \dot{x}(s) ds \\
& = x^T(t) dS_{11} x(t) + x^T(t-d) dS_{12}^T x(t) \\
& \quad + \int_{t-d}^t \dot{x}^T(s) ds S_{13}^T x(t) \\
& \quad + x^T(t) dS_{12} x(t-d) + x^T(t-d) dS_{22} x(t-d) \\
& \quad + \int_{t-d}^t \dot{x}^T(s) ds S_{23}^T x(t-d) \\
& \quad + x^T(t) S_{13} \int_{t-d}^t \dot{x}(s) ds \\
& \quad + x^T(t-d) S_{23} \int_{t-d}^t \dot{x}(s) ds \\
& = x^T(t) dS_{11} x(t) + x^T(t-d) dS_{12}^T x(t) \\
& \quad + x^T(t) dS_{12} x(t-d) \\
& \quad + x^T(t-d) dS_{22} x(t-d) + [x(t) - x(t-d)]^T \\
& \quad \times \left[S_{13}^T x(t) + S_{23}^T x(t-d) \right] \\
& \quad + \left[x^T(t) S_{13} + x^T(t-d) S_{23} \right] \\
& \quad \times [x(t) - x(t-d)],
\end{aligned}$$

By (H2), we have

$$\begin{aligned}
& - e^{-2kd} x^T(t-d(t)) W_1 x(t-d(t)) \\
& \leq -e^{-2kd} f^T(x(t-d(t))) F^{-1} W_1 \\
& \quad \times F^{-1} f(x(t-d(t))), \\
& - e^{-2kh} y^T(t-h(t)) W_2 y(t-h(t)) \\
& \leq -e^{-2kh} g^T(y(t-h(t))) M^{-1} W_2 \\
& \quad \times M^{-1} g(y(t-h(t))). \tag{24}
\end{aligned}$$

By (24), we conclude that

$$\begin{aligned}
\dot{V}_4(t) &\leq e^{2kt} \left\{ x^T(t) W_1 x(t) - (1-\tau) e^{-2kd} \right. \\
&\quad \times f^T(x(t-d(t))) F^{-1} W_1 F^{-1} \\
&\quad \times f(x(t-d(t))) \\
&\quad + y^T(t) W_2 y(t) - (1-\mu) e^{-2kh} \\
&\quad \times g^T(y(t-h(t))) M^{-1} W_2 M^{-1} \\
&\quad \left. \times g(y(t-h(t))) \right\}. \tag{25}
\end{aligned}$$

$$\begin{aligned}
\dot{V}_5(t) &= d e^{2kt} \dot{x}^T(t) Z_1 \dot{x}(t) \\
&\quad - \int_{t-d}^t e^{2ks} \dot{x}^T(s) Z_1 \dot{x}(s) ds \\
&\quad + h e^{2kt} \dot{y}^T(t) Z_2 \dot{y}(t) \\
&\quad - \int_{t-h}^t e^{2ks} \dot{y}^T(s) Z_2 \dot{y}(s) ds \\
&\leq e^{2kt} \left\{ d \dot{x}^T(t) Z_1 \dot{x}(t) \right. \\
&\quad \left. - e^{-2kd} \int_{t-d}^t \dot{x}^T(s) Z_1 \dot{x}(s) ds \right\}
\end{aligned}$$

