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Research Article

Best Proximity Points for Generalized α-ψ-Proximal Contractive Type Mappings

Mohamed Jleli,[1] Erdal Karapınar,[2] and Bessem Samet[1]

[1] *Department of Mathematics, King Saud University, Riyadh, Saudi Arabia*
[2] *Department of Mathematics, Atılım University, İncek, 06836 Ankara, Turkey*

Correspondence should be addressed to Erdal Karapınar; erdalkarapinar@yahoo.com

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We introduce a new class of non-self-contractive mappings. For such mappings, we study the existence and uniqueness of best proximity points. Several applications and interesting consequences of our obtained results are derived.

1. Introduction and Preliminaries

Let A and B be two nonempty subsets of a metric space (X, d). An element $x \in A$ is said to be a fixed point of a given map $T : A \to B$ if $Tx = x$. Clearly, $T(A) \cap A \neq \emptyset$ is a necessary (but not sufficient) condition for the existence of a fixed point of T. If $T(A) \cap A = \emptyset$, then $d(x, Tx) > 0$ for all $x \in A$ that is, the set of fixed points of T is empty. In a such situation, one often attempts to find an element x which is in some sense closest to Tx. Best proximity point analysis has been developed in this direction.

An element $a \in A$ is called a best proximity point of T if

$$d(a, Ta) = d(A, B), \tag{1}$$

where

$$d(A, B) = \inf \{d(x, y) : x \in A, \ y \in B\}. \tag{2}$$

Because of the fact that $d(x, Tx) \geq d(A, B)$ for all $x \in A$, the global minimum of the mapping $x \mapsto d(x, Tx)$ is attained at a best proximity point. Clearly, if the underlying mapping is a self-mapping, then it can be observed that a best proximity point is essentially a fixed point. The goal of best proximity point theory is to furnish sufficient conditions that assure the existence of such points. For more details on this approach, we refer the reader to [1–12] and references therein.

Recently, Samet et al. [13] introduced a new class of contractive mappings called α-ψ-contractive type mappings. Let (X, d) be a metric space.

Definition 1. A self-mapping $T : X \to X$ is said to be an α-ψ-contraction, where $\alpha : X \times X \to [0, \infty)$ and ψ is a (c)-comparison function, if

$$\alpha(x, y) d(Tx, Ty) \leq \psi(d(x, y)), \quad \forall x, y \in X. \tag{3}$$

Definition 2. A self-mapping $T : X \to X$ is said to be α-admissible, where $\alpha : X \times X \to [0, \infty)$, if

$$x, y \in X, \quad \alpha(x, y) \geq 1 \Longrightarrow \alpha(Tx, Ty) \geq 1. \tag{4}$$

The main results obtained in [13] are the following fixed point theorems.

Theorem 3. *Let (X, d) be a complete metric space and $T : X \to X$ be an α-ψ-contractive mapping satisfying the following conditions:*

(i) *T is α-admissible;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*

(iii) *T is continuous.*

Then, T has a fixed point; that is, there exists $x \in X$ such that $Tx = x$.

Theorem 4. *Let (X, d) be a complete metric space and $T : X \to X$ be an α-ψ-contractive mapping satisfying the following conditions:*

(i) *T is α-admissible;*

(ii) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$;

(iii) *if* $\{x_n\}$ *is a sequence in* X *such that* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* n *and* $x_n \to x \in X$ *as* $n \to \infty$, *then* $\alpha(x_n, x) \geq 1$ *for all* n.

Then, T *has a fixed point.*

Theorem 5. *In addition to the hypotheses of Theorem 3 (resp., Theorem 4), suppose that for all* $(x, y) \in X \times X$, *there exists* $z \in X$ *such that* $\alpha(x, z) \geq 1$ *and* $\alpha(y, z) \geq 1$. *Then we have a unique fixed point.*

It was shown in [13, 14] that various types of contractive mappings belong to the class of α-ψ-contractive type mappings (classical contractive mappings, contractive mappings on ordered metric spaces, cyclic contractive mappings, etc.). For other works in this direction, we refer the reader to [15, 16].

In a very recent paper, Jleli and Samet [17] established some best proximity point results for α-ψ-contractive type mappings. Before presenting the main results obtained in [17], we need to fix some notations and recall some definitions.

Let A and B, two nonempty subsets of a metric space (X, d). We will use the following notations:

$$\begin{aligned} d(A, B) &:= \inf \{d(a, b) : a \in A,\ b \in B\}, \\ A_0 &:= \{a \in A : d(a, b) = d(A, B) \text{ for some } b \in B\}, \\ B_0 &:= \{b \in B : d(a, b) = d(A, B) \text{ for some } a \in A\}. \end{aligned} \tag{5}$$

Definition 6. An element $x^* \in A$ is said to be a best proximity point of the non-self-mapping $T : A \to B$ if it satisfies the condition that

$$d(x^*, Tx^*) = d(A, B). \tag{6}$$

The following concept was introduced in [11].

Definition 7. Let (A, B) be a pair of nonempty subsets of a metric space (X, d) with $A_0 \neq \emptyset$. Then, the pair (A, B) is said to have the P-property if and only if

$$\left.\begin{aligned} d(x_1, y_1) = d(A, B) \\ d(x_2, y_2) = d(A, B) \end{aligned}\right\} \Longrightarrow d(x_1, x_2) = d(y_1, y_2), \tag{7}$$

where $x_1, x_2 \in A$ and $y_1, y_2 \in B$.

The following concepts were introduced in [17].

