

## Research Article

# Strong Convergence of Iterative Algorithm for a New System of Generalized $H(\cdot, \cdot) - \eta$ -Cocoercive Operator Inclusions in Banach Spaces

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We introduce and study a new system of generalized  $H(\cdot, \cdot) - \eta$ -cocoercive operator inclusions in Banach spaces. Using the resolvent operator technique associated with  $H(\cdot, \cdot) - \eta$ -cocoercive operators, we suggest and analyze a new generalized algorithm of nonlinear set-valued variational inclusions and establish strong convergence of iterative sequences produced by the method. We highlight the applicability of our results by examples in function spaces.

## 1. Introduction

The resolvent operator technique is a powerful tool to study the approximation solvability of nonlinear variational inequalities and variational inclusions, which have been applied widely to optimization and control, mechanics and physics, economics and transportation equilibrium, and engineering sciences, see, for example, [1–4] and the references therein.

In a series of papers [5–8], the authors investigated  $(A, \eta)$ -accretive and  $H(\cdot, \cdot)$ -accretive operators for solving variational inclusions in Banach spaces. Convergence and stability of iterative algorithms for the systems of  $(A, \eta)$ -accretive operators have been studied in [9, 10]. The notion of  $(H, \phi) - \eta$ -monotone operators has been introduced and investigated by the authors in [11]. Generalized mixed variational inclusions involving  $(H(\cdot, \cdot), \eta)$ -monotone operators have been discussed in [12]. Some results on  $H((\cdot, \cdot), \eta)$ -accretive operators and application for solving set-valued variational inclusions in Banach spaces have been proved in [7]. Some other related articles on the variational inclusion problems can be found in [13–22].

Very recently, Ahmad et al. [23] introduced a new  $H(\cdot, \cdot) - \eta$ -cocoercive operator and its resolvent operator in the setting of Banach spaces. The authors proposed concrete examples in support of  $H(\cdot, \cdot) - \eta$ -cocoercive operators and they also proved the Lipschitz continuity of resolvent operator associated with  $H(\cdot, \cdot) - \eta$ -cocoercive operator. Motivated and inspired by the research works mentioned above, in this paper, we introduce and study a new system of  $H(\cdot, \cdot) - \eta$ -cocoercive mapping inclusions in Banach spaces. Using the resolvent operator associated with  $H(\cdot, \cdot) - \eta$ -cocoercive mapping, we suggest and analyze a new general algorithm and establish the existence and uniqueness of solutions for this system of  $H(\cdot, \cdot) - \eta$ -cocoercive mappings.

## 2. Preliminaries

Throughout this paper, we denote the set of positive integers by  $\mathbb{N}$ . Let  $X$  be a Banach space with the norm  $\|\cdot\|$  and the dual space  $X^*$ . For any  $x \in X$ , we denote the value of  $x^* \in X^*$  at  $x$  by  $\langle x, x^* \rangle$ . When  $\{x_n\}$  is a sequence in  $X$ , we denote the strong convergence of  $\{x_n\}$  to  $x \in X$  by  $x_n \rightarrow x$  as  $n \rightarrow \infty$ . We denote by  $2^X$  the family of all nonempty subsets of  $X$ . Let

$CB(X)$  be the family of all nonempty, closed, and bounded subsets of  $X$ . The Hausdörff metric on  $CB(X)$  [24] is defined by

$$D(A, B) = \max \left\{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(A, y) \right\}, \quad (1)$$

$$A, B \in CB(X),$$

where  $d(x, B) = \inf_{b \in B} \|x - b\|$  and  $d(A, y) = \inf_{a \in A} \|a - y\|$ .

*Definition 1* (see [25]). A continuous and strictly increasing function  $\phi : [0, +\infty) \rightarrow [0, \infty)$  such that  $\phi(0) = 0$  and  $\lim_{t \rightarrow \infty} \phi(t) = \infty$  is called a gauge function.

*Definition 2* (see [25]). Let  $X$  be a Banach space. Given a gauge function  $\phi$ , the mapping  $J_\phi : X \rightarrow 2^{X^*}$  corresponding to  $\phi$  defined by

$$J_\phi(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\| \|x^*\|, \quad (2)$$

$$\|x^*\| = \phi(\|x\|)\}, \quad \forall x \in X,$$

is called the duality mapping with gauge function  $\phi$ .

In particular, if  $\phi(t) = t$ , the duality map  $J = J_\phi$  is called the normalized duality mapping.

**Lemma 3** (see [26]). Let  $X$  be a real Banach space and  $J : X \rightarrow 2^{X^*}$  be the normalized duality mapping. Then, for any  $x, y \in X$ ,

$$\|x + y\|^2 \leq \|x\|^2 + 2 \langle y, j(x + y) \rangle, \quad (3)$$

for all  $j(x + y) \in J(x + y)$ .

*Definition 4.* Let  $X$  be a Banach space. Let  $A : X \rightarrow X$  and  $\eta : X \times X \rightarrow X$  be two mappings and  $J : X \rightarrow 2^{X^*}$  be the normalized duality mapping. Then,  $A$  is called

(i)  $\eta$ -cocoercive, if there exists a constant  $\mu_1 > 0$  such that

$$\langle Ax - Ay, j(\eta(x, y)) \rangle \geq \mu_1 \|Ax - Ay\|^2, \quad \forall x, y \in X, \quad (4)$$

$$j(\eta(x, y)) \in J(\eta(x, y)),$$

(ii)  $\eta$ -accretive, if

$$\langle Ax - Ay, j(\eta(x, y)) \rangle \geq 0, \quad \forall x, y \in X, \quad (5)$$

$$j(\eta(x, y)) \in J(\eta(x, y)),$$

(iii)  $\eta$ -strongly accretive, if there exists a constant  $\beta_1 > 0$  such that

$$\langle Ax - Ay, j(\eta(x, y)) \rangle \geq \beta_1 \|x - y\|^2, \quad \forall x, y \in X, \quad (6)$$

$$j(\eta(x, y)) \in J(\eta(x, y)),$$

(iv)  $\eta$ -relaxed cocoercive, if there exists a constant  $\gamma_1 > 0$  such that

$$\langle Ax - Ay, j(\eta(x, y)) \rangle \geq (-\gamma_1) \|Ax - Ay\|^2, \quad \forall x, y \in X, \quad (7)$$

$$j(\eta(x, y)) \in J(\eta(x, y)),$$

(v) Lipschitz continuous, if there exists a constant  $\lambda_A > 0$  such that

$$\|Ax - Ay\| \leq \lambda_A \|x - y\|, \quad \forall x, y \in X, \quad (8)$$

(vi)  $\alpha$ -expansive, if there exists a constant  $\alpha > 0$  such that

$$\|Ax - Ay\| \geq \alpha \|x - y\|, \quad \forall x, y \in X, \quad (9)$$

(vii)  $\eta$  is said to be Lipschitz continuous, if there exists a constant  $\tau > 0$  such that

$$\|\eta(x, y)\| \leq \tau \|x - y\|, \quad \forall x, y \in X. \quad (10)$$

*Definition 5.* Let  $X$  be a Banach space. Let  $A, B : X \rightarrow X, H : X \times X \rightarrow X, \eta : X \times X \rightarrow X$  be four single-valued mappings and  $J : X \rightarrow 2^{X^*}$  be the normalized duality mapping. Then,

