Research Article

Uniform Asymptotic Stability of Solutions of Fractional Functional Differential Equations

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Some global existence and uniform asymptotic stability results for fractional functional differential equations are proved. It is worth mentioning that when $\alpha = 1$ the initial value problem $D^{\alpha}[y(t)e^{\beta t}] = f(t, y_t)e^{\beta t}$, $t \in [t_0, \infty)$, $t_0 \geq 0$, $0 < \alpha < 1$, $y(t) = \phi(t)$, $t_0 - h \leq t \leq t_0$, reduces to a classical dissipative differential equation with delays as in Caraballo et al.'s work (2005).

1. Introduction

Consider the initial value problem (IVP for short) of the following fractional functional differential equation:

$$D^{\alpha}\left[y(t)e^{\beta t} \right] = f(t, y_t)e^{\beta t}, \quad t \in [t_0, \infty), t_0 \geq 0,$$
$$0 < \alpha < 1, \quad (1)$$
$$y(t) = \phi(t), \quad t_0 - h \leq t \leq t_0,$$

where D^{α} is the Caputo fractional derivative, $\beta > 0$, $f : J \times C([-h, 0], \mathbb{R}) \rightarrow \mathbb{R}$, where $J = [t_0, \infty)$, is a given function satisfying some assumptions that will be specified later, $h > 0$, and $\phi \in C([t_0 - h, t_0], \mathbb{R})$. If $y \in C([t_0 - h, \infty), \mathbb{R})$, then for any $t \in [t_0, \infty)$, define y_t by

$$y_t(\theta) = y(t + \theta), \quad \theta \in [-h, 0]. \quad (2)$$

The study of retarded differential equations is an important area of applied mathematics due to physical reasons, noninstant transmission phenomena, memory processes, and specially biological motivations (see, e.g., [1–4]). Fractional differential equations have attracted much attention recently (see, e.g., [5–11] and the references cited therein for the applications in various sciences such as physics, mechanics, chemistry, and engineering).

Some attractive results for fractional functional differential equations and nonlinear functional integral equations are obtained by using the fixed point theory; see [12–16] and references therein. Global asymptotic stability of solutions of a functional integral equation is discussed in [17]; however, there is no work on uniform asymptotic stability of solutions of fractional functional differential equation. It is our intention here to show the global existence and uniform asymptotic stability of the fractional functional differential equation (1).

We organize the paper as follows. In Section 2, we recall some necessary concepts and results. In Section 3 we give the global existence and uniform asymptotic stability of fractional functional differential equations. Finally, two examples are given to illustrate our main results.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

We consider $BC := BC([t_0 - h, \infty), \mathbb{R})$ the Banach space of all bounded and continuous functions from $[t_0 - h, \infty)$ into $\mathbb{R}$ with the norm

$$\|y\|_{\infty} := \sup\left\{ |y(t)| : t \in [t_0 - h, \infty) \right\}. \quad (3)$$

Let $\|y_t\| = \sup_{-h \leq \theta \leq 0} |y(t + \theta)|$ for $t \in J$.

Throughout this paper, we always assume that $f(t, x_t)$ satisfies the following condition:

(H_0) $f(t, x_t)$ is Lebesgue measurable with respect to t on $[t_0, \infty)$, and $f(t, \varphi)$ is continuous with respect to φ on $C([-h, 0], \mathbb{R})$.

By condition (H_0) and the technique used in [7], we get the equivalent form of IVP (1) as

$$
y(t) = \begin{cases} y(t_0) e^{-\beta(t-t_0)} \\ \quad + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{t_0}^{t} (t-s)^{\alpha-1} e^{-\beta(t-s)} \\ \qquad \times f(s, y_s) \, ds, \qquad t \geq t_0, \\ \phi(t), \qquad\qquad\qquad t \in [t_0 - h, t_0], \end{cases}
\tag{4}
$$

where $\Gamma(\cdot)$ is the gamma function.

Definition 1. We say that the solutions y and x of IVP (1) are uniformly asymptotically stable if for any bounded subset B of $C([-h, 0], \mathbb{R})$ and $\varepsilon > 0$, there exists a $T > 0$ such that

$$
\begin{aligned}
\left| y(t, t_0, \phi) - x(t, t_0, \psi) \right| &\leq \varepsilon \quad \forall t \geq T, \\
&\phi, \psi \in B.
\end{aligned}
\tag{5}
$$

We recall the following generalization of Gronwall's lemma for singular kernels [18], which will be used in the sequel.