Definition 8. Let $T : A \to B$ and $\alpha : A \times A \to [0, \infty)$. We say that T is α-proximal admissible if

$$\left.\begin{aligned} \alpha(x_1, x_2) \geq 1 \\ d(u_1, Tx_1) = d(A, B) \\ d(u_2, Tx_2) = d(A, B) \end{aligned}\right\} \Longrightarrow \alpha(u_1, u_2) \geq 1, \tag{8}$$

for all $x_1, x_2, u_1, u_2 \in A$.

Clearly, if $A = B$, T is α-proximal admissible implies that T is α-admissible.

Definition 9. A non-self-mapping $T : A \to B$ is said to be an α-ψ-proximal contraction, where $\alpha : A \times A \to [0, \infty)$ and ψ is a (c)-comparison function, if

$$\alpha(x, y) d(Tx, Ty) \leq \psi(d(x, y)), \quad \forall x, y \in A. \tag{9}$$

The main results obtained in [17] are the following.

Theorem 10. *Let* A *and* B *be nonempty closed subsets of a complete metric space* (X, d) *such that* A_0 *is nonempty. Let* $\alpha : A \times A \to [0, \infty)$ *and* ψ *be a (c)-comparison function. Suppose that* $T : A \to B$ *is a non-self-mapping satisfying the following conditions:*

(i) $T(A_0) \subseteq B_0$, *and* (A, B) *satisfies the* P*-property;*

(ii) T *is* α*-proximal admissible;*

(iii) *there exist elements* x_0 *and* x_1 *in* A_0 *such that*

$$d(x_1, Tx_0) = d(A, B), \qquad \alpha(x_0, x_1) \geq 1; \tag{10}$$

(iv) T *is a continuous* α*-*ψ*-proximal contraction.*

Then, there exists an element $x^* \in A_0$ *such that*

$$d(x^*, Tx^*) = d(A, B). \tag{11}$$

Theorem 11. *Let* A *and* B *be nonempty closed subsets of a complete metric space* (X, d) *such that* A_0 *is nonempty. Let* $\alpha : A \times A \to [0, \infty)$ *and* ψ *be a (c)-comparison function. Suppose that* $T : A \to B$ *is a non-self-mapping satisfying the following conditions:*

(i) $T(A_0) \subseteq B_0$, *and* (A, B) *satisfies the* P*-property;*

(ii) T *is* α*-proximal admissible;*

(iii) *there exist elements* x_0 *and* x_1 *in* A_0 *such that*

$$d(x_1, Tx_0) = d(A, B), \qquad \alpha(x_0, x_1) \geq 1; \tag{12}$$

(iv) T *is an* α*-*ψ*-proximal contraction;*

(v) *if* $\{x_n\}$ *is a sequence in* A *such that* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* n *and* $x_n \to x \in A$ *as* $n \to \infty$, *then there exists a subsequence* $\{x_{n(k)}\}$ *of* $\{x_n\}$ *such that* $\alpha(x_{n(k)}, x) \geq 1$ *for all* k.

Then, there exists an element $x^* \in A_0$ *such that*

$$d(x^*, Tx^*) = d(A, B). \tag{13}$$

Theorem 12. *In addition to the hypotheses of Theorem 10 (resp., Theorem 11), suppose that for all* $(x, y) \in \alpha^{-1}([0, 1[)$, *there exists* $z \in A_0$ *such that* $\alpha(x, z) \geq 1$ *and* $\alpha(y, z) \geq 1$. *Then,* T *has a unique best proximity point.*

In this paper, we extend and generalize the above results by introducing a new family of non-self-contractive mappings that will be called the class of generalized α-ψ-proximal contractive type mappings. For such mappings, we discuss the existence and uniqueness of best proximity points. Various applications and interesting consequences are derived from our main results.

2. Main Results

All the notations presented in the previous section will be used through this paper.

We denote by Ψ the set of nondecreasing functions $\psi : [0,\infty) \to [0,\infty)$ such that

$$\sum_{n=1}^{\infty} \psi^n(t) < \infty, \quad \forall t > 0, \tag{14}$$

where ψ^n is the nth iterate of ψ. These functions are known in the literature as (c)-comparison functions. It is easily proved that if ψ is a (c)-comparison function, then $\psi(t) < t$ for all $t > 0$.

We introduce the following concept.

Definition 13. A non-self-mapping $T : A \to B$ is said to be a generalized α-ψ-proximal contraction, where $\alpha : A \times A \to [0,\infty)$ and $\psi \in \Psi$, if

$$\alpha(x,y)\, d(Tx,Ty) \le \psi(M(x,y)), \quad \forall x, y \in A, \tag{15}$$

where

$$M(x,y) = \max\Big\{d(x,y), \frac{1}{2}[d(x,Tx)+d(y,Ty)] - d(A,B), \frac{1}{2}[d(y,Tx)+d(x,Ty)] - d(A,B)\Big\}. \tag{16}$$

Our first main result is the following best proximity point theorem.