(i)  $H(A, \cdot)$  is said to be  $\eta$ -cocoercive with respect to  $A$ , if there exists a constant  $\mu > 0$  such that

$$\langle H(Ax, u) - H(Ay, u), j(\eta(x, y)) \rangle$$

$$\geq \mu \|Ax - Ay\|^2, \quad \forall x, y, u \in X, \quad (11)$$

$$j(\eta(x, y)) \in J(\eta(x, y)),$$

(ii)  $H(\cdot, B)$  is said to be  $\eta$ -relaxed cocoercive with respect to  $B$ , if there exists a constant  $\gamma > 0$  such that

$$\langle H(u, Bx) - H(u, By), j(\eta(x, y)) \rangle$$

$$\geq (-\gamma) \|Bx - By\|^2, \quad \forall x, y, u \in X, \quad (12)$$

$$j(\eta(x, y)) \in J(\eta(x, y)),$$

(iii)  $H(A, \cdot)$  is said to be  $r_1$ -Lipschitz continuous with respect to  $A$ , if there exists a constant  $r_1 > 0$  such that

$$\|H(Ax, u) - H(Ay, u)\| \leq r_1 \|x - y\|, \quad \forall x, y, u \in X, \quad (13)$$

(iv)  $H(\cdot, B)$  is said to be  $r_2$ -Lipschitz continuous with respect to  $B$ , if there exists a constant  $r_2 > 0$  such that

$$\|H(u, Bx) - H(u, By)\| \leq r_2 \|x - y\|, \quad \forall x, y, u \in X. \quad (14)$$

*Definition 6.* Let  $X$  be a Banach space. A set-valued mapping  $M : X \rightarrow 2^X$  is said to be  $\eta$ -cocoercive, if there exists a constant  $\mu_2 > 0$  such that

$$\langle u - v, j(\eta(x, y)) \rangle \geq \mu_2 \|u - v\|^2, \quad \forall x, y \in X,$$

$$u \in M(x), \quad v \in M(y), \quad j(\eta(x, y)) \in J(\eta(x, y)). \quad (15)$$

*Definition 7.* Let  $X$  be a Banach space. A mapping  $T : X \rightarrow CB(X)$  is said to be  $\mathcal{D}$ -Lipschitz continuous, if there exists a constant  $\lambda_T > 0$  such that

$$\mathcal{D}(Tx, Ty) \leq \lambda_T \|x - y\|, \quad x, y \in X. \quad (16)$$

*Definition 8.* Let  $X$  be a Banach space. Let  $T, Q : X \rightarrow CB(X)$  be the mappings. A mapping  $N : X \times X \rightarrow X$  is said to be

(i) Lipschitz continuous in the first argument with respect to  $T$ , if there exists a constant  $t_1 > 0$  such that

$$\|N(w_1, \cdot) - N(w_2, \cdot)\| \leq t_1 \|w_1 - w_2\|, \quad \forall u_1, u_2 \in X, \\ w_1 \in T(u_1), \quad w_2 \in T(u_2), \quad (17)$$

(ii) Lipschitz continuous in the second argument with respect to  $Q$ , if there exists a constant  $t_2 > 0$  such that

$$\|N(\cdot, v_1) - N(\cdot, v_2)\| \leq t_2 \|v_1 - v_2\|, \quad \forall u_1, u_2 \in X, \\ v_1 \in Q(u_1), \quad v_2 \in Q(u_2), \quad (18)$$

(iii)  $\eta$ -relaxed Lipschitz in the first argument with respect to  $T$ , if there exists a constant  $\tau_1 > 0$  such that

$$\langle N(w_1, \cdot) - N(w_2, \cdot), j(\eta(u_1, u_2)) \rangle \\ \leq (-\tau_1) \|u_1 - u_2\|^2, \quad \forall u_1, u_2 \in X, \\ w_1 \in T(u_1), \quad w_2 \in T(u_2), \\ j(\eta(u_1, u_2)) \in J(\eta(u_1, u_2)), \quad (19)$$

(iv)  $\eta$ -relaxed Lipschitz in the second argument with respect to  $Q$ , if there exists a constant  $\tau_2 > 0$  such that

$$\langle N(\cdot, v_1) - N(\cdot, v_2), j(\eta(u_1, u_2)) \rangle \\ \leq (-\tau_2) \|u_1 - u_2\|^2, \quad \forall u_1, u_2 \in X, \quad v_1 \in Q(u_1), \quad (20) \\ v_2 \in Q(u_2), \quad j(\eta(u_1, u_2)) \in J(\eta(u_1, u_2)).$$

*Definition 9.* Let  $X$  be a Banach space. Let  $A, B : X \rightarrow X$ ,  $H : X \times X \rightarrow X$ ,  $\eta : X \times X \rightarrow X$  be four single-valued mappings. Let  $M : X \rightarrow 2^X$  be a set-valued mapping.  $M$  is said to be  $H(\cdot, \cdot) - \eta$ -cocoercive operator with respect to  $A$  and  $B$ , if  $M$  is  $\eta$ -cocoercive and  $(H(A, B) + \lambda M)(X) = X$ , for every  $\lambda > 0$ .

*Example 10.* Let  $X = \mathbb{R} \times \mathbb{R}$  and  $A, B : X \rightarrow X$  be defined by

$$A(x_1, x_2) = (2x_1 - x_2, x_1 - x_2), \\ B(y_1, y_2) = (-2y_2, y_1 - y_2), \quad \forall (x_1, x_2), (y_1, y_2) \in X. \quad (21)$$

Assume now that  $H(A, B), \eta : X \times X \rightarrow X$  are defined by

$$H(Ax, By) = Ax + By, \quad \eta(x, y) = x - y, \quad \forall x, y \in X. \quad (22)$$

Let  $M = I$ , where  $I$  is the identity mapping. Then,  $M$  is  $H(\cdot, \cdot) - \eta$ -cocoercive with respect to  $A$  and  $B$ .

*Example 11.* Let  $X = C[0, 1]$ , the space of all real valued continuous functions defined on closed interval  $[0, 1]$  with the norm

$$\|f\| = \max_{t \in [0, 1]} |f(t)|. \quad (23)$$

Let  $A, B : X \rightarrow X$  be defined by

$$A(f) = \sin^2(f), \quad B(g) = \cos^2(g), \quad \forall f, g \in X. \quad (24)$$

Let  $H(A, B) : X \times X \rightarrow X$  be defined by

$$H(A(f), B(g)) = A(f) + B(g), \quad \forall f, g \in X. \quad (25)$$

Suppose that  $M(f) = f^2$ , where  $f^2(t) = f(t)f(t)$  for all  $t \in [0, 1]$ . Then, for  $\lambda = 1$ , we conclude that

$$\| (H(A, B) + M)(f) \| = \| A(f) + B(f) + f^2 \| \\ = \max_{t \in [0, 1]} | \sin^2(f(t)) + \cos^2(f(t)) + f^2(t) | \\ = 1 + f^2(t) > 0. \quad (26)$$

This proves that  $0 \notin (H(A, B) + M)(X)$  and  $M$  is not  $H(A, B) - \eta$ -cocoercive with respect to  $A$  and  $B$ .

**Proposition 12** (see [23]). *Let  $H(A, B)$  be  $\eta$ -cocoercive with respect to  $A$  with constant  $\mu > 0$  and  $\eta$ -relaxed cocoercive with respect to  $B$  with constant  $\gamma > 0$ ,  $A$  be  $\alpha$ -expansive and  $B$  be  $\beta$ -Lipschitz continuous  $\mu > \gamma$  and  $\alpha > \beta$ . Let  $M : X \rightarrow 2^X$  be  $H(A, B) - \eta$ -cocoercive operator. Suppose that*

$$\langle x - y, j(\eta(u, v)) \rangle \geq 0, \quad \forall (v, y) \in \text{Graph}(M), \\ j(\eta(u, v)) \in J(\eta(u, v)). \quad (27)$$

Then,  $x \in Mu$ , where  $\text{Graph}(M) = \{(u, x) \in X \times X : x \in Mu\}$ .