$$\begin{aligned}
 & - \int_{t-h}^t \dot{y}^T(s) Z_2 \dot{y}(s) ds \\
 & \leq \int_{t-h}^t \left[y^T(t) \quad y^T(t-h) \quad \dot{y}^T(s) \right] \\
 & \quad \times \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{12}^T & T_{22} & T_{23} \\ T_{13} & T_{23} & 0 \end{bmatrix} \\
 & \quad \times \begin{bmatrix} y(t) \\ y(t-h) \\ \dot{y}(s) \end{bmatrix} ds \\
 & = \int_{t-h}^t \left(y^T(t) T_{11} + y^T(t-h) T_{12}^T + \dot{y}^T(s) T_{13}^T \right) y(t) ds \\
 & \quad + \int_{t-h}^t \left(y^T(t) T_{12} + y^T(t-h) T_{22} + \dot{y}^T(s) T_{23}^T \right) \\
 & \quad \quad \times y(t-h) ds \\
 & \quad + \int_{t-h}^t \left(y^T(t) T_{13} + y^T(t-h) T_{23} \right) \dot{y}(s) ds \\
 & = y^T(t) h T_{11} y(t) + y^T(t-h) h T_{12}^T y(t) \\
 & \quad + \int_{t-h}^t \dot{y}^T(s) ds T_{13}^T y(t) \\
 & \quad + y^T(t) h T_{12} y(t-h) + y^T(t-h) h T_{22} y(t-h) \\
 & \quad + \int_{t-h}^t \dot{y}^T(s) ds T_{23}^T y(t-h) \\
 & \quad + y^T(t) T_{13} \int_{t-h}^t \dot{y}(s) ds \\
 & \quad + y^T(t-h) T_{23} \int_{t-h}^t \dot{y}(s) ds \\
 & = y^T(t) h T_{11} y(t) + y^T(t-h) h T_{12}^T y(t) \\
 & \quad + y^T(t) h T_{12} y(t-h) \\
 & \quad + y^T(t-h) h T_{22} y(t-h) + [y(t) - y(t-h)]^T \\
 & \quad \times [T_{13}^T y(t) + T_{23}^T y(t-h)] \\
 & \quad + [y^T(t) T_{13} + y^T(t-h) T_{23}] \\
 & \quad \times [y(t) - y(t-h)]. \tag{27}
 \end{aligned}$$

$$\begin{aligned}
 & + 2dx^T(t) S_{12}^T x(t-d) \\
 & + 2[x(t) - x(t-d)]^T \\
 & \quad \times [S_{13}^T x(t) + S_{23}^T x(t-d)] \\
 & + dx^T(t-d) S_{22} x(t-d) \\
 & + h\dot{y}^T(t) Z_2 \dot{y}(t) \\
 & + e^{-2kh} [hy^T(t) T_{11} y(t) \\
 & \quad + 2hy^T(t) T_{12} y(t-h) \\
 & \quad + 2[y(t) - y(t-h)]^T \\
 & \quad \times [T_{13}^T y(t) + T_{23}^T y(t-h)] \\
 & \quad + hy^T(t-h) T_{22} y(t-h)] \}. \tag{28}
 \end{aligned}$$

From (22), (25), and (28) we obtain

$$\dot{V}(t) \leq e^{2kt} \xi^T(t) \Xi \xi(t), \tag{29}$$

where Ξ is defined as in (16), and $\xi(t) = [x(t), x(t-d), f(x(t)), f(x(t-d)), y(t), y(t-h), g(y(t)), g(y(t-h))]$. Since the matrix Ξ given in Theorem 5 is the negative definite matrix, we have $\dot{V}(t) \leq 0$, for all $t \geq 0$ which implies that $V(t) \leq V(0)$. From the definition of $V(t)$ in (20), we obtain

$$\begin{aligned}
 V(0) & = x^T(0) P_1 x(0) \\
 & + 2 \sum_{i=1}^n k_i \int_0^{x_i(0)} [f_i(s) - E_i s] ds \\
 & + \int_{-d}^0 e^{2ks} x^T(s) Q_1 x(s) ds \\
 & + \int_{-d(0)}^0 e^{2ks} f^T(x(s)) Q_3 f(x(s)) ds \\
 & + \int_{-d(0)}^0 e^{2ks} x^T(s) W_1 x(s) ds \\
 & + y^T(0) P_2 y(0) \\
 & + 2 \sum_{j=1}^m r_j \int_0^{y_j(0)} [g_j(s) - N_j s] ds \\
 & + \int_{-h}^0 e^{2ks} y^T(s) Q_2 y(s) ds
 \end{aligned}$$

Substituting (27) into (26), we have

$$\begin{aligned}
 \dot{V}_5(t) & \leq e^{2kt} \{ d\dot{x}^T(t) Z_1 \dot{x}(t) \\
 & \quad + e^{-2kd} [dx^T(t) S_{11} x(t)
 \end{aligned}$$