Theorem 14. *Let A and B be nonempty closed subsets of a complete metric space (X,d) such that A_0 is nonempty. Let $\alpha : A \times A \to [0,\infty)$ and $\psi \in \Psi$. Suppose that $T : A \to B$ is a non-self-mapping satisfying the following conditions:*

(i) *$T(A_0) \subseteq B_0$, and (A,B) satisfies the P-property;*

(ii) *T is α-proximal admissible;*

(iii) *there exist elements x_0 and x_1 in A_0 such that*

$$d(x_1,Tx_0) = d(A,B), \qquad \alpha(x_0,x_1) \ge 1; \tag{17}$$

(iv) *T is a continuous generalized α-ψ-proximal contraction.*

Then, there exists an element $x^* \in A_0$ such that

$$d(x^*,Tx^*) = d(A,B). \tag{18}$$

Proof. From condition (iii), there exist elements x_0 and x_1 in A_0 such that

$$d(x_1,Tx_0) = d(A,B), \qquad \alpha(x_0,x_1) \ge 1. \tag{19}$$

Since $T(A_0) \subseteq B_0$, there exists $x_2 \in A_0$ such that

$$d(x_2,Tx_1) = d(A,B). \tag{20}$$

Now, we have

$$\begin{gathered}\alpha(x_0,x_1) \ge 1,\\ d(x_1,Tx_0) = d(A,B),\\ d(x_2,Tx_1) = d(A,B).\end{gathered} \tag{21}$$

Since T is α-proximal admissible, this implies that $\alpha(x_1,x_2) \ge 1$. Thus, we have

$$d(x_2,Tx_1) = d(A,B), \qquad \alpha(x_1,x_2) \ge 1. \tag{22}$$

Again, since $T(A_0) \subseteq B_0$, there exists $x_3 \in A_0$ such that

$$d(x_3,Tx_2) = d(A,B). \tag{23}$$

Now, we have

$$\begin{gathered}\alpha(x_1,x_2) \ge 1,\\ d(x_2,Tx_1) = d(A,B),\\ d(x_3,Tx_2) = d(A,B).\end{gathered} \tag{24}$$

Since T is α-proximal admissible, this implies that $\alpha(x_2,x_3) \ge 1$. Thus, we have

$$d(x_3,Tx_2) = d(A,B), \qquad \alpha(x_2,x_3) \ge 1. \tag{25}$$

Continuing this process, by induction, we can construct a sequence $\{x_n\} \subset A_0$ such that

$$\begin{gathered}d(x_{n+1},Tx_n) = d(A,B),\\ \alpha(x_n,x_{n+1}) \ge 1, \quad \forall n \in \mathbb{N} \cup \{0\}.\end{gathered} \tag{26}$$

Since (A,B) satisfies the P-property, we conclude from (26) that

$$d(x_n,x_{n+1}) = d(Tx_{n-1},Tx_n), \quad \forall n \in \mathbb{N}. \tag{27}$$

From condition (iv), that is, T is a generalized α-ψ-proximal contraction, for all $n \in \mathbb{N}$, we have

$$\alpha(x_{n-1},x_n)\, d(Tx_{n-1},Tx_n) \le \psi(M(x_{n-1},x_n)). \tag{28}$$

On the other hand, using (26) and (27), we have

$$\begin{aligned}
&M(x_{n-1},x_n)\\
&= \max\Big\{d(x_{n-1},x_n),\\
&\qquad \frac{1}{2}\left[d(x_{n-1},Tx_{n-1})+d(x_n,Tx_n)\right]-d(A,B),\\
&\qquad \frac{1}{2}\left[d(x_n,Tx_{n-1})+d(x_{n-1},Tx_n)\right]-d(A,B)\Big\}\\
&\le \max\Big\{d(x_{n-1},x_n),\\
&\qquad \frac{1}{2}\left[d(x_{n-1},x_n)+d(x_n,Tx_{n-1})\right.\\
&\qquad\quad \left.+d(x_n,x_{n+1})+d(x_{n+1},Tx_n)\right]-d(A,B),\\
&\qquad \frac{1}{2}\left[d(A,B)+d(x_{n-1},x_n)\right.\\
&\qquad\quad \left.+d(x_n,Tx_{n-1})+d(Tx_{n-1},Tx_n)\right]-d(A,B)\Big\}\\
&= \max\Big\{d(x_{n-1},x_n),\\
&\qquad \frac{1}{2}\left[d(x_{n-1},x_n)+d(A,B)\right.\\
&\qquad\quad \left.+d(x_n,x_{n+1})+d(A,B)\right]-d(A,B),\\
&\qquad \frac{1}{2}\left[d(A,B)+d(x_{n-1},x_n)\right.\\
&\qquad\quad \left.+d(A,B)+d(x_n,x_{n+1})\right]-d(A,B)\Big\}\\
&= \max\Big\{d(x_{n-1},x_n),\\
&\qquad \frac{1}{2}\left[d(x_{n-1},x_n)+d(x_n,x_{n+1})\right],\\
&\qquad \frac{1}{2}\left[d(x_{n-1},x_n)+d(x_n,x_{n+1})\right]\Big\}\\
&= \max\left\{d(x_{n-1},x_n),\frac{1}{2}\left[d(x_{n-1},x_n)+d(x_n,x_{n+1})\right]\right\}\\
&\le \max\left\{d(x_{n-1},x_n),d(x_n,x_{n+1})\right\}.
\end{aligned} \tag{29}$$

Thus, we proved that

$$M(x_{n-1},x_n) \le \max\left\{d(x_{n-1},x_n),d(x_n,x_{n+1})\right\}, \quad \forall n\in\mathbb{N}. \tag{30}$$

Using the above inequality, (26), (27), and (28), and taking in consideration that ψ is a nondecreasing function, we get that

$$d(x_{n+1},x_n) \le \psi\left(\max\left\{d(x_n,x_{n-1}),d(x_n,x_{n+1})\right\}\right), \quad \forall n\in\mathbb{N}. \tag{31}$$