**Theorem 13** (see [23]). *Let  $H(A, B)$  be  $\eta$ -cocoercive with respect to  $A$  with constant  $\mu > 0$  and  $\eta$ -relaxed cocoercive with respect to  $B$  with constant  $\gamma > 0$ ,  $A$  be  $\alpha$ -expansive and  $B$  be  $\beta$ -Lipschitz continuous,  $\mu > \gamma$  and  $\alpha > \beta$ . Let  $M$  be an  $H(\cdot, \cdot) - \eta$ -cocoercive operator with respect to  $A$  and  $B$ . Then, for each  $\lambda > 0$ , the operator  $(H(A, B) + \lambda M)^{-1}$  is single-valued.*

*Definition 14.* Let  $X$  be a Banach space. Let  $H(A, B)$  be  $\eta$ -cocoercive with respect to  $A$  with constant  $\mu > 0$  and  $\eta$ -relaxed cocoercive with respect to  $B$  with constant  $\gamma > 0$ ,  $A$  be  $\alpha$ -expansive  $B$  be  $\beta$ -Lipschitz continuous and  $\eta$  be  $\beta$ -Lipschitz continuous,  $\mu > \gamma$ , and  $\alpha > \beta$ . Let  $M$  be a  $H(\cdot, \cdot) - \eta$ -cocoercive operator with respect to  $A$  and  $B$ . Then, the resolvent  $R_{\lambda, M}^{H(\cdot, \cdot) - \eta} : X \rightarrow X$  is defined by

$$R_{\lambda, M}^{H(\cdot, \cdot) - \eta}(u) = (H(A, B) + \lambda M)^{-1}(u), \quad \forall u \in X. \quad (28)$$

**Theorem 15** (see [23]). *Let  $X$  be a Banach space. Let  $H(A, B)$  be  $\eta$ -cocoercive with respect to  $A$  with constant  $\mu > 0$  and  $\eta$ -relaxed cocoercive with respect to  $B$  with constant  $\gamma > 0$ ,  $A$  be  $\alpha$ -expansive  $B$  be  $\beta$ -Lipschitz continuous, and  $\eta$  be  $\rho$ -Lipschitz continuous;  $\mu > \gamma$  and  $\alpha > \beta$ . Let  $M$  be  $H(\cdot, \cdot) - \eta$ -cocoercive*

operator with respect to  $A$  and  $B$ . Then, the resolvent operator  $R_{\lambda, M}^{H(\cdot, \cdot) - \eta} : X \rightarrow X$  is  $\rho/(\mu\alpha^2 - \gamma\beta^2)$ -Lipschitz continuous, that is,

$$\begin{aligned} & \left\| R_{\lambda, M}^{H(\cdot, \cdot) - \eta}(u) - R_{\lambda, M}^{H(\cdot, \cdot) - \eta}(v) \right\| \\ & \leq \frac{\rho}{\mu\alpha^2 - \gamma\beta^2} \|u - v\|, \quad \forall u, v \in X. \end{aligned} \tag{29}$$

### 3. Strong Convergence Theorem

In this section, using the resolvent operator technique associated with  $H(\cdot, \cdot) - \eta$ -cocoercive operators, we propose a new generalized algorithm of nonlinear set-valued variational inclusions and establish strong convergence of iterative sequences produced by the method.

For  $i = 1, 2$ , let  $X_i$  be real Banach spaces with the norm  $\|\cdot\|_i$ . Let  $A_i, B_i : X_i \rightarrow X_i, H_i : X_i \times X_i \rightarrow X_i, \eta_i : X_i \times X_i \rightarrow X_i, F : X_1 \times X_2 \rightarrow X_1$ , and  $G : X_1 \times X_2 \rightarrow X_2$  be single-valued mappings, and  $T : X_1 \rightarrow CB(X_1), Q : X_2 \rightarrow CB(X_2)$  be set-valued mappings. Let  $M : X_1 \times X_1 \rightarrow 2^{X_1}, N : X_2 \times X_2 \rightarrow 2^{X_2}$  be  $H_1(\cdot, \cdot) - \eta_1$ -cocoercive and  $H_2(\cdot, \cdot) - \eta_2$ -cocoercive operators with respect to  $(A_1, B_1)$  and  $(A_2, B_2)$ , respectively. We consider the following problem.

Find  $(x, y) \in X_1 \times X_2, w \in T(x),$  and  $v \in Q(y)$  such that

$$\begin{aligned} 0 & \in M(x, x) + F(w, v), \\ 0 & \in N(y, y) + G(w, v). \end{aligned} \tag{30}$$

We call problem (30) a system of generalized  $H(\cdot, \cdot) - \eta$ -cocoercive operator inclusions.

Under the assumptions mentioned above, we have the following key and simple lemma.

**Lemma 16.**  $(x, y) \in X_1 \times X_2, w \in T(x), v \in Q(y)$  is a solution of problem (30) if and only if

$$\begin{aligned} x & = R_{\lambda_1, M(\cdot, x)}^{H_1(\cdot, \cdot) - \eta_1} [H_1(A_1x, B_1x) - \lambda_1 F(w, v)], \\ y & = R_{\lambda_2, N(\cdot, y)}^{H_2(\cdot, \cdot) - \eta_2} [H_2(A_2y, B_2y) - \lambda_2 G(w, v)], \end{aligned} \tag{31}$$

where  $R_{\lambda_1, M(\cdot, x)}^{H_1(\cdot, \cdot) - \eta_1} = (H_1(A_1x, B_1x) + \lambda_1 M(\cdot, x))^{-1}, R_{\lambda_1, N(\cdot, y)}^{H_2(\cdot, \cdot) - \eta_2} = (H_2(A_2y, B_2y) + \lambda_2 N(\cdot, y))^{-1}$ , and  $\lambda_1, \lambda_2 > 0$  are constants.

*Proof.* This is an easy and direct consequence of Definition 14.  $\square$

*Algorithm 17.* For  $i = 1, 2$ , let  $X_i$  be real Banach spaces with the norm  $\|\cdot\|_i$ . Let  $A_i, B_i : X_i \rightarrow X_i, H_i : X_i \times X_i \rightarrow X_i, \eta_i : X_i \times X_i \rightarrow X_i, F : X_1 \times X_2 \rightarrow X_1$ , and  $G : X_1 \times X_2 \rightarrow X_2$  be single-valued mappings, and  $T : X_1 \rightarrow CB(X_1), Q : X_2 \rightarrow CB(X_2)$  be set-valued mappings. Let  $M : X_1 \times X_1 \rightarrow 2^{X_1}, N : X_2 \times X_2 \rightarrow 2^{X_2}$  be such that, for each fixed  $x \in X_1, y \in X_2, M(\cdot, x)$  and  $N(\cdot, y)$  are  $H_1(\cdot, \cdot) - \eta_1$ -cocoercive and  $H_2(\cdot, \cdot) - \eta_2$ -cocoercive operators with respect to  $(A_1, B_1)$  and  $(A_2, B_2)$ , respectively. For any given constants  $\lambda_i > 0$

( $i = 1, 2$ ), define the mappings  $S_1 : X_1 \times X_2 \rightarrow X_1$  and  $S_2 : X_1 \times X_2 \rightarrow X_2$  by

$$\begin{aligned} S_1(x, y) & = \bigcup_{w \in T(x)} \bigcup_{v \in Q(y)} R_{\lambda_1, M(\cdot, x)}^{H_1(\cdot, \cdot) - \eta_1} [H_1(A_1x, B_1x) - \lambda_1 F(w, v)], \\ S_2(x, y) & = \bigcup_{w \in T(x)} \bigcup_{v \in Q(y)} R_{\lambda_2, N(\cdot, y)}^{H_2(\cdot, \cdot) - \eta_2} [H_2(A_2y, B_2y) - \lambda_2 G(w, v)]. \end{aligned} \tag{32}$$