$$\begin{aligned}
& + \int_{-h(0)}^0 e^{2ks} g^T(y(s)) Q_4 g(y(s)) ds \\
& + \int_{-h(0)}^0 y^T(s) W_2 y(s) ds \\
& + \int_{-d}^0 \int_s^0 e^{2k\theta} \dot{x}^T(\theta) Z_1 \dot{x}(\theta) d\theta ds \\
& + \int_{-h}^0 \int_s^0 e^{2k\theta} \dot{y}^T(\theta) Z_2 \dot{y}(\theta) d\theta ds \\
\leq & \lambda_M(P_1) \|\phi\|^2 + 2K_M(F-E) \|\phi\|^2 \\
& + [\lambda_M(Q_1) + \lambda_M(Q_3)(F-E) + \lambda_M(W_1)] \\
& \times \int_{-d}^0 e^{2ks} x^T(s) x(s) ds + \lambda_M(P_2) \|\phi\|^2 \\
& + 2R_M(M-N) \|\phi\|^2 \\
& + [\lambda_M(Q_2) + \lambda_M(Q_4)(M-N) + \lambda_M(W_2)] \\
& \times \int_{-h}^0 e^{2ks} y^T(s) y(s) ds \\
& + \lambda_M(Z_1) \int_{-d}^0 \int_s^0 \dot{x}^T(\theta) \dot{x}(\theta) d\theta ds \\
& + \lambda_M(Z_2) \int_{-h}^0 \int_s^0 \dot{y}^T(\theta) \dot{y}(\theta) d\theta ds, \\
\end{aligned} \tag{30}$$

$$\begin{aligned}
& \leq 3x^T(\theta) C^T C x(\theta) \\
& + 3g^T(y(\theta)) A_1^T A_1 g(y(\theta)) \\
& + 3g^T(y(\theta-h(\theta))) A_2^T \\
& \times A_2 g(y(\theta-h(\theta))), \\
& \dot{y}^T(\theta) \dot{y}(\theta) \\
& = [-Dy(\theta) + B_1 f(x(\theta)) + B_2 f(x(\theta-d(\theta)))]^T \\
& \times [-Dy(\theta) + B_1 f(x(\theta)) + B_2 f(x(\theta-d(\theta)))] \\
& = y^T(\theta) D^T D y(\theta) + f^T(x(\theta)) B_1^T B_1 f(x(\theta)) \\
& + f^T(x(\theta-d(\theta))) \\
& \times B_2^T B_2 f(x(\theta-d(\theta))) \\
& - 2y^T(\theta) D^T B_1 f(x(\theta)) - 2y^T(\theta) D^T B_2 \\
& \times f(x(\theta-d(\theta))) \\
& + 2f^T(x(\theta)) B_1^T B_2 f(x(\theta-d(\theta))) \\
& \leq 3y^T(\theta) D^T D y(\theta) + 3f^T(x(\theta)) B_1^T B_1 f(x(\theta)) \\
& + 3f^T(x(\theta-d(\theta))) B_2^T \times B_2 g(y(\theta-h(\theta))). \\
\end{aligned} \tag{31}$$

Substituting (31) into (30), we obtain the bound of $V(0)$ as follows:

$$\begin{aligned}
V(0) \leq & \lambda_M(P_1) \|\phi\|^2 + 2K_M(F-E) \|\phi\|^2 \\
& + \left(\frac{1-e^{-2kd}}{2k} \right) \\
& \times [\lambda_M(Q_1) + \lambda_M(Q_3) \times (F-E) + \lambda_M(W_1)] \\
& + \lambda_M(P_2) \|\phi\|^2 + 2R_M(M-N) \|\phi\|^2 \\
& + \left(\frac{1-e^{-2kh}}{2k} \right) \\
& \times [\lambda_M(Q_2) + \lambda_M(Q_4)(M-N) + \lambda_M(W_2)] \\
& + \frac{d^2}{2} \lambda_M(Z_1) [3\lambda_M(C^T C) \|\phi\|^2 \\
& + 3\lambda_M(A_1^T A_1) \|\phi\|^2 \\
& + 3\lambda_M(A_2^T A_2) \|\phi\|^2]
\end{aligned}$$

where $K_M = \max_{1 \leq i \leq n} (k_i)$, $F-E = \max_{1 \leq i \leq n} (F_i - E_i)$, $R_M = \max_{1 \leq j \leq m} (r_j)$ and $M-N = \max_{1 \leq j \leq n} (M_j - N_j)$.