Lemma 2. Let $v : [t_0, b] \to [0, +\infty)$ be a real function and $w(\cdot)$ is a nonnegative, locally integrable function on $[t_0, b]$ and there are constants $a > 0$ and $0 < \alpha < 1$ such that

$$
v(t) \leq w(t) + a \int_{t_0}^{t} \frac{v(s)}{(t-s)^{\alpha}} \, ds.
\tag{6}
$$

Then there exists a constant $K = K(\alpha)$ such that

$$
v(t) \leq w(t) + Ka \int_{t_0}^{t} \frac{w(s)}{(t-s)^{\alpha}} \, ds,
\tag{7}
$$

for every $t \in [t_0, b]$.

Theorem 3 (Leray-Schauder fixed-point theorem). *Let P be a continuous and compact mapping of a Banach space X into itself, such that the set*

$$
\{ x \in X : x = \lambda P x \;\; \text{for some} \;\; 0 \leq \lambda \leq 1 \},
\tag{8}
$$

is bounded. Then, P has a fixed point.

3. FDEs of Fractional Order

In this section, we will investigate the IVP (1). Our first global existence and uniform asymptotic stability result for the IVP (1) is based on the Banach contradiction principle and Lemma 2.

Theorem 4. *Assume that $f(t, y_t)$ satisfies conditions (H_0) and*

(H_1) there exists $l > 0$ such that

$$
\left| f(t, u_t) - f(t, v_t) \right| \leq l \left\| u_t - v_t \right\|,
\tag{9}
$$

for $t \in J$ and every $u_t, v_t \in C([-h, 0], \mathbb{R})$. Moreover, the function $t \mapsto f(t, 0)$ is bounded with $f_0 = \sup_{t \geq t_0} |f(t, 0)|$.
If

$$
l \left(\frac{(t_0 + h)^{\alpha-1} e^{-\beta(t_0+h)}}{\beta\Gamma(\alpha)} + \frac{(t_0 + h)^{\alpha}}{\Gamma(\alpha+1)} \right) < 1,
\tag{10}
$$

then the IVP (1) has a unique solution in the space BC. Moreover, solutions of IVP (1) are uniformly asymptotically stable.

Proof. We divide the proof into two steps.

Step 1. We define the operator $P : C([t_0 - h, \infty), \mathbb{R}) \to C([t_0 - h, \infty), \mathbb{R})$ by

$$
(Py)(t) = \begin{cases} y(t_0) e^{-\beta(t-t_0)} \\ \quad + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{t_0}^{t} (t-s)^{\alpha-1} e^{-\beta(t-s)} \\ \qquad \times f(s, y_s) \, ds, \qquad t \geq t_0, \\ \phi(t), \qquad\qquad\qquad t \in [t_0 - h, t_0]. \end{cases}
\tag{11}
$$

The operator P maps BC into itself. Indeed for each $y \in BC$, and for each $t \geq 2t_0 + h$, it follows from (H_1) that

$$
\begin{aligned}
&\left| (Py)(t) \right| \\
&\leq \left| y(t_0) \right| e^{-\beta(t-t_0)} \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} e^{-\beta(t-s)} \left(f_0 + l \left\| y_s \right\| \right) ds \\
&\leq \left\| \phi \right\| e^{-\beta(t-t_0)} + \frac{f_0 + l \left\| y \right\|_\infty}{\Gamma(\alpha)} \\
&\quad \times \left(\int_{t_0}^{t-(t_0+h)} (t_0+h)^{\alpha-1} e^{-\beta(t-s)} ds + \int_{t-(t_0+h)}^{t} (t-s)^{\alpha-1} ds \right) \\
&\leq \left\| \phi \right\| + \left(f_0 + l \left\| y \right\|_\infty \right) \left(\frac{(t_0+h)^{\alpha-1} e^{-\beta(t_0+h)}}{\beta\Gamma(\alpha)} + \frac{(t_0+h)^{\alpha}}{\Gamma(\alpha+1)} \right).
\end{aligned}
\tag{12}
$$

For each $t \in [t_0, 2t_0 + h]$, we have

$$
\left| (Py)(t) \right| \leq \left\| \phi \right\| + \frac{\left(f_0 + l \left\| y \right\|_\infty \right) (t_0+h)^{\alpha}}{\Gamma(\alpha+1)},
\tag{13}
$$

and consequently $P(y) \in BC$.