For any given  $(x_0, y_0) \in X_1 \times X_2, w_0 \in T(x_0), v_0 \in Q(y_0)$ , let

$$\begin{aligned} z_0 & = R_{\lambda_1, M(\cdot, x_0)}^{H_1(\cdot, \cdot) - \eta_1} \\ & \quad \times [H_1(A_1x_0, B_1x_0) - \lambda_1 F(w_0, v_0)] \in S_1(x_0, y_0), \\ u_0 & = R_{\lambda_2, N(\cdot, y_0)}^{H_2(\cdot, \cdot) - \eta_2} \\ & \quad \times [H_2(A_2y_0, B_2y_0) - \lambda_2 G(w_0, v_0)] \in S_2(x_0, y_0). \end{aligned} \tag{33}$$

Since  $w_0 \in T(x_0) \subset CB(X_1)$  and  $v_0 \in Q(y_0) \subset CB(X_2)$ , in view of Nadler's theorem [24], there exist  $w_1 \in T(x_1)$  and  $v_1 \in Q(y_1)$  such that

$$\begin{aligned} \|w_1 - w_0\|_1 & \leq (1 + 1) D(T(x_1), T(x_0)), \\ \|v_1 - v_0\|_2 & \leq (1 + 1) D(Q(y_1), Q(y_0)). \end{aligned} \tag{34}$$

By induction, we define iterative sequences  $\{x_n\}, \{y_n\}, \{w_n\}$ , and  $\{v_n\}$  as follows:

$$\begin{aligned} x_{n+1} & = R_{\lambda_1, M(\cdot, x_n)}^{H_1(\cdot, \cdot) - \eta_1} \\ & \quad \times [H_1(A_1x_n, B_1x_n) - \lambda_1 F(w_n, v_n)] \in S_1(x_n, y_n), \\ y_{n+1} & = R_{\lambda_2, N(\cdot, y_n)}^{H_2(\cdot, \cdot) - \eta_2} \\ & \quad \times [H_2(A_2y_n, B_2y_n) - \lambda_2 G(w_n, v_n)] \in S_2(x_n, y_n), \end{aligned}$$

$$w_n \in T(x_n),$$

$$\|w_{n+1} - w_n\|_1 \leq \left(1 + \frac{1}{n+1}\right) D(T(x_{n+1}), T(x_n)),$$

$$v_n \in Q(y_n),$$

$$\|v_{n+1} - v_n\|_2 \leq \left(1 + \frac{1}{n+1}\right) D(Q(y_{n+1}), Q(y_n)), \tag{35}$$

where  $n = 0, 1, 2, \dots$ , and  $\lambda_1, \lambda_2 > 0$  are constants.

**Theorem 18.** For  $i = 1, 2$ , let  $X_i$  be real Banach spaces with the norm  $\|\cdot\|_i$ . Let  $A_i, B_i : X_i \rightarrow X_i, H_i : X_i \times X_i \rightarrow X_i, \eta_i : X_i \times X_i \rightarrow X_i, F : X_1 \times X_2 \rightarrow X_1$ , and  $G : X_1 \times X_2 \rightarrow X_2$  be single-valued mappings, and  $T : X_1 \rightarrow CB(X_1), Q : X_2 \rightarrow$

$CB(X_2)$  be set-valued mappings. Let  $M : X_1 \times X_1 \rightarrow 2^{X_1}$  and  $N : X_2 \times X_2 \rightarrow 2^{X_2}$  be such that, for each fixed  $x \in X_1, y \in X_2, M(\cdot, x)$  and  $N(\cdot, y)$  are  $H_1(\cdot, \cdot) - \eta_1$ -cocoercive and  $H_2(\cdot, \cdot) - \eta_2$ -cocoercive operators with respect to  $(A_1, B_1)$  and  $(A_2, B_2)$ , respectively. Suppose that the following conditions are satisfied.

- (i)  $H_i(A_i, B_i)$  is  $\eta_i$ -cocoercive with respect to  $A_i$  with constant  $\mu_i$  and  $\eta_i$ -relaxed cocoercive with respect to  $B_i$  with constant  $\gamma_i, i = 1, 2$ .
- (ii)  $A_i$  is  $\alpha_i$ -expansive and  $B_i$  is  $\beta_i$ -Lipschitz continuous,  $i = 1, 2$ .
- (iii)  $H_i(A_i, B_i)$  is  $r_i$ -Lipschitz continuous with respect to  $A_i$  and  $s_i$ -Lipschitz continuous with respect to  $B_i, i = 1, 2$ .
- (iv)  $T$  is  $D$ -Lipschitz continuous with constant  $\lambda_T$  and  $Q$  is  $D$ -Lipschitz continuous with constant  $\lambda_Q$ .
- (v)  $F$  is  $t_1$ -Lipschitz continuous with respect to  $T$  in the first argument and  $t_2$ -Lipschitz continuous with respect to  $Q$  in the second argument.
- (vi)  $G$  is  $l_1$ -Lipschitz continuous with respect to  $T$  in the first argument and  $l_2$ -Lipschitz continuous with respect to  $Q$  in the second argument.
- (vii)  $\eta_i$  is  $\rho_i$ -Lipschitz continuous,  $i = 1, 2$ .
- (viii)  $F$  is  $\eta_1$ -relaxed Lipschitz continuous with respect to  $T$  in the first argument and  $\eta_1$ -relaxed Lipschitz continuous with respect to  $Q$  in the second argument with constants  $\tau_1$  and  $\tau_2$ , respectively.
- (ix)  $G$  is  $\eta_2$ -relaxed Lipschitz continuous with respect to  $T$  in the first argument and  $\eta_2$ -relaxed Lipschitz continuous with respect to  $Q$  in the second argument with constants  $\epsilon_1$  and  $\epsilon_2$ , respectively. Furthermore, assume that there exist constants  $\sigma_1, \sigma_2 > 0$  such that

$$\left\| R_{\lambda_1, M(\cdot, x_1)}^{H_1(\cdot, \cdot) - \eta_1}(x) - R_{\lambda_1, M(\cdot, x_2)}^{H_1(\cdot, \cdot) - \eta_1}(x) \right\|_1 \leq \sigma_1 \|x_1 - x_2\|_1, \quad \forall x, x_1, x_2 \in X_1, \tag{36}$$

$$\left\| R_{\lambda_2, N(\cdot, y_1)}^{H_2(\cdot, \cdot) - \eta_2}(y) - R_{\lambda_2, N(\cdot, y_2)}^{H_2(\cdot, \cdot) - \eta_2}(y) \right\|_2 \leq \sigma_2 \|y_1 - y_2\|_2, \quad \forall y, y_1, y_2 \in X_2, \tag{37}$$

and  $\lambda_1, \lambda_2 > 0$  are constants satisfying the following conditions:

$$\begin{aligned} p_0 &= \sqrt{r_1^2 + 2\lambda_1 t_1 \lambda_T [r_1 + \lambda_1 t_1 \lambda_T + \rho_1] - 2\lambda_1 \tau_1}, \\ q_0 &= \sqrt{r_2^2 + 2\lambda_2 l_2 \lambda_Q [r_2 + \lambda_2 l_2 \lambda_Q + \rho_2] - 2\lambda_2 \epsilon_2}, \\ r_1^2 + 2\lambda_1 t_1 \lambda_T [r_1 + \lambda_1 t_1 \lambda_T + \rho_1] &> 2\lambda_1 \tau_1, \\ r_2^2 + 2\lambda_2 l_2 \lambda_Q [r_2 + \lambda_2 l_2 \lambda_Q + \rho_2] &> 2\lambda_2 \epsilon_2, \end{aligned}$$

$$\begin{aligned} \mu_i &> \gamma_i, \quad \alpha_i > \beta_i, \quad i = 1, 2, \\ \theta_1 &= \frac{\rho_1}{\mu_1 \alpha_1^2 - \gamma_1 \beta_1^2}, \quad \theta_2 = \frac{\rho_2}{\mu_2 \alpha_2^2 - \gamma_2 \beta_2^2}, \\ \sigma_1 + \theta_1 p_0 + \theta_1 s_1 + \lambda_2 \theta_2 l_1 \lambda_T &< 1, \\ \sigma_2 + \theta_2 q_0 + \theta_2 s_2 + \lambda_1 \theta_1 t_2 \lambda_Q &< 1. \end{aligned} \tag{38}$$