If for some $N\in\mathbb{N}\cup\{0\}$, we have $x_{N+1}=x_N$, from (26), we get that $d(x_N,Tx_N)=d(A,B)$; that is, x_N is a best proximity point. So, we can suppose that

$$d(x_{n+1},x_n)>0, \quad \forall n\in\mathbb{N}\cup\{0\}. \tag{32}$$

Suppose that $\max\{d(x_n,x_{n-1}),d(x_n,x_{n+1})\} = d(x_n,x_{n+1})$. Using (32) and since $\psi(t)<t$ for all $t>0$, we have

$$d(x_{n+1},x_n) \le \psi(d(x_n,x_{n+1})) < d(x_n,x_{n+1}), \tag{33}$$

which is a contradiction. Thus, we have

$$\max\left\{d(x_n,x_{n-1}),d(x_n,x_{n+1})\right\} = d(x_n,x_{n-1}), \quad \forall n\in\mathbb{N}. \tag{34}$$

Now, from (31), we get that

$$d(x_{n+1},x_n) \le \psi(d(x_n,x_{n-1})), \quad \forall n\in\mathbb{N}. \tag{35}$$

Using the monotony of ψ, by induction, it follows from (35) that

$$d(x_{n+1},x_n) \le \psi^n(d(x_1,x_0)), \quad \forall n\in\mathbb{N}\cup\{0\}. \tag{36}$$

Now, we shall prove that $\{x_n\}$ is a Cauchy sequence in the metric space (X,d). Let $\varepsilon>0$ be fixed. Since $\sum_{n=1}^{\infty}\psi^n(d(x_1,x_0))<\infty$, there exists some positive integer $h=h(\varepsilon)$ such that

$$\sum_{k\ge h}\psi^k(d(x_1,x_0)) < \varepsilon. \tag{37}$$

Let $m>n>h$, using the triangular inequality, (36) and (37), we obtain

$$\begin{aligned}
d(x_n,x_m) &\le \sum_{k=n}^{m-1} d(x_k,x_{k+1})\\
&\le \sum_{k=n}^{m-1}\psi^k(d(x_1,x_0))\\
&\le \sum_{k\ge h}\psi^k(d(x_1,x_0))\\
&< \varepsilon.
\end{aligned} \tag{38}$$

Thus, $\{x_n\}$ is a Cauchy sequence in the metric space (X,d). Since (X,d) is complete and A is closed, there exists some $x^*\in A$ such that $x_n\to x^*$ as $n\to\infty$. On the other hand, T is a continuous mapping. Then, we have $Tx_n\to Tx^*$ as $n\to\infty$. The continuity of the metric function d implies that $d(A,B)=d(x_{n+1},Tx_n)\to d(x^*,Tx^*)$ as $n\to\infty$. Therefore, $d(x^*,Tx^*)=d(A,B)$. This completes the proof of the theorem. □

In the next result, we remove the continuity hypothesis, assuming the following condition in A:

(H) If $\{x_n\}$ is a sequence in A such that $\alpha(x_n,x_{n+1})\ge 1$ for all n and $x_n\to x\in A$ as $n\to\infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)},x)\ge 1$ for all k.

Theorem 15. *Let A and B be nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty. Let $\alpha : A \times A \to [0, \infty)$ and $\psi \in \Psi$. Suppose that $T : A \to B$ is a non-self-mapping satisfying the following conditions:*

(i) *$T(A_0) \subseteq B_0$, and (A, B) satisfies the P-property;*

(ii) *T is α-proximal admissible;*

(iii) *there exist elements x_0 and x_1 in A_0 such that*

$$d(x_1, Tx_0) = d(A, B), \qquad \alpha(x_0, x_1) \geq 1; \tag{39}$$

(iv) *(H) holds, and T is a generalized α-ψ-proximal contraction.*

Then, there exists an element $x^* \in A_0$ such that

$$d(x^*, Tx^*) = d(A, B). \tag{40}$$

Proof. Following the proof of Theorem 14, there exists a Cauchy sequence $\{x_n\} \subset A$ such that (26) holds, and $x_n \to x^* \in A$ as $n \to \infty$. From the condition (H), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x^*) \geq 1$ for all k. Since T is a generalized α-ψ-proximal contraction, we get that

$$\begin{aligned} d(Tx_{n(k)}, Tx^*) &\leq \alpha(x_{n(k)}, x^*)\, d(Tx_{n(k)}, Tx^*) \\ &\leq \psi(M(x_{n(k)}, x^*)), \quad \forall k, \end{aligned} \tag{41}$$

where

$$\begin{aligned} M(x_{n(k)}, x^*) = \max\Bigg\{ & d(x_{n(k)}, x^*), \\ & \frac{d(x_{n(k)}, Tx_{n(k)}) + d(x^*, Tx^*)}{2} - d(A, B), \\ & \frac{d(x_{n(k)}, Tx^*) + d(x^*, Tx_{n(k)})}{2} - d(A, B) \Bigg\}. \end{aligned} \tag{42}$$

On the other hand, from (26), for all k, we have

$$\begin{aligned} d(x^*, Tx^*) &\leq d(x^*, x_{n(k)+1}) + d(x_{n(k)+1}, Tx_{n(k)}) \\ &\quad + d(Tx_{n(k)}, Tx^*) \\ &= d(x^*, x_{n(k)+1}) + d(A, B) \\ &\quad + d(Tx_{n(k)}, Tx^*). \end{aligned} \tag{43}$$

Thus, we have

$$\begin{aligned} & d(Tx_{n(k)}, Tx^*) \\ & \quad \geq d(x^*, Tx^*) - d(x^*, x_{n(k)+1}) - d(A, B), \quad \forall k. \end{aligned} \tag{44}$$

Combining (41) with (44), we get that

$$\begin{aligned} & d(x^*, Tx^*) - d(x^*, x_{n(k)+1}) - d(A, B) \\ & \quad \leq \psi(M(x_{n(k)}, x^*)), \quad \forall k. \end{aligned} \tag{45}$$