Since $BC := BC([t_0 - h, \infty), \mathbb{R})$ is a Banach space with norm $\|\cdot\|_\infty$, we will show that $P : BC \to BC$ is a contraction map. Let $y_1, y_2 \in BC$. Then, we have for each $t \geqslant t_0$,

$$
\begin{aligned}
&\left|(Py_1)(t) - (Py_2)(t)\right| \\
&\quad \leqslant \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} e^{-\beta(t-s)} \\
&\qquad \times \left|f(s, y_{1s}) - f(s, y_{2s})\right| ds \\
&\quad \leqslant \frac{l}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} e^{-\beta(t-s)} \|y_{1s} - y_{2s}\| ds.
\end{aligned}
\tag{14}
$$

Therefore, for any $t \geqslant 2t_0 + h$,

$$
\begin{aligned}
&\left|(Py_1)(t) - (Py_2)(t)\right| \\
&\quad \leqslant \frac{l}{\Gamma(\alpha)} \|y_1(\cdot) - y_2(\cdot)\|_\infty \\
&\qquad \times \left(\int_{t_0}^{t-(t_0+h)} (t_0+h)^{\alpha-1} e^{-\beta(t-s)} ds \right. \\
&\qquad\quad \left. + \int_{t-(t_0+h)}^{t} (t-s)^{\alpha-1} ds \right) \\
&\quad \leqslant l \left(\frac{(t_0+h)^{\alpha-1} e^{-\beta(t_0+h)}}{\beta \Gamma(\alpha)} + \frac{(t_0+h)^\alpha}{\Gamma(\alpha+1)} \right) \\
&\qquad \times \|y_1(\cdot) - y_2(\cdot)\|_\infty,
\end{aligned}
\tag{15}
$$

and for $t_0 - h \leqslant t \leqslant 2t_0 + h$,

$$
\begin{aligned}
&\left|(Py_1)(t) - (Py_2)(t)\right| \\
&\quad \leqslant \frac{l}{\Gamma(\alpha)} \|y_1(\cdot) - y_2(\cdot)\|_\infty \int_{t_0}^{t} (t-s)^{\alpha-1} ds \\
&\quad \leqslant \frac{l(t_0+h)^\alpha}{\Gamma(\alpha+1)} \|y_1(\cdot) - y_2(\cdot)\|_\infty,
\end{aligned}
\tag{16}
$$

and thus

$$
\begin{aligned}
&\|(Py_1)(\cdot) - (Py_2)(\cdot)\|_\infty \\
&\quad \leqslant l \left(\frac{(t_0+h)^{\alpha-1} e^{-\beta(t_0+h)}}{\beta \Gamma(\alpha)} + \frac{(t_0+h)^\alpha}{\Gamma(\alpha+1)} \right) \\
&\qquad \times \|y_1(\cdot) - y_2(\cdot)\|_\infty.
\end{aligned}
\tag{17}
$$

Hence, (10) and (17) imply that the operator P is a contraction. Therefore, P has a unique fixed point by Banach's contraction principle.

Step 2. For any two solutions $x = x(t)$ and $y = y(t)$ of IVP (1) corresponding to initial values ψ and ϕ, by (4) we can deduce that for all $t \geqslant t_0 + h$ and all $\theta \in [-h, 0]$,

$$
\begin{aligned}
&\left|x(t+\theta) - y(t+\theta)\right| \\
&\quad \leqslant \left|x(t_0) - y(t_0)\right| e^{-\beta(t+\theta-t_0)} \\
&\qquad + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t+\theta} (t+\theta-s)^{\alpha-1} e^{-\beta(t+\theta-s)} \\
&\qquad\quad \times \left|f(s, x_s) - f(s, y_s)\right| ds \\
&\quad \leqslant \left|x(t_0) - y(t_0)\right| e^{-\beta(t+\theta-t_0)} \\
&\qquad + \frac{l}{\Gamma(\alpha)} \int_{t_0}^{t+\theta} (t+\theta-s)^{\alpha-1} e^{-\beta(t+\theta-s)} \|x_s - y_s\| ds.
\end{aligned}
\tag{18}
$$

Then, it follows that

$$
\begin{aligned}
&e^{\beta t} \|x_t - y_t\| \\
&\quad \leqslant \left|x(t_0) - y(t_0)\right| e^{\beta(h+t_0)} \\
&\qquad + \frac{l e^{\beta h}}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} e^{\beta s} \|x_s - y_s\| ds.
\end{aligned}
\tag{19}
$$

Let $w(t) = e^{\beta t} \|x_t - y_t\|$. Then, we have

$$
\begin{aligned}
w(t) &\leqslant \left|x(t_0) - y(t_0)\right| e^{\beta(h+t_0)} \\
&\quad + \frac{l e^{\beta h}}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} w(s) ds.
\end{aligned}
\tag{20}
$$