Then, the iterative sequences  $\{x_n\}, \{y_n\}, \{w_n\}$ , and  $\{v_n\}$  generated by Algorithm 17 converge strongly to  $x, y, w$ , and  $v$ , respectively, and  $(x, y, w, v)$  is a solution of problem (30).

*Proof.* In view of Theorem 13, the resolvent operator  $R_{\lambda_1, M}^{H_1(\cdot, \cdot) - \eta_1}$  is  $\theta_1$ -Lipschitz continuous. This, together with Algorithm 17 and (36), implies that

$$\begin{aligned} &\|x_{n+1} - x_n\|_1 \\ &= \left\| R_{\lambda_1, M(\cdot, x_n)}^{H_1(\cdot, \cdot) - \eta_1} [H_1(A_1 x_n, B_1 x_n) - \lambda_1 F(w_n, v_n)] \right. \\ &\quad \left. - R_{\lambda_1, M(\cdot, x_{n-1})}^{H_1(\cdot, \cdot) - \eta_1} [H_1(A_1 x_{n-1}, B_1 x_{n-1}) - \lambda_1 F(w_{n-1}, v_{n-1})] \right\|_1 \\ &\leq \left\| R_{\lambda_1, M(\cdot, x_n)}^{H_1(\cdot, \cdot) - \eta_1} [H_1(A_1 x_n, B_1 x_n) - \lambda_1 F(w_n, v_n)] \right. \\ &\quad \left. - R_{\lambda_1, M(\cdot, x_{n-1})}^{H_1(\cdot, \cdot) - \eta_1} [H_1(A_1 x_n, B_1 x_n) - \lambda_1 F(w_n, v_n)] \right\|_1 \\ &\quad + \left\| R_{\lambda_1, M(\cdot, x_{n-1})}^{H_1(\cdot, \cdot) - \eta_1} [H_1(A_1 x_n, B_1 x_n) - \lambda_1 F(w_n, v_n)] \right. \\ &\quad \left. - R_{\lambda_1, M(\cdot, x_{n-1})}^{H_1(\cdot, \cdot) - \eta_1} [H_1(A_1 x_{n-1}, B_1 x_{n-1}) - \lambda_1 F(w_{n-1}, v_{n-1})] \right\|_1 \\ &\leq \sigma_1 \|x_n - x_{n-1}\|_1 \\ &\quad + \theta_1 \|H_1(A_1 x_n, B_1 x_n) - H_1(A_1 x_{n-1}, B_1 x_{n-1}) - \lambda_1 (F(w_n, v_n) - F(w_{n-1}, v_{n-1}))\|_1 \\ &\leq \sigma_1 \|x_n - x_{n-1}\|_1 \\ &\quad + \theta_1 \|H_1(A_1 x_n, B_1 x_n) - H_1(A_1 x_{n-1}, B_1 x_{n-1}) - \lambda_1 (F(w_n, v_n) - F(w_{n-1}, v_{n-1}))\|_1 \\ &\quad + \lambda_1 \theta_1 \|F(w_{n-1}, v_n) - F(w_{n-1}, v_{n-1})\|_1 \\ &\leq \sigma_1 \|x_n - x_{n-1}\|_1 \\ &\quad + \theta_1 \|H_1(A_1 x_n, B_1 x_n) - H_1(A_1 x_{n-1}, B_1 x_{n-1}) - \lambda_1 (F(w_n, v_n) - F(w_{n-1}, v_n))\|_1 \\ &\quad + \lambda_1 \theta_1 \|F(w_{n-1}, v_n) - F(w_{n-1}, v_{n-1})\|_1 \\ &\leq \sigma_1 \|x_n - x_{n-1}\|_1 \\ &\quad + \theta_1 \|H_1(A_1 x_n, B_1 x_n) - H_1(A_1 x_{n-1}, B_1 x_{n-1}) - \lambda_1 (F(w_n, v_n) - F(w_{n-1}, v_n))\|_1 \end{aligned}$$

$$\begin{aligned}
 & + \theta_1 \|H_1(A_1x_{n-1}, B_1x_n) - H_1(A_1x_{n-1}, B_1x_{n-1})\|_1 \\
 & + \lambda_1 \theta_1 \|F(w_{n-1}, v_n) - F(w_{n-1}, v_{n-1})\|_1.
 \end{aligned} \tag{39}$$

Since  $F$  is  $t_1$ -Lipschitz continuous with respect to  $T$  in the first argument and  $t_2$ -Lipschitz continuous in the second argument,  $T$  is  $\lambda_T$ -Lipschitz continuous, and  $Q$  is  $\lambda_Q$ -Lipschitz continuous, by Algorithm 17, we get

$$\begin{aligned}
 & \|F(w_n, v_n) - F(w_{n-1}, v_n)\|_1 \\
 & \leq t_1 \|w_n - w_{n-1}\|_1 \\
 & \leq t_1 \left(1 + \frac{1}{n}\right) D(T(x_n), T(x_{n-1})) \\
 & \leq t_1 \lambda_T \left(1 + \frac{1}{n}\right) \|x_n - x_{n-1}\|_1, \\
 & \|F(w_{n-1}, v_n) - F(w_{n-1}, v_{n-1})\|_1 \\
 & \leq t_2 \|v_n - v_{n-1}\|_2 \\
 & \leq t_2 \left(1 + \frac{1}{n}\right) D(Q(y_n), Q(y_{n-1})) \\
 & \leq t_2 \lambda_Q \left(1 + \frac{1}{n}\right) \|y_n - y_{n-1}\|_2.
 \end{aligned} \tag{40}$$

As  $H_1(\cdot, \cdot)$  is  $r_1$ -Lipschitz continuous with respect to  $A_1$ , we obtain

$$\|H_1(A_1x_n, B_1x_n) - H_1(A_1x_{n-1}, B_1x_n)\|_1 \leq r_1 \|x_n - x_{n-1}\|_1. \tag{42}$$

Since  $\eta_1$  is  $\rho_1$ -Lipschitz continuous, we conclude that

$$\|\eta_1(x_n, x_{n-1})\|_1 \leq \rho_1 \|x_n - x_{n-1}\|_1. \tag{43}$$

Since  $H_1(\cdot, \cdot)$  is  $\eta_1$ -relaxed Lipschitz continuous with respect to  $T$  and  $\eta_2$ -relaxed Lipschitz continuous with respect to  $Q$  in the first and second arguments with constants  $\tau_1$  and  $\tau_2$ , respectively, we have

$$\begin{aligned}
 & \langle F(w_n, v_n) - F(w_{n-1}, v_n), j(\eta_1(x_n, x_{n-1})) \rangle \\
 & \leq -\tau_1 \|x_n - x_{n-1}\|_1^2.
 \end{aligned} \tag{44}$$