From (26), for all k, we have

$$\begin{aligned} & M(x_{n(k)}, x^*) \\ & = \max\Bigg\{ d(x_{n(k)}, x^*), \\ & \qquad \frac{d(x_{n(k)}, Tx_{n(k)}) + d(x^*, Tx^*)}{2} - d(A, B), \\ & \qquad \frac{d(x_{n(k)}, Tx^*) + d(x^*, Tx_{n(k)})}{2} - d(A, B) \Bigg\} \\ & \leq \max\Bigg\{ d(x_{n(k)}, x^*), \\ & \qquad \frac{d(x_{n(k)}, x_{n(k)+1}) + d(x_{n(k)+1}, Tx_{n(k)}) + d(x^*, Tx^*)}{2} \\ & \qquad - d(A, B), \\ & \qquad (d(x_{n(k)}, x^*) + d(x^*, Tx^*) + d(x^*, x_{n(k)+1}) \\ & \qquad + d(x_{n(k)+1}, Tx_{n(k)})) \times (2)^{-1} - d(A, B) \Bigg\} \\ & = \max\Bigg\{ d(x_{n(k)}, x^*), \\ & \qquad \frac{d(x_{n(k)}, x_{n(k)+1}) + d(A, B) + d(x^*, Tx^*)}{2} \\ & \qquad - d(A, B), \\ & \qquad \frac{d(x_{n(k)}, x^*) + d(x^*, Tx^*) + d(x^*, x_{n(k)+1}) + d(A, B)}{2} \\ & \qquad - d(A, B) \Bigg\} \\ & := \zeta(x_{n(k)}, x^*). \end{aligned} \tag{46}$$

Since ψ is a nondecreasing function, we get from (45) that

$$\begin{aligned} & d(x^*, Tx^*) - d(x^*, x_{n(k)+1}) - d(A, B) \\ & \quad \leq \psi(\zeta(x_{n(k)}, x^*)), \quad \forall k. \end{aligned} \tag{47}$$

Suppose that $d(x^*, Tx^*) - d(A, B) > 0$. In this case, we have

$$\lim_{k\to\infty} \zeta(x_{n(k)}, x^*) = \max\left\{0, \frac{d(x^*, Tx^*) - d(A, B)}{2}, \frac{d(x^*, Tx^*) - d(A, B)}{2}\right\}, \quad (48)$$

that is,

$$\lim_{k\to\infty} \zeta(x_{n(k)}, x^*) = \frac{d(x^*, Tx^*) - d(A, B)}{2}. \quad (49)$$

Since

$$\frac{d(x^*, Tx^*) - d(A, B)}{2} > 0, \quad (50)$$

for k large enough, we have $\zeta(x_{n(k)}, x^*) > 0$. On the other hand, we have $\psi(t) < t$ for all $t > 0$. Then, from (47), we get that

$$d(x^*, Tx^*) - d(x^*, x_{n(k)+1}) - d(A, B) < \zeta(x_{n(k)}, x^*), \quad \text{for } k \text{ large enough.} \quad (51)$$

Using (49) and letting $k \to \infty$ in the above inequality, we obtain that

$$d(x^*, Tx^*) - d(A, B) \le \frac{d(x^*, Tx^*) - d(A, B)}{2}, \quad (52)$$

which is a contradiction. Thus, we deduce that x^* is a best proximity point of T; that is, $d(x^*, Tx^*) = d(A, B)$. □

The next result gives us a sufficient condition that assures the uniqueness of the best proximity point. We need the following definition.

Definition 16. Let $T : A \to B$ be a non-self-mapping and $\alpha : A \times A \to [0, \infty)$. We say that T is (α, d) regular if for all $(x, y) \in \alpha^{-1}([0, 1[)$, there exists $z \in A_0$ such that

$$\alpha(x, z) \ge 1, \qquad \alpha(y, z) \ge 1. \quad (53)$$

Theorem 17. *In addition to the hypotheses of Theorem 14 (resp., Theorem 15), suppose that T is (α, d) regular. Then, T has a unique best proximity point.*

Proof. From the proof of Theorem 14 (resp., Theorem 15), we know that the set of best proximity points of T is nonempty ($x^* \in A_0$ is a best proximity point). Suppose that $y^* \in A_0$ is another best proximity point of T, that is,

$$d(Tx^*, x^*) = d(Ty^*, y^*) = d(A, B). \quad (54)$$

Using the P-property and (54), we get that

$$d(Tx^*, Ty^*) = d(x^*, y^*). \quad (55)$$

We distinguish two cases.

Case 1. If $\alpha(x^*, y^*) \ge 1$.

Since T is a generalized α-ψ-proximal contraction, using (55), we obtain that

$$\begin{aligned} d(x^*, y^*) &= d(Tx^*, Ty^*) \\ &\le \alpha(x^*, y^*)\, d(Tx^*, Ty^*) \\ &\le \psi(M(x^*, y^*)), \end{aligned} \quad (56)$$

where from (54) and (55), we have

$$\begin{aligned} &M(x^*, y^*) \\ &= \max\Big\{d(x^*, y^*), \\ &\qquad \frac{1}{2}[d(x^*, Tx^*) + d(y^*, Ty^*)] - d(A, B), \\ &\qquad \frac{1}{2}[d(y^*, Tx^*) + d(x^*, Ty^*)] - d(A, B)\Big\} \\ &= \max\Big\{d(x^*, y^*), 0, \\ &\qquad \frac{1}{2}[d(y^*, Tx^*) + d(x^*, Ty^*)] - d(A, B)\Big\} \\ &\le \max\Big\{d(x^*, y^*), \frac{1}{2}[d(y^*, Ty^*) + d(Ty^*, Tx^*) \\ &\qquad + d(x^*, Tx^*) + d(Tx^*, Ty^*)] - d(A, B)\Big\} \\ &= \max\{d(x^*, y^*), d(x^*, y^*)\} = d(x^*, y^*). \end{aligned} \quad (57)$$

This equality with (56) imply that

$$d(x^*, y^*) \le \psi(d(x^*, y^*)). \quad (58)$$

Since $\psi(t) < t$ for all $t > 0$, the above inequality holds only if $d(x^*, y^*) = 0$, that is, $x^* = y^*$.