It follows from Lemma 3 that

$$\begin{aligned}
& \dot{x}^T(\theta) \dot{x}(\theta) \\
& = [-Cx(\theta) + A_1 g(y(\theta)) + A_2 g(y(\theta-h(\theta)))]^T \\
& \times [-Cx(\theta) + A_1 g(y(\theta)) + A_2 g(y(\theta-h(\theta)))] \\
& = x^T(\theta) C^T C x(\theta) + g^T(y(\theta)) A_1^T A_1 g(y(\theta)) \\
& + g^T(y(\theta-h(\theta))) \times A_2^T A_2 g(y(\theta-h(\theta))) \\
& - 2x^T(\theta) C^T A_1 g(y(\theta)) - 2x^T(\theta) C^T A_2 \\
& \times g(y(\theta-h(\theta))) \\
& + 2g^T(y(\theta)) A_1^T A_2 g(y(\theta-h(\theta)))
\end{aligned}$$

$$\begin{aligned}
 & + \frac{h^2}{2} \lambda_M(Z_2) [3\lambda_M(D^T D) \|\varphi\|^2 \\
 & \quad + 3\lambda_M(B_1^T B_1) \|\phi\|^2 \\
 & \quad + 3\lambda_M(B_2^T B_2) \|\phi\|^2].
 \end{aligned} \tag{32}$$

Thus,

$$V(0) \leq \chi_1 \|\phi\|^2 + \chi_2 \|\varphi\|^2, \tag{33}$$

where

$$\begin{aligned}
 \chi_1 & = \lambda_M(P_1) + 2K_M(F - E) + \left(\frac{1 - e^{-2kd}}{2k}\right) \\
 & \quad \times [\lambda_M(Q_1) + \lambda_M(Q_3)(F - E) + \lambda_M(W_1)] \\
 & \quad + \frac{3d^2}{2} \lambda_M(Z_1) \lambda_M(C^T C) \\
 & \quad + \frac{h^2}{2} \lambda_M(Z_2) [3\lambda_M(B_1^T B_1) + 3\lambda_M(B_2^T B_2)], \\
 \chi_2 & = \lambda_M(P_2) + 2R_M(M - N) + \left(\frac{1 - e^{-2kh}}{2k}\right) \\
 & \quad \times [\lambda_M(Q_2) + \lambda_M(Q_4)(M - N) + \lambda_M(W_2)] \\
 & \quad + \frac{3h^2}{2} \lambda_M(Z_2) \lambda_M(D^T D) \\
 & \quad + \frac{d^2}{2} \lambda_M(Z_1) [3\lambda_M(A_1^T A_1) + 3\lambda_M(A_2^T A_2)].
 \end{aligned} \tag{34}$$

On the other hand, we have

$$V(t) \geq e^{2kt} \{ \lambda_m(P_1) \|x(t)\|^2 + \lambda_m(P_2) \|y(t)\|^2 \}. \tag{35}$$

Therefore,

$$\|x(t)\|^2 + \|y(t)\|^2 \leq \rho e^{-2kt} \{ \|\phi\|^2 + \|\varphi\|^2 \}, \tag{36}$$

where $\rho = \max(\chi_1, \chi_2) / \min(\lambda_m(P_1), \lambda_m(P_2)) \geq 1$. Therefore, the system (7) is global exponentially stable with the convergent rate $k > 0$. This completes the proof. \square

Remark 6. In hypothesis (H2), lower bounds E_i, N_j and upper bounds $F_i, M_j, i = 1, 2, \dots, n, j = 1, 2, \dots, m$, of activation functions are allowed to be either positive, negative, or zero. Clearly, hypothesis (H2) in our paper is more general than those given in [14, 15, 18, 21, 22, 24–27]. Hence, our result is less conservative than some existing results given in the literature.

4. Numerical Examples

Example 1. Consider the BAM neural networks in (7) with $C = \text{diag}(1, 1, 1), D = \text{diag}(2, 2, 2), A_1 = B_1 = 0,$

$$\begin{aligned}
 A_2 & = \begin{bmatrix} 0.05 & 0.25 & 0.05 \\ 0.1 & 0.05 & 0.15 \\ 0.15 & 0.15 & 0.05 \end{bmatrix}, \\
 B_2 & = \begin{bmatrix} 0.75 & 0.75 & 0.95 \\ 0 & 0.50 & 0.15 \\ 0.15 & 0.15 & 0.05 \end{bmatrix}.
 \end{aligned} \tag{37}$$