Employing Lemma 3 and taking into account (39)–(44), we obtain

$$\begin{aligned}
 & \|H_1(A_1x_n, B_1x_n) - H_1(A_1x_{n-1}, B_1x_n) \\
 & - \lambda_1 (F(w_n, v_n) - F(w_{n-1}, v_n))\|_1^2 \\
 & \leq \|H_1(A_1x_n, B_1x_n) - H_1(A_1x_{n-1}, B_1x_n)\|_1^2 \\
 & - 2\lambda_1 \langle F(w_n, v_n) - F(w_{n-1}, v_n),
 \end{aligned}$$

$$\begin{aligned}
 & j [H_1(A_1x_n, B_1x_n) - H_1(A_1x_{n-1}, B_1x_n) \\
 & - \lambda_1 (F(w_n, v_n) - F(w_{n-1}, v_n))] \rangle \\
 & = \|H_1(A_1x_n, B_1x_n) - H_1(A_1x_{n-1}, B_1x_n)\|_1^2 \\
 & - 2\lambda_1 \langle F(w_n, v_n) - F(w_{n-1}, v_n), \\
 & j [H_1(A_1x_n, B_1x_n) - H_1(A_1x_{n-1}, B_1x_n) \\
 & - \lambda_1 (F(w_n, v_n) - F(w_{n-1}, v_n))] \\
 & + j(\eta_1(x_n, x_{n-1})) \rangle \\
 & + 2\lambda_1 \langle F(w_n, v_n) - F(w_{n-1}, v_n), j(\eta_1(x_n, x_{n-1})) \rangle \\
 & \leq \|H_1(A_1x_n, B_1x_n) - H_1(A_1x_{n-1}, B_1x_n)\|_1^2 \\
 & + 2\lambda_1 \|F(w_n, v_n) - F(w_{n-1}, v_n)\|_1 \\
 & \times [\|H_1(A_1x_n, B_1x_n) - H_1(A_1x_{n-1}, B_1x_n)\|_1 \\
 & + \lambda_1 \|F(w_n, v_n) - F(w_{n-1}, v_n)\|_1 \\
 & + \|\eta_1(x_n, x_{n-1})\|_1] \\
 & + 2\lambda_1 \langle F(w_n, v_n) - F(w_{n-1}, v_n), j(\eta_1(x_n, x_{n-1})) \rangle \\
 & \leq r_1^2 \|x_n - x_{n-1}\|_1^2 + 2\lambda_1 t_1 \lambda_T \left(1 + \frac{1}{n}\right) \|x_n - x_{n-1}\|_1 \\
 & \times \left[ r_1 \|x_n - x_{n-1}\|_1 + \lambda_1 t_1 \lambda_T \left(1 + \frac{1}{n}\right) \right. \\
 & \quad \left. \times \|x_n - x_{n-1}\|_1 + \rho_1 \|x_n - x_{n-1}\|_1 \right] \\
 & - 2\lambda_1 \tau_1 \|x_n - x_{n-1}\|_1^2 \\
 & = \left[ r_1^2 + 2\lambda_1 t_1 \lambda_T \left(1 + \frac{1}{n}\right) \right. \\
 & \quad \left. \times \left[ r_1 + \lambda_1 t_1 \lambda_T \left(1 + \frac{1}{n}\right) + \rho_1 \right] - 2\lambda_1 \tau_1 \right] \|x_n - x_{n-1}\|_1^2.
 \end{aligned} \tag{45}$$

This implies that

$$\begin{aligned}
 & \|H_1(A_1x_n, B_1x_n) - H_1(A_1x_{n-1}, B_1x_n) \\
 & - \lambda_1 (F(w_n, v_n) - F(w_{n-1}, v_n))\|_1 \\
 & \leq \sqrt{r_1^2 + 2\lambda_1 t_1 \lambda_T \left(1 + \frac{1}{n}\right) \left[ r_1 + \lambda_1 t_1 \lambda_T \left(1 + \frac{1}{n}\right) + \rho_1 \right] - 2\lambda_1 \tau_1} \\
 & \quad \times \|x_n - x_{n-1}\|_1 \\
 & = p_n \|x_n - x_{n-1}\|_1,
 \end{aligned} \tag{46}$$

where

$$p_n = \sqrt{r_1^2 + 2\lambda_1 t_1 \lambda_T \left(1 + \frac{1}{n}\right) \left[ \left[ r_1 + \lambda_1 t_1 \lambda_T \left(1 + \frac{1}{n}\right) + \rho_1 \right] - 2\lambda_1 \tau_1 \right]}. \tag{47}$$

Using  $s_1$ -Lipschitz continuity of  $H_1(\cdot, \cdot)$  with respect to  $B_1$ , we deduce that

$$\begin{aligned} & \|H_1(A_1x_{n-1}, B_1x_n) - H_1(A_1x_{n-1}, B_1x_{n-1})\|_1 \\ & \leq s_1 \|x_n - x_{n-1}\|_1. \end{aligned} \tag{48}$$

In view of (41), (46), (48), (39) becomes

$$\begin{aligned} & \|x_{n+1} - x_n\|_1 \\ & \leq \sigma_1 \|x_n - x_{n-1}\|_1 + \theta_1 p_n \|x_n - x_{n-1}\|_1 \\ & \quad + \theta_1 s_1 \|x_n - x_{n-1}\|_1 \\ & \quad + \lambda_1 \theta_1 t_2 \lambda_Q \left(1 + \frac{1}{n}\right) \|y_n - y_{n-1}\|_2 \tag{49} \\ & = (\sigma_1 + \theta_1 p_n + \theta_1 s_1) \|x_n - x_{n-1}\|_1 \\ & \quad + \lambda_1 \theta_1 t_2 \lambda_Q \left(1 + \frac{1}{n}\right) \|y_n - y_{n-1}\|_2. \end{aligned}$$

Similarly, we have

$$\begin{aligned} & \|y_{n+1} - y_n\|_2 \\ & = \left\| R_{\lambda_2, N(\cdot, y_n)}^{H_2(\cdot, \cdot) - \eta_2} [H_2(A_2y_n, B_2y_n) - \lambda_2 G(w_n, v_n)] \right. \\ & \quad \left. - R_{\lambda_2, N(\cdot, y_{n-1})}^{H_2(\cdot, \cdot) - \eta_2} [H_2(A_2y_{n-1}, B_2y_{n-1}) \right. \\ & \quad \left. - \lambda_2 G(w_{n-1}, v_{n-1})] \right\|_2 \\ & \leq \left\| R_{\lambda_2, N(\cdot, y_n)}^{H_2(\cdot, \cdot) - \eta_2} [H_2(A_2y_n, B_2y_n) - \lambda_2 G(w_n, v_n)] \right. \\ & \quad \left. - R_{\lambda_2, N(\cdot, y_{n-1})}^{H_2(\cdot, \cdot) - \eta_2} [H_2(A_2y_n, B_2y_n) - \lambda_2 G(w_n, v_n)] \right\|_2 \\ & \quad + \left\| R_{\lambda_2, N(\cdot, y_{n-1})}^{H_2(\cdot, \cdot) - \eta_2} [H_2(A_2y_n, B_2y_n) - \lambda_2 G(w_n, v_n)] \right. \\ & \quad \left. - R_{\lambda_2, N(\cdot, y_{n-1})}^{H_2(\cdot, \cdot) - \eta_2} [H_2(A_2y_{n-1}, B_2y_{n-1}) \right. \\ & \quad \left. - \lambda_2 G(w_{n-1}, v_{n-1})] \right\|_2 \\ & \leq \sigma_2 \|y_n - y_{n-1}\|_2 \\ & \quad + \theta_2 \|H_2(A_2y_n, B_2y_n) - H_2(A_2y_{n-1}, B_2y_{n-1}) \\ & \quad - \lambda_2 (G(w_n, v_n) - G(w_{n-1}, v_{n-1}))\|_2 \\ & \leq \sigma_2 \|y_n - y_{n-1}\|_2 \\ & \quad + \theta_2 \|H_2(A_2y_n, B_2y_n) - H_2(A_2y_{n-1}, B_2y_{n-1}) \\ & \quad - \lambda_2 (G(w_n, v_n) - G(w_n, v_{n-1}))\|_2 \\ & \quad + \lambda_2 \theta_2 \|G(w_n, v_{n-1}) - G(w_{n-1}, v_{n-1})\|_2 \end{aligned}$$