Case 2. If $\alpha(x^*, y^*) < 1$.

By hypothesis, there exists $z_0 \in A_0$ such that $\alpha(x^*, z_0) \ge 1$ and $\alpha(y^*, z_0) \ge 1$. Since $T(A_0) \subseteq B_0$, there exists $z_1 \in A_0$ such that

$$d(z_1, Tz_0) = d(A, B). \quad (59)$$

Now, we have

$$\begin{aligned} \alpha(x^*, z_0) &\ge 1, \\ d(x^*, Tx^*) &= d(A, B), \\ d(z_1, Tz_0) &= d(A, B). \end{aligned} \quad (60)$$

Since T is α-proximal admissible, we get that $\alpha(x^*, z_1) \ge 1$. Thus, we have

$$d(z_1, Tz_0) = d(A, B), \qquad \alpha(x^*, z_1) \ge 1. \quad (61)$$

Continuing this process, by induction, we can construct a sequence $\{z_n\}$ in A_0 such that

$$d(z_{n+1}, Tz_n) = d(A, B), \quad \alpha(x^*, z_n) \geq 1, \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{62}$$

Using the P-property and (62), we get that

$$d(z_{n+1}, x^*) = d(Tz_n, Tx^*), \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{63}$$

Since T is a generalized α-ψ-proximal contraction, we have

$$\begin{aligned}&\alpha(z_{n+1}, x^*) d(Tz_n, Tx^*)\\ &\quad \leq \psi(M(z_n, x^*)), \quad \forall n \in \mathbb{N} \cup \{0\}.\end{aligned} \tag{64}$$

Combining the above inequality with (63), we get that

$$\begin{aligned}&\alpha(z_{n+1}, x^*) d(z_{n+1}, x^*)\\ &\quad \leq \psi(M(z_n, x^*)), \quad \forall n \in \mathbb{N} \cup \{0\}.\end{aligned} \tag{65}$$

This implies from (62) that

$$d(z_{n+1}, x^*) \leq \psi(M(z_n, x^*)), \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{66}$$

On the other hand, from (63), for all $n \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned}
M&(z_n, x^*)\\
&= \max\Big\{d(z_n, x^*),\\
&\qquad \frac{1}{2}[d(z_n, Tz_n) + d(x^*, Tx^*)] - d(A, B),\\
&\qquad \frac{1}{2}[d(x^*, Tz_n) + d(z_n, Tx^*)] - d(A, B)\Big\}\\
&= \max\Big\{d(z_n, x^*),\\
&\qquad \frac{1}{2}[d(z_n, x^*) + d(x^*, Tx^*)\\
&\qquad\quad + d(Tx^*, Tz_n) + d(A, B)] - d(A, B),\\
&\qquad \frac{1}{2}[d(x^*, Tx^*) + d(Tx^*, Tz_n)\\
&\qquad\quad + d(z_n, x^*) + d(x^*, Tx^*)] - d(A, B)\Big\}\\
&= \max\Big\{d(z_n, x^*), \frac{1}{2}[d(z_n, x^*) + d(z_{n+1}, x^*)]\Big\}\\
&\leq \max\{d(z_n, x^*), d(z_{n+1}, x^*)\}.
\end{aligned} \tag{67}$$

Combining the above inequality with (71), we get that

$$\begin{aligned}&d(z_{n+1}, x^*)\\ &\quad \leq \psi(\max\{d(z_n, x^*), d(z_{n+1}, x^*)\}),\\ &\qquad\qquad \forall n \in \mathbb{N} \cup \{0\}.\end{aligned} \tag{68}$$

Suppose that for some N, we have $z_N = x^*$. From (63), we get that $z_n = x^*$ for all $n \geq N$. This implies that $z_n \to x^*$ as $n \to \infty$. Now, suppose that $d(z_n, x^*) > 0$ for all $n \in \mathbb{N} \cup \{0\}$. Since $\psi(t) < t$ for all $t > 0$, the inequality (68) holds only if

$$\begin{aligned}&\max\{d(z_n, x^*), d(z_{n+1}, x^*)\}\\ &\quad = d(z_n, x^*), \quad \forall n \in \mathbb{N} \cup \{0\}.\end{aligned} \tag{69}$$

Now, we have

$$d(z_{n+1}, x^*) \leq \psi(d(z_n, x^*)), \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{70}$$

By induction, we then derive

$$d(z_n, x^*) \leq \psi^n(d(z_0, x^*)), \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{71}$$

Letting $n \to \infty$ in (71), we obtain that $z_n \to x^*$ as $n \to \infty$. So, in all cases, we have $z_n \to x^*$ as $n \to \infty$. Similarly, we can prove that $z_n \to y^*$ as $n \to \infty$. By uniqueness of the limit, we obtain that $x^* = y^*$. □