In this example, the activation function and time delay are given as follows: $f_1(x) = f_2(x) = f_3(x) = (1/2)(|x + 1| - |x - 1|), g_1(y) = g_2(y) = g_3(y) = (1/2)(|y + 1| - |y - 1|), d(t) = 0.5, h(t) = 1$. It follows that $d = 0.5, \tau = 0.3, h = 1,$ and $\mu = 0.3$. The assumption (H2) is satisfied with $F = \text{diag}(1, 1, 1), M = \text{diag}(1, 1, 1), E = N = 0$. Let $k = 0.1$. By using the LMI Toolbox in MATLAB, the LMI (16) of Theorem 5 is feasible with $k = 0.1$ and a set of solutions of (16) is given by

$$\begin{aligned}
 P_1 & = \begin{bmatrix} 28.6382 & 0.6843 & 0.7501 \\ 0.6843 & 28.2061 & 1.0981 \\ 0.7501 & 1.0981 & 27.7326 \end{bmatrix}, \\
 P_2 & = \begin{bmatrix} 11.7631 & 0.5671 & 0.4170 \\ 0.5671 & 25.8296 & 0.7934 \\ 0.4170 & 0.7934 & 26.2860 \end{bmatrix}, \\
 Q_1 & = \begin{bmatrix} 14.4157 & -0.0495 & -0.0610 \\ -0.0495 & 14.4217 & -0.0911 \\ -0.0610 & -0.0911 & 14.4462 \end{bmatrix}, \\
 Q_2 & = \begin{bmatrix} 12.8030 & -0.9294 & -0.9704 \\ -0.9294 & 16.9982 & -0.4287 \\ -0.9704 & -0.4287 & 17.2419 \end{bmatrix}, \\
 Q_3 & = \begin{bmatrix} 16.7725 & 0.4495 & 0.2528 \\ 0.4495 & 15.0669 & 0.5969 \\ 0.2528 & 0.5969 & 13.1708 \end{bmatrix}, \\
 Q_4 & = \begin{bmatrix} 11.9146 & -0.2010 & -0.3870 \\ -0.2010 & 15.1767 & -0.6757 \\ -0.3870 & -0.6757 & 15.3739 \end{bmatrix}, \\
 W_1 & = \begin{bmatrix} 14.6218 & 1.0858 & 0.9997 \\ 1.0858 & 13.0260 & 1.3001 \\ 0.9997 & 1.3001 & 11.6739 \end{bmatrix}, \\
 W_2 & = \begin{bmatrix} 8.4065 & -0.0639 & -0.2523 \\ -0.0639 & 11.9959 & 0.1677 \\ -0.2523 & 0.1677 & 11.6957 \end{bmatrix}, \\
 Z_1 & = \begin{bmatrix} 13.3993 & 0.2142 & 0.3899 \\ 0.2142 & 13.5746 & 0.3941 \\ 0.3899 & 0.3941 & 13.7729 \end{bmatrix}, \\
 Z_2 & = \begin{bmatrix} 2.3511 & 0.7468 & 0.6736 \\ 0.7468 & 11.2627 & 0.3500 \\ 0.6736 & 0.3500 & 11.7724 \end{bmatrix},
 \end{aligned}$$