$$\begin{aligned} & \leq \sigma_2 \|y_n - y_{n-1}\|_2 \\ & \quad + \theta_2 \|H_2(A_2y_n, B_2y_n) - H_2(A_2y_{n-1}, B_2y_n) \\ & \quad - \lambda_2 (G(w_n, v_n) - G(w_n, v_{n-1}))\|_2 \\ & \quad + \theta_2 \|H_2(A_2y_{n-1}, B_2y_n) - H_2(A_2y_{n-1}, B_2y_{n-1})\|_2 \\ & \quad + \lambda_2 \theta_2 \|G(w_n, v_{n-1}) - G(w_{n-1}, v_{n-1})\|_2 \\ & \leq \sigma_2 \|y_n - y_{n-1}\|_2 + \theta_2 q_n \|y_n - y_{n-1}\|_2 + \theta_2 s_2 \|y_n - y_{n-1}\|_2 \\ & \quad + \lambda_2 \theta_2 t_1 \lambda_T \left(1 + \frac{1}{n}\right) \|x_n - x_{n-1}\|_1 \\ & = (\sigma_2 + \theta_2 q_n + \theta_2 s_2) \|y_n - y_{n-1}\|_2 \\ & \quad + \lambda_2 \theta_2 l_1 \lambda_T \left(1 + \frac{1}{n}\right) \|x_n - x_{n-1}\|_1, \end{aligned} \tag{50}$$

where

$$\begin{aligned} q_n & = \sqrt{r_2^2 + 2\lambda_2 l_2 \lambda_Q \left(1 + \frac{1}{n}\right) \left[r_2 + \lambda_2 l_2 \lambda_Q \left(1 + \frac{1}{n}\right) + \rho_2\right] - 2\lambda_2 \epsilon_2}. \end{aligned} \tag{51}$$

In view of (49) and (50), we obtain

$$\begin{aligned} & \|x_{n+1} - x_n\|_1 + \|y_{n+1} - y_n\|_2 \\ & \leq c_n \|x_n - x_{n-1}\|_1 + d_n \|y_n - y_{n-1}\|_2 \tag{52} \\ & \leq k_n (\|x_n - x_{n-1}\|_1 + \|y_n - y_{n-1}\|_2), \end{aligned}$$

where  $c_n = \sigma_1 + \theta_1 p_n + \theta_1 s_1 + \lambda_2 \theta_2 l_1 \lambda_T (1 + 1/n)$ ,  $d_n = \sigma_2 + \theta_2 q_n + \theta_2 s_2 + \lambda_1 \theta_1 t_2 \lambda_Q (1 + 1/n)$ , and  $k_n = \max\{c_n, d_n\}$ .

Letting  $n \rightarrow \infty$ , we obtain  $k_n \rightarrow k$ , where

$$\begin{aligned} k & = \max \{ \sigma_1 + \theta_1 p_0 + \theta_1 s_1 + \lambda_2 \theta_2 l_1 \lambda_T, \\ & \quad \sigma_2 + \theta_2 q_0 + \theta_2 s_2 + \lambda_1 \theta_1 t_2 \lambda_Q \}. \end{aligned} \tag{53}$$

Next, we define the norm  $\|\cdot\|$  on  $X_1 \times X_2$  by

$$\|(x, y)\| = \|x\|_1 + \|y\|_2, \quad (x, y) \in X_1 \times X_2. \tag{54}$$

One can easily check that  $(X_1 \times X_2, \|\cdot\|)$  is a Banach space.

Define  $a_{n+1} = (x_{n+1}, y_{n+1})$ . Then, we have

$$\|a_{n+1} - a_n\| = \|x_{n+1} - x_n\|_1 + \|y_{n+1} - y_n\|_2. \tag{55}$$

In view of (38), we conclude that  $0 < k < 1$ . This implies that there exist  $n_0 \in \mathbb{N}$  and  $k_0 \in (0, 1)$  such that  $k_n \leq k_0$  for all  $n \geq n_0$ . It follows from (52) and (54) that

$$\|a_{n+1} - a_n\| \leq k_0 \|a_n - a_{n-1}\|, \quad \forall n \geq n_0. \tag{56}$$

In view of (56), we obtain

$$\|a_{n+1} - a_n\| \leq k_0^{n-n_0} \|a_{n_0+1} - a_{n_0}\|, \quad \forall n \geq n_0. \tag{57}$$

This implies that for any  $m \geq n \geq n_0$ ,

$$\begin{aligned} \|x_m - x_n\|_1 &\leq \|a_m - a_n\| \\ &\leq \sum_{i=n}^{m-1} \|a_{i+1} - a_i\| \leq \sum_{i=n}^{m-1} k_0^{i-n_0} \|a_{n_0+1} - a_{n_0}\|. \end{aligned} \tag{58}$$

Since  $0 < k_0 < 1$ , it follows from (58) that  $\|x_m - x_n\|_1 \rightarrow 0$  and  $n \rightarrow \infty$ . This proves that  $\{x_n\}$  is a Cauchy sequence in  $X_1$ . Similarly, we conclude that  $\{y_n\}$  is a Cauchy sequence in  $X_2$ . Thus, there exist  $x \in X_1$  and  $y \in X_2$  such that  $x_n \rightarrow x$  and  $y_n \rightarrow y$  as  $n \rightarrow \infty$ .

Next, we prove that  $w_n \rightarrow w \in T(x)$  and  $v_n \rightarrow v \in Q(y)$ . In view of Lipschitz continuity of  $T$  and  $Q$  and Algorithm 17, we obtain

$$\begin{aligned} \|w_n - w_{n-1}\|_1 &\leq \left(1 + \frac{1}{n}\right) \lambda_T \|x_n - x_{n-1}\|_1, \\ \|v_n - v_{n-1}\|_2 &\leq \left(1 + \frac{1}{n}\right) \lambda_Q \|y_n - y_{n-1}\|_2. \end{aligned} \tag{59}$$

From (59), we deduce that  $\{w_n\}, \{v_n\}$  are Cauchy sequences in  $X_1$  and  $X_2$ , respectively. Thus, there exist  $w \in T(x)$  and  $v \in Q(y)$  such that  $w_n \rightarrow w$  and  $v_n \rightarrow v$  as  $n \rightarrow \infty$ . Since  $T$  is  $D$ -Lipschitz continuous with constant  $\lambda_T$ , it is obvious that

$$\begin{aligned} d(w, T(x)) &\leq \|w - w_n\|_1 + d(w_n, T(x)) \\ &\leq \|w - w_{n-1}\|_1 + D(T(x_n), T(x)) \\ &\leq \|w - w_n\|_1 \\ &\quad + \lambda_T \|x_n - x\|_1 \rightarrow 0 \quad (n \rightarrow \infty). \end{aligned} \tag{60}$$

By the closedness of  $T(x)$ , we conclude that  $w \in T(x)$ . Similarly, we have  $v \in Q(y)$ .