3. Applications

3.1. Standard Best Proximity Point Results. We have the following best proximity point result.

Corollary 18. *Let A and B be nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty. Let $\psi \in \Psi$ and suppose that $T : A \to B$ is a non-self-mapping satisfying the following conditions:*

(i) *$T(A_0) \subseteq B_0$, and (A, B) satisfies the P-property;*

(ii) *$d(Tx, Ty) \leq \psi(M(x, y))$, for all $x, y \in A$.*

Then, there exists a unique element $x^* \in A_0$ such that

$$d(x^*, Tx^*) = d(A, B). \tag{72}$$

Proof. Consider the mapping $\alpha : A \times A \to [0, \infty)$ defined by:

$$\alpha(x, y) = 1, \quad \forall x, y \in A. \tag{73}$$

From the definition of α, clearly T is α-proximal admissible and also it is an α-ψ-proximal contraction. On the other hand, for any $x \in A_0$, since $T(A_0) \subseteq B_0$, there exists $y \in A_0$ such that $d(Tx, y) = d(A, B)$. Moreover, from condition (ii), T is a continuous mapping. Now, all the hypotheses of Theorem 14 are satisfied and the existence of the best proximity point follows from Theorem 14. The uniqueness is an immediate consequence of the definition of α and Theorem 17. □

Taking in Corollary 18 $\psi(t) = kt$, where $k \in (0, 1)$, we obtain the following best proximity point result.

Corollary 19. *Let A and B be nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty. Suppose* *that $T : A \to B$ is a non-self-mapping satisfying the following conditions:*

(i) *$T(A_0) \subseteq B_0$, and (A, B) satisfies the P-property;*

(ii) *there exists $k \in (0,1)$ such that $d(Tx, Ty) \le kM(x, y)$, for all $x, y \in A$.*

Then, there exists a unique element $x^* \in A_0$ such that

$$d(x^*, Tx^*) = d(A, B). \tag{74}$$

3.2. Best Proximity Points on a Metric Space Endowed with an Arbitrary Binary Relation. Before presenting our results, we need a few preliminaries.

Let (X, d) be a metric space and $\mathcal{R}$ be a binary relation over X. Denote

$$\mathcal{S} = \mathcal{R} \cup \mathcal{R}^{-1}; \tag{75}$$

this is the symmetric relation attached to R. Clearly,

$$x, y \in X, \quad x\mathcal{S}y \Longleftrightarrow x\mathcal{R}y \text{ or } y\mathcal{R}x. \tag{76}$$

Definition 20. We say that $T : A \to B$ is a proximal comparative mapping if

$$\left.\begin{array}{c} x_1 \mathcal{S} x_2 \\ d(u_1, Tx_1) = d(A, B) \\ d(u_2, Tx_2) = d(A, B) \end{array}\right\} \Longrightarrow u_1 \mathcal{S} u_2, \tag{77}$$

for all $x_1, x_2, u_1, u_2 \in A$.

We have the following best proximity point result.

Corollary 21. *Let A and B be nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty. Let $\mathcal{R}$ be a binary relation over X. Suppose that $T : A \to B$ is a continuous non-self-mapping satisfying the following conditions:*

(i) *$T(A_0) \subseteq B_0$, and (A, B) satisfies the P-property;*

(ii) *T is a proximal comparative mapping;*

(iii) *there exist elements x_0 and x_1 in A_0 such that*

$$d(x_1, Tx_0) = d(A, B), \quad x_0 \mathcal{S} x_1; \tag{78}$$

(iv) *there exists $\psi \in \Psi$ such that*

$$x, y \in A, \quad x\mathcal{S}y \Longrightarrow d(Tx, Ty) \le \psi(M(x, y)). \tag{79}$$

Then, there exists an element $x^* \in A_0$ such that

$$d(x^*, Tx^*) = d(A, B). \tag{80}$$

Proof. Define the mapping $\alpha : A \times A \to [0, \infty)$ by:

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x\mathcal{S}y, \\ 0 & \text{otherwise.} \end{cases} \tag{81}$$

Suppose that

$$\begin{gathered} \alpha(x_1, x_2) \ge 1, \\ d(u_1, Tx_1) = d(A, B), \\ d(u_2, Tx_2) = d(A, B), \end{gathered} \tag{82}$$

for some $x_1, x_2, u_1, u_2 \in A$. By the definition of α, we get that

$$\begin{gathered} x_1 \mathcal{S} x_2, \\ d(u_1, Tx_1) = d(A, B), \\ d(u_2, Tx_2) = d(A, B). \end{gathered} \tag{83}$$

Condition (ii) implies that $u_1 \mathcal{S} u_2$, which gives us from the definition of α that $\alpha(u_1, u_2) \ge 1$. Thus, we proved that T is α-proximal admissible. Condition (iii) implies that

$$d(x_1, Tx_0) = d(A, B), \qquad \alpha(x_0, x_1) \ge 1. \tag{84}$$

Finally, condition (iv) implies that

$$\alpha(x, y) d(Tx, Ty) \le \psi(M(x, y)), \quad \forall x, y \in A, \tag{85}$$

that is, T is a generalized α-ψ-proximal contraction. Now, all the hypotheses of Theorem 14 are satisfied, and the desired result follows immediately from this theorem. □

In order to remove the continuity assumption, we need the following condition:

($\mathcal{H}$) if the sequence $\{x_n\}$ in X and the point $x \in X$ are such that $x_n \mathcal{S} x_{n+1}$ for all n and $\lim_{n\to\infty} d(x_n, x) = 0$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \mathcal{S} x$ for all k.