$$\begin{aligned}
 S_{11} &= \begin{bmatrix} 12.4617 & -0.0242 & -0.0266 \\ -0.0242 & 12.4732 & -0.0399 \\ -0.0266 & -0.0399 & 12.4949 \end{bmatrix}, \\
 S_{12} &= \begin{bmatrix} -1.1535 & -0.0211 & -0.0257 \\ -0.0211 & -1.1444 & -0.0335 \\ -0.0257 & -0.0335 & -1.1320 \end{bmatrix}, \\
 S_{13} &= \begin{bmatrix} -0.9075 & -0.0415 & -0.0544 \\ -0.0415 & -0.9093 & -0.0805 \\ -0.0544 & -0.0805 & -0.9019 \end{bmatrix}, \\
 S_{22} &= \begin{bmatrix} 12.2314 & -0.0115 & -0.0101 \\ -0.0115 & 12.2463 & -0.0163 \\ -0.0101 & -0.0163 & 12.2626 \end{bmatrix}, \\
 S_{23} &= \begin{bmatrix} 1.6306 & 0.0129 & 0.0208 \\ 0.0129 & 1.6430 & 0.0270 \\ 0.0208 & 0.0270 & 1.6490 \end{bmatrix}, \\
 T_{11} &= \begin{bmatrix} 8.6800 & -0.5650 & -0.6089 \\ -0.5650 & 12.0158 & -0.3052 \\ -0.6089 & -0.3052 & 12.2653 \end{bmatrix}, \\
 T_{12} &= \begin{bmatrix} -1.8596 & -0.2702 & -0.2795 \\ -0.2702 & -1.2902 & -0.1294 \\ -0.2795 & -0.1294 & -1.2317 \end{bmatrix}, \\
 T_{13} &= \begin{bmatrix} -1.7089 & -0.3795 & -0.4069 \\ -0.3795 & -0.6642 & -0.2437 \\ -0.4069 & -0.2437 & -0.5078 \end{bmatrix}, \\
 T_{22} &= \begin{bmatrix} 8.2813 & -0.2103 & -0.2276 \\ -0.2103 & 10.2975 & -0.1027 \\ -0.2276 & -0.1027 & 10.4221 \end{bmatrix}, \\
 T_{23} &= \begin{bmatrix} 1.9707 & 0.1871 & 0.1870 \\ 0.1871 & 2.3062 & 0.0783 \\ 0.1870 & 0.0783 & 2.3346 \end{bmatrix}, \\
 K &= \begin{bmatrix} 14.9427 & 0 & 0 \\ 0 & 14.2734 & 0 \\ 0 & 0 & 13.4467 \end{bmatrix}, \\
 R &= \begin{bmatrix} 5.8315 & 0 & 0 \\ 0 & 8.1963 & 0 \\ 0 & 0 & 8.3562 \end{bmatrix}.
 \end{aligned} \tag{38}$$

Thus, the system (7) is 0.1-exponentially stable and the value $\rho = 13.4606$. The solution of the closed-loop system satisfies

$$\begin{aligned}
 &\|x(t)\|^2 + \|y(t)\|^2 \\
 &\leq 13.4606e^{-2(0.1)t} \{ \|\phi\|^2 + \|\varphi\|^2 \}, \quad \forall t \in \mathbb{R}^+.
 \end{aligned} \tag{39}$$

By applying Theorem 5 and by solving the LMI (16) using MATLAB LMI Toolbox, we obtain the convergence rate k which guarantees that the global exponential stability is 0.998. In Table 1, we give comparison of maximum allowable convergence rate k obtained by Theorem 5 and by other

TABLE 1: Maximum allowable convergence rate.

| $\tau = \mu$ | 0 | 0.3 | 0.5 | 0.7 | 0.9 |
|--------------|-------|-------|-------|-------|-------|
| [24] | — | 0.459 | 0.455 | 0.455 | 0.455 |
| [29] | — | 0.445 | 0.424 | 0.408 | 0.407 |
| [25] | — | 0.52 | 0.47 | 0.39 | 0.21 |
| This paper | 0.998 | 0.998 | 0.998 | 0.998 | 0.998 |

methods in some previous existing results. From Table 1, it is shown that the proposed global exponential stability criterion is less conservative than those obtained in [24, 25, 29].

Example 2. Consider the BAM neural networks in (7) with

$$\begin{aligned}
 C &= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \\
 A_1 &= \begin{bmatrix} -0.72 & -0.44 & -0.21 \\ -0.72 & -0.83 & -0.1 \\ -0.01 & 0.01 & -0.04 \end{bmatrix}, \\
 A_2 &= \begin{bmatrix} -0.01 & -0.12 & -0.24 \\ 0.17 & -0.33 & -0.43 \\ -0.25 & 0.33 & -0.05 \end{bmatrix}, \\
 D &= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \\
 B_1 &= \begin{bmatrix} -0.31 & -0.31 & 0.92 \\ 0.34 & -0.33 & -0.78 \\ 0.34 & 0.47 & 0.25 \end{bmatrix}, \\
 B_2 &= \begin{bmatrix} -0.83 & -0.12 & -0.52 \\ -0.65 & 0.5 & -0.14 \\ -0.05 & -0.14 & -0.65 \end{bmatrix}.
 \end{aligned} \tag{40}$$