Assume now that

$$\begin{aligned} x_0 &= R_{\lambda_1, M(\cdot, x)}^{H_1(\cdot, \cdot) - \eta_1} [H_1(A_1 x, B_1 x) - \lambda_1 F(w, v)], \\ y_0 &= R_{\lambda_2, N(\cdot, y)}^{H_2(\cdot, \cdot) - \eta_2} [H_2(A_2 y, B_2 y) - \lambda_2 G(w, v)]. \end{aligned} \tag{61}$$

Then, we have

$$\begin{aligned} \|x_{n+1} - x_0\|_1 &\leq \left\| R_{\lambda_1, M(\cdot, x_n)}^{H_1(\cdot, \cdot) - \eta_1} [H_1(A_1 x_n, B_1 x_n) - \lambda_1 F(w_n, v_n)] \right. \\ &\quad \left. - R_{\lambda_1, M(\cdot, x)}^{H_1(\cdot, \cdot) - \eta_1} [H_1(A_1 x, B_1 x) - \lambda_1 F(w, v)] \right\|_1 \\ &\leq \left\| R_{\lambda_1, M(\cdot, x_n)}^{H_1(\cdot, \cdot) - \eta_1} [H_1(A_1 x_n, B_1 x_n) - \lambda_1 F(w_n, v_n)] \right. \\ &\quad \left. - R_{\lambda_1, M(\cdot, x)}^{H_1(\cdot, \cdot) - \eta_1} [H_1(A_1 x_n, B_1 x_n) - \lambda_1 F(w_n, v_n)] \right\|_1 \end{aligned}$$

$$\begin{aligned} &+ \left\| R_{\lambda_1, M(\cdot, x)}^{H_1(\cdot, \cdot) - \eta_1} [H_1(A_1 x_n, B_1 x_n) - \lambda_1 F(w_n, v_n)] \right. \\ &\quad \left. - R_{\lambda_1, M(\cdot, x)}^{H_1(\cdot, \cdot) - \eta_1} [H_1(A_1 x, B_1 x) - \lambda_1 F(w, v)] \right\|_1 \\ &\leq \sigma_1 \|x_n - x\|_1 \\ &\quad + \theta_1 \|H_1(A_1 x_n, B_1 x_n) - \lambda_1 F(w_n, v_n) \\ &\quad - [H_1(A_1 x, B_1 x) - \lambda_1 F(w, v)]\|_1 \\ &\leq \sigma_1 \|x_n - x\|_1 \\ &\quad + \theta_1 [\|H_1(A_1 x_n, B_1 x_n) - H_1(A_1 x, B_1 x)\|_1 \\ &\quad + \lambda_1 \|F(w_n, v_n) - F(w, v)\|_1] \\ &\leq \sigma_1 \|x_n - x\|_1 \\ &\quad + \theta_1 [\|H_1(A_1 x_n, B_1 x_n) - H_1(A_1 x, B_1 x)\|_1 \\ &\quad + \|H_1(A_1 x, B_1 x_n) - H_1(A_1 x, B_1 x)\|_1 \\ &\quad + \lambda_1 \theta_1 [\|F(w_n, v_n) - F(w, v_n)\|_1 \\ &\quad + \|F(w, v_n) - F(w, v)\|_1] \\ &\leq \sigma_1 \|x_n - x\|_1 + \theta_1 [r_1 + s_1] \|x_n - x\|_1 \\ &\quad + \lambda_1 \theta_1 [t_1 \|w_n - w\|_1 + t_2 \|v_n - v\|_2] \\ &= (\sigma_1 + \theta_1 r_1 + \theta_1 s_1) \|x_n - x\|_1 \\ &\quad + \lambda_1 \theta_1 [t_1 \|w_n - w\|_1 + t_2 \|v_n - v\|_2]. \end{aligned} \tag{62}$$

Since  $x_n \rightarrow x, w_n \rightarrow w$ , and  $v_n \rightarrow v$  as  $n \rightarrow \infty$ , it follows from (62) that

$$\lim_{n \rightarrow \infty} \|x_n - x_0\|_1 = 0, \tag{63}$$

and hence  $x_0 = x$ .

A similar argument shows that  $y_0 = y$ . Therefore,

$$\begin{aligned} x &= x_0 = R_{\lambda_1, M(\cdot, x)}^{H_1(\cdot, \cdot) - \eta_1} [H_1(A_1 x, B_1 x) - \lambda_1 F(w, v)], \\ y &= y_0 = R_{\lambda_2, N(\cdot, y)}^{H_2(\cdot, \cdot) - \eta_2} [H_2(A_2 y, B_2 y) - \lambda_2 G(w, v)]. \end{aligned} \tag{64}$$

In view of Lemma 16, we conclude that  $(x, y, w, v)$  is a solution of problem (30), which completes the proof.  $\square$

At the end of this paper, we include the following simple example in support of Theorem 18.

*Example 19.* Let  $X = \mathbb{R}^2$  with the usual inner product. We define two mappings  $A, B : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  by

$$\begin{aligned} A(x) &:= \left(\frac{1}{4}x_1 - x_2, x_1 + \frac{1}{4}x_2\right), \\ B(x) &:= \left(-\frac{1}{4}x_1 + \frac{1}{4}x_2, -\frac{1}{4}x_1 - \frac{1}{4}x_2\right), \end{aligned} \tag{65}$$

$$\forall x = (x_1, x_2) \in \mathbb{R}^2.$$

Let a mapping  $H : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be defined by

$$H(Ax, By) := Ax + By, \quad \forall x, y \in \mathbb{R}^2. \quad (66)$$

By similar arguments, as in Example 4.1 of [27], we can prove the following.

- (1)  $H(A, B)$  is  $4/17$ -cocoercive with respect to  $A$  and  $1$ -relaxed cocoercive with respect to  $B$ .
- (2)  $A$  is  $\sqrt{17}/n$ -expansive, for  $n = 4, 5$ .
- (3)  $B$  is  $1/\sqrt{n}$ -expansive, for  $n = 1, 2$ .
- (4)  $H(A, B)$  is  $\sqrt{17}/n$ -Lipschitz continuous with constant  $\sqrt{17}/n$  with respect to  $A$  and  $B$ , for  $n = 1, 2, \dots, 15, 16$ .
- (5) Let  $f, g : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be defined by

$$\begin{aligned} f(x) &:= \left( 8x_1 - \frac{8}{5}x_2, \frac{8}{5}x_1 + 8x_2 \right), \\ g(x) &:= \left( -\frac{17}{8}x_1 + \frac{5}{8}x_2, \frac{5}{8}x_1 + \frac{17}{8}x_2 \right), \end{aligned} \quad (67)$$

$$\forall x = (x_1, x_2) \in \mathbb{R}^2.$$

- (6) Now, we define a mapping  $M : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$  by

$$M(fx, gy) := fx - gy, \quad \forall x, y \in \mathbb{R}^2. \quad (68)$$

Let  $R, S, T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be the identity mappings. It is obvious that these mappings are  $D$ -Lipschitz continuous.

- (7) Assume that  $F, G : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  are defined by

$$\begin{aligned} F(x) &:= \left( -\frac{1}{4}x_1 - \frac{1}{8}x_2, \frac{1}{8}x_1 - \frac{1}{4}x_2 \right), \\ G(x) &:= \left( -\frac{1}{8}x_1 + \frac{1}{5}x_2, -\frac{1}{5}x_1 - \frac{1}{8}x_2 \right), \end{aligned} \quad (69)$$

$$\forall x = (x_1, x_2) \in \mathbb{R}^2.$$

It could easily be seen that all the aspects of the hypotheses of Theorem 18 are satisfied, so we have the desired conclusion.

### Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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