Corollary 22. *Let A and B be nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty. Let $\mathcal{R}$ be a binary relation over X. Suppose that $T : A \to B$ is a non-self-mapping satisfying the following conditions:*

(i) *$T(A_0) \subseteq B_0$, and (A, B) satisfies the P-property;*

(ii) *T is a proximal comparative mapping;*

(iii) *there exist elements x_0 and x_1 in A_0 such that*

$$d(x_1, Tx_0) = d(A, B), \quad x_0 \mathcal{S} x_1; \tag{86}$$

(iv) *there exists $\psi \in \Psi$ such that*

$$x, y \in A, \quad x\mathcal{S}y \Longrightarrow d(Tx, Ty) \le \psi(M(x, y)), \tag{87}$$

(v) *($\mathcal{H}$) holds.*

Then, there exists an element $x^* \in A_0$ such that

$$d(x^*, Tx^*) = d(A, B). \tag{88}$$

Proof. The result follows from Theorem 15 by considering the mapping α given by (81), and by observing that, condition ($\mathcal{H}$) implies condition (H). □

Corollary 23. *In addition to the hypotheses of Corollary 21 (resp., Corollary 22), suppose that the following condition holds: for all $(x, y) \in A \times A$ with $(x, y) \notin \mathcal{S}$, there exists $z \in A_0$ such that $x\mathcal{S}z$ and $y\mathcal{S}z$. Then, T has a unique best proximity point.*

Proof. The result follows from Theorem 17 by considering the mapping α given by (81). □

3.3. Related Fixed Point Theorems

3.3.1. Fixed Points for Generalized α-ψ Contractive Type Mappings. The concept of generalized α-ψ contractive type mappings was introduced recently in [14].

Definition 24. Let A be a nonempty subset of a metric space (X, d) and $T : A \to A$ be a self-mapping. We say that T is a generalized α-ψ contractive mapping if there exist two functions $\alpha : A \times A \to [0, \infty)$ and $\psi \in \Psi$ such that for all $x, y \in A$, we have

$$\alpha(x, y) d(Tx, Ty) \leq \psi\left(\max\left\{d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}, \frac{d(y, Tx) + d(x, Ty)}{2}\right\}\right). \quad (89)$$

Taking $A = B$ in Theorems 14–17, we obtain the following fixed point results established in [14].

Corollary 25. *Let A be a nonempty closed subset of a complete metric space (X, d). Let $T : A \to A$ be a generalized α-ψ contractive mapping satisfying the following conditions:*

(i) *T is α-admissible;*

(ii) *there exists $x_0 \in A$ such that $\alpha(x_0, Tx_0) \geq 1$;*

(iii) *T is continuous.*

Then, T has a fixed point.

Corollary 26. *Let A be a nonempty closed subset of a complete metric space (X, d). Let $T : A \to A$ be a generalized α-ψ contractive mapping satisfying the following conditions:*

(i) *T is α-admissible;*

(ii) *there exists $x_0 \in A$ such that $\alpha(x_0, Tx_0) \geq 1$;*

(iii) *condition (H) holds.*

Then, T has a fixed point.

Corollary 27. *In addition to the hypotheses of Corollary 25 (resp., Corollary 26), suppose that for all $(x, y) \in \alpha^{-1}([0, 1[)$, there exists $z \in A$ such that*

$$\alpha(x, z) \geq 1, \qquad \alpha(y, z) \geq 1. \quad (90)$$

Then, T has a unique fixed point.

3.3.2. Fixed Points on a Metric Space Endowed with an Arbitrary Binary Relation. We recall the following concept introduced in [18].

Let A be a nonempty closed subset of a complete metric space (X, d). Suppose that X is endowed with an arbitrary binary relation $\mathcal{R}$. We denote by $\mathcal{S}$ the symmetric relation attached to $\mathcal{R}$. Let $T : A \to A$ be a given mapping.

Definition 28. We say that $T : A \to A$ is a comparative mapping if T maps comparable elements into comparable elements, that is,

$$x, y \in A, \quad x\mathcal{S}y \Longrightarrow Tx\mathcal{S}Ty. \quad (91)$$

We have the following fixed point theorem.

Corollary 29. *Assume that $T : A \to A$ is a continuous comparative map, and*

$$x, y \in A, \quad x\mathcal{S}y \Longrightarrow \psi\left(\max\left\{d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}, \frac{d(y, Tx) + d(x, Ty)}{2}\right\}\right), \quad (92)$$

where $\psi \in \Psi$. Suppose also that there exists $x_0 \in X$ such that $x_0\mathcal{S}Tx_0$. Then, T has a fixed point.

Proof. It follows from Corollary 21 by taking $A = B$ and remarking that if $A = B$, a comparative map is a proximal comparative map. □

Remark that a self-mapping $T : A \to A$ satisfying the property (92) is not necessarily continuous (see Example 2.2 in [18]).

Similarly, Taking $A = B$ in Corollary 22, we obtain the following fixed point result.

Corollary 30. *Assume that $T : A \to A$ is a comparative map satisfying (92) for some $\psi \in \Psi$. Suppose also that there exists $x_0 \in X$ such that $x_0\mathcal{S}Tx_0$. If ($\mathcal{H}$) holds, then T has a fixed point.*

The uniqueness of the fixed point follows from Corollary 23 by taking $A = B$.

Corollary 31. *In addition to the hypotheses of Corollary 29 (resp., Corollary 30), suppose that the following condition holds: for all $(x, y) \in A \times A$ with $(x, y) \notin \mathcal{S}$, there exists $z \in A$ such that $x\mathcal{S}z$ and $y\mathcal{S}z$. Then, T has a unique fixed point.*

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