In this example, the activation function and time-varying delay are given as follows: $f_1(x) = f_2(x) = f_3(x) = (1/2)(|x + 1| - |x - 1|)$, $g_1(y) = g_2(y) = g_3(y) = (1/2)(|y + 1| - |y - 1|)$, $d(t) = \sin^2(0.5t)$, $h(t) = 0.1\cos^2(t)$; the assumption (H2) is satisfied with $E = \text{diag}(-0.2, -0.25, -0.2)$, $F = \text{diag}(0.3, 0.4, 0.5)$, $N = \text{diag}(-0.2, -0.25, -0.2)$, $M = \text{diag}(0.3, 0.4, 0.5)$. It follows that $d = 0.5$, $\tau = 0.3$, $h = 0.1$, and $\mu = 0.3$. Let $k = 0.1$. By using the LMI Toolbox in MATLAB, the LMI (16) is feasible with $k = 0.1$ and a set of solutions of (16) is given by

$$\begin{aligned}
 P_1 &= \begin{bmatrix} 110.3704 & -30.1135 & -5.4152 \\ -30.1135 & 90.3013 & 8.8948 \\ -5.4152 & 8.8948 & 115.0609 \end{bmatrix}, \\
 P_2 &= \begin{bmatrix} 43.9625 & -2.3031 & 7.3341 \\ -2.3031 & 38.1671 & 4.9969 \\ 7.3341 & 4.9969 & 62.3183 \end{bmatrix},
 \end{aligned}$$

$$Q_1 = \begin{bmatrix} 125.4489 & -1.8802 & 0.9399 \\ -1.8802 & 127.5495 & -1.0063 \\ 0.9399 & -1.0063 & 130.0399 \end{bmatrix},$$

$$Q_2 = \begin{bmatrix} 100.3079 & -4.1254 & -0.7526 \\ -4.1254 & 101.8490 & 1.0897 \\ -0.7526 & 1.0897 & 114.9297 \end{bmatrix},$$

$$Q_3 = \begin{bmatrix} 136.0956 & -4.2919 & 38.6712 \\ -4.2919 & 90.4808 & 19.2022 \\ 38.6712 & 19.2022 & 113.1213 \end{bmatrix},$$

$$Q_4 = \begin{bmatrix} 42.6986 & -34.8491 & -15.2241 \\ -34.8491 & 96.7728 & 2.0801 \\ -15.2241 & 2.0801 & 145.8684 \end{bmatrix},$$

$$W_1 = \begin{bmatrix} 140.8059 & 25.3914 & -0.9357 \\ 25.3914 & 146.4301 & -4.4760 \\ -0.9357 & -4.4760 & 11.6128 \end{bmatrix},$$

$$W_2 = \begin{bmatrix} 105.4681 & -0.1405 & 5.1590 \\ -0.1405 & 107.4419 & 16.7637 \\ 5.1590 & 16.7637 & 134.1991 \end{bmatrix},$$

$$Z_1 = \begin{bmatrix} 69.1695 & -24.4444 & -4.5155 \\ -24.4444 & 48.7177 & 7.8545 \\ -4.5155 & 7.8545 & 65.4657 \end{bmatrix},$$

$$Z_2 = \begin{bmatrix} 99.9490 & 1.3357 & 35.2470 \\ 1.3357 & 80.2179 & 10.6682 \\ 35.2470 & 10.6682 & 111.8243 \end{bmatrix},$$

$$S_{11} = \begin{bmatrix} 103.3199 & -2.9895 & 0.4043 \\ -2.9895 & 103.2019 & -0.1404 \\ 0.4043 & -0.1404 & 106.3073 \end{bmatrix},$$

$$S_{12} = \begin{bmatrix} -2.9235 & -0.6314 & 0.1584 \\ -0.6314 & -2.4889 & -0.1672 \\ 0.1584 & -0.1672 & -1.9572 \end{bmatrix},$$

$$S_{13} = \begin{bmatrix} 1.0569 & -2.4649 & 0.0237 \\ -2.4649 & 0.4090 & 0.2113 \\ 0.0237 & 0.2113 & 2.4210 \end{bmatrix},$$

$$S_{22} = \begin{bmatrix} 97.0839 & -2.1772 & 0.0865 \\ -2.1772 & 96.3284 & 0.1980 \\ 0.0865 & 0.1980 & 98.4407 \end{bmatrix},$$

$$S_{23} = \begin{bmatrix} 8.9358 & -1.7429 & -0.3501 \\ -1.7429 & 7.0242 & 0.7119 \\ -0.3501 & 0.7119 & 8.3850 \end{bmatrix},$$

$$T_{11} = \begin{bmatrix} 105.0332 & -0.4193 & 0.0331 \\ -0.4193 & 105.2266 & 0.3043 \\ 0.0331 & 0.3043 & 107.5064 \end{bmatrix},$$

$$T_{12} = \begin{bmatrix} -1.5261 & -0.6096 & -0.1149 \\ -0.6096 & -1.2415 & 0.0894 \\ -0.1149 & 0.0894 & 0.3340 \end{bmatrix},$$