Applying Lemma 2, one can see that there exists a constant K such that

$$
\begin{aligned}
w(t) &\leqslant \left|x(t_0) - y(t_0)\right| e^{\beta(h+t_0)} \\
&\quad + \frac{K l e^{\beta h}}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \left|x(t_0) - y(t_0)\right| e^{\beta(h+t_0)} ds \\
&\leqslant \left|x(t_0) - y(t_0)\right| e^{\beta(h+t_0)} \left(1 + \frac{K l e^{\beta h}}{\Gamma(\alpha+1)} (t-t_0)^\alpha \right).
\end{aligned}
\tag{21}
$$

Hence, we obtain

$$
\begin{aligned}
e^{\beta t} \|x_t - y_t\| = w(t) &\leqslant \left|x(t_0) - y(t_0)\right| e^{\beta(h+t_0)} \\
&\quad \times \left(1 + \frac{K l e^{\beta h}}{\Gamma(\alpha+1)} (t-t_0)^\alpha \right),
\end{aligned}
\tag{22}
$$

and thus for all $t \geqslant t_0 + h$,

$$
\begin{aligned}
&\left|x(t) - y(t)\right| \\
&\quad \leqslant \left|x(t_0) - y(t_0)\right| e^{-\beta(t-h-t_0)} \left(1 + \frac{K l e^{\beta h}}{\Gamma(\alpha+1)} (t-t_0)^\alpha \right),
\end{aligned}
\tag{23}
$$

which implies that the solutions of IVP (1) are uniformly asymptotically stable. □

Now we give global existence and uniform asymptotic stability results based on the nonlinear alternative of Leray-Schauder type.

Theorem 5. *Assume that the following hypotheses hold:*

(H_2) *f is a continuous function;*

(H_3) *there exist positive functions $k_1, k_2 \in BC([t_0, \infty), \mathbb{R}_+)$ such that*

$$|f(t, u_t)| \le k_1(t) + k_2(t) \|u_t\| \tag{24}$$

for $t \in J$ and every $u_t \in C([-h, 0], \mathbb{R})$;

(H_4) *moreover, assume that*

$$\lim_{t \to \infty} \int_{t_0}^{t} (t - s)^{\alpha - 1} e^{-\beta(t-s)} k_1(s) \, ds = 0,$$
$$\lim_{t \to \infty} \int_{t_0}^{t} (t - s)^{\alpha - 1} e^{-\beta(t-s)} k_2(s) \, ds = 0. \tag{25}$$

Then the IVP (1) admits a solution in the space BC. Moreover, solutions of IVP (1) are uniformly asymptotically stable.

Proof. Let $P : C([t_0 - h, \infty), \mathbb{R}) \to C([t_0 - h, \infty), \mathbb{R})$ be defined as in (11). First, we show that P maps BC into itself. Let $K_1 = \sup_{t \ge t_0} k_1(t)$, $K_2 = \sup_{t \ge t_0} k_2(t)$. Indeed, the map $P(y)$ is continuous on $[t_0 - h, +\infty)$ for each $y \in BC$, and for each $t \ge 2t_0 + h$, (H_2) implies that

$$|(Py)(t)|$$
$$\le |y(t_0)| e^{-\beta(t-t_0)}$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t - s)^{\alpha - 1} e^{-\beta(t-s)} (K_1 + K_2 \|y_s\|) \, ds$$
$$\le \|\phi\| e^{-\beta(t-t_0)} + \frac{K_1 + K_2 \|y\|_\infty}{\Gamma(\alpha)}$$
$$\times \left(\int_{t_0}^{t-(t_0+h)} (t_0 + h)^{\alpha - 1} e^{-\beta(t-s)} ds \right.$$
$$\left. + \int_{t-(t_0+h)}^{t} (t - s)^{\alpha - 1} ds \right) \tag{26}$$
$$\le \|\phi\| + (K_1 + K_2 \|y\|_\infty)$$
$$\times \left(\frac{(t_0 + h)^{\alpha - 1} e^{-\beta(t_0+h)}}{\beta \Gamma(\alpha)} + \frac{(t_0 + h)^\alpha}{\Gamma(\alpha + 1)} \right),$$

for each $t \in [t_0, 2t_0 + h]$, we have

$$|(Py)(t)| \le \|\phi\| + \frac{(K_1 + K_2 \|y\|_\infty)(t_0 + h)^\alpha}{\Gamma(\alpha + 1)}, \tag{27}$$

and for any $t \in [t_0 - h, t_0]$,

$$|(Py)(t)| \le \|\phi\|. \tag{28}$$

Thus,

$$\|P(y)\|_\infty$$
$$\le \|\phi\| + (K_1 + K_2 \|y\|_\infty)$$
$$\times \left(\frac{(t_0 + h)^{\alpha - 1} e^{-\beta(t_0+h)}}{\beta \Gamma(\alpha)} + \frac{(t_0 + h)^\alpha}{\Gamma(\alpha + 1)} \right), \tag{29}$$

and consequently $P(y) \in BC$.