TABLE 2: Maximum allowable convergence rate.

| $\tau = \mu$ | 0 | 0.3 | 0.5 | 0.7 |
|----------------------|------|-------|------|-------|
| Convergence rate k | 0.81 | 0.638 | 0.48 | 0.221 |

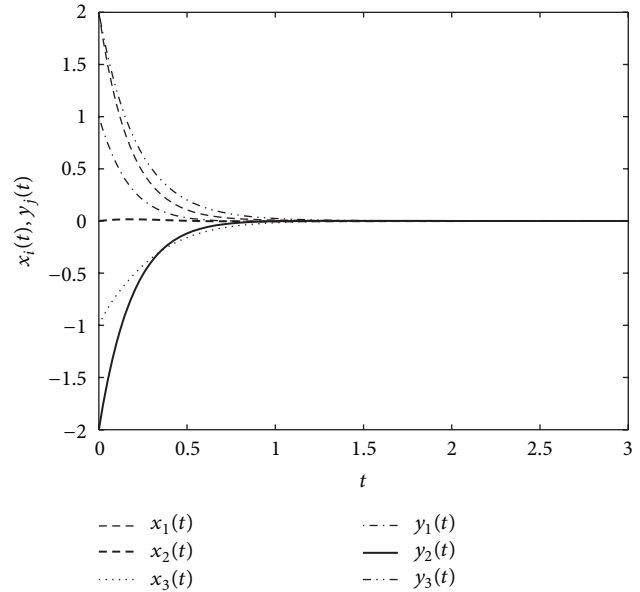


FIGURE 1: Time responses of state variables.

$$T_{13} = \begin{bmatrix} -7.6562 & -3.2414 & -0.6934 \\ -3.2414 & -5.7650 & 0.5527 \\ -0.6934 & 0.5527 & 4.8321 \end{bmatrix},$$

$$T_{22} = \begin{bmatrix} 104.9648 & -0.5090 & -0.0471 \\ -0.5090 & 105.1940 & 0.0964 \\ -0.0471 & 0.0964 & 106.5444 \end{bmatrix},$$

$$T_{23} = \begin{bmatrix} 2.6072 & 0.1420 & 0.2077 \\ 0.1420 & 2.6111 & -0.0404 \\ 0.2077 & -0.0404 & 2.2053 \end{bmatrix},$$

$$K = \begin{bmatrix} 106.5957 & 0 & 0 \\ 0 & 63.3921 & 0 \\ 0 & 0 & 6.3380 \end{bmatrix},$$

$$R = \begin{bmatrix} 86.0059 & 0 & 0 \\ 0 & 72.0231 & 0 \\ 0 & 0 & 71.2919 \end{bmatrix}.$$

(41)

Thus, the system (7) is 0.1-exponentially stable and the value $\rho = 14.1811$. The solution of the closed-loop system satisfies

$$\begin{aligned} & \|x(t)\|^2 + \|y(t)\|^2 \\ & \leq 14.1811e^{-2(0.1)t} \{ \|\phi\|^2 + \|\varphi\|^2 \}, \quad \forall t \in \mathbb{R}^+. \end{aligned} \tag{42}$$

The maximum allowable convergence rate k for different values of $\tau = \mu$ is given in Table 2. The trajectory of solutions of BAM neural networks with time-varying delays is shown

in Figure 1, where the initial conditions are chosen as $\phi_1 = \cos(s)$, $\phi_2 = \sin(s)$, $\phi_3 = \sin(s) - 1$, $\varphi_1 = \cos(s) + 1$, $\varphi_2 = \sin(s) - 2$, $\varphi_3 = \cos(s) + 1$, $s \in [-0.5, 0]$.

5. Conclusion

This paper has proposed a new sufficient condition guaranteeing the global exponential stability criteria for bidirectional associative memory neural networks with time-varying delays and generalized activation functions. The developed stability condition is in terms of LMI, which can be easily solved by some existing software packages. Furthermore, the proposed stability conditions are less conservative than some works in the literature.

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