Next, we show that the operator P is continuous and completely continuous, and there exists an open set $U \subset BC$ with $y \ne \lambda P(y)$ for $\lambda \in (0, 1)$ and $y \in \partial U$.

Step 1 (P is continuous). Let $\{y_n\}$ be a sequence such that $y_n \to y$ in BC. Then, there exist $R > 0$ and $N > 0$ such that

$$\|y_n\|_\infty + \|y\|_\infty < R, \quad \forall n \ge N. \tag{30}$$

Let $\varepsilon > 0$ be given. Since (H_4) holds, there is a real number $T > 0$ such that

$$\frac{2}{\Gamma(\alpha)} \int_{t_0}^{t} (t - s)^{\alpha - 1} e^{-\beta(t-s)} (k_1(s) + k_2(s) R) \, ds < \varepsilon, \tag{31}$$

for all $t \ge T$. Now we consider the following two cases.

Case 1. If $t \ge T$, then it follows from (H_3) and (30)-(31) that for n sufficiently large

$$|Py_n(t) - Py(t)|$$
$$\le |y_n(t_0) - y(t_0)| e^{-\beta(t-t_0)}$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t - s)^{\alpha - 1} e^{-\beta(t-s)} |f(s, y_{ns}) - f(s, y_s)| \, ds$$
$$\le |y_n(t_0) - y(t_0)|$$
$$+ \frac{2}{\Gamma(\alpha)} \int_{t_0}^{t} (t - s)^{\alpha - 1} e^{-\beta(t-s)}$$
$$\times (k_1(s) + k_2(s) R) \, ds < 2\varepsilon. \tag{32}$$

Case 2. If $t_0 \le t \le T$, since f is a continuous function, one has

$$|Py_n(t) - Py(t)|$$
$$\le |y_n(t_0) - y(t_0)|$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t - s)^{\alpha - 1} e^{-\beta(t-s)} |f(s, y_{ns}) - f(s, y_s)| \, ds$$
$$\le |y_n(t_0) - y(t_0)|$$
$$+ \frac{(T - t_0)^\alpha}{\Gamma(\alpha + 1)} \sup_{s \in [t_0, T]} |f(s, y_{ns}) - f(s, y_s)|. \tag{33}$$

Note that $y_n \to y$ in BC. Hence, (32) and (33) imply that

$$\|P(y_n) - P(y)\|_\infty \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty. \tag{34}$$

Step 2 (P maps bounded sets into bounded sets in BC). Indeed, it is enough to show that for any $\eta > 0$, there exists a positive constant ℓ such that for each $y \in B_\eta = \{y \in BC : \|y\|_\infty \le \eta\}$ one has $\|P(y)\|_\infty \le \ell$. Let $y \in B_\eta$. Then, we have for each $t \ge 2t_0 + h$,

$$|(Py)(t)| \le |y(t_0)| e^{-\beta(t-t_0)}$$

$$+ \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} e^{-\beta(t-s)} |f(s, y_s)| ds$$

$$\le \eta + \frac{K_1 + K_2 \|y\|_\infty}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} e^{-\beta(t-s)} ds \tag{35}$$

$$\le \eta + (K_1 + K_2 \eta)$$

$$\times \left(\frac{(t_0 + h)^{\alpha-1} e^{-\beta(t_0 + h)}}{\beta \Gamma(\alpha)} + \frac{(t_0 + h)^\alpha}{\Gamma(\alpha + 1)} \right)$$

$$=: \ell,$$

and for each t with $t_0 \le t \le 2t_0 + h$,

$$|(Py)(t)| \le \eta + (K_1 + K_2 \eta) \frac{(t_0 + h)^\alpha}{\Gamma(\alpha + 1)}. \tag{36}$$

Hence, $\|P(y)\|_\infty \le \ell$.

Step 3 (P maps bounded sets into equicontinuous sets on every compact subset $[t_0 - h, b]$ of $[t_0 - h, \infty)$). Let $t_1, t_2 \in [t_0, b]$, $t_1 < t_2$, and let B_η be a bounded set of BC as in Step 2. Let $y \in B_\eta$. Then, we have

$$|(Py)(t_2) - (Py)(t_1)|$$

$$\le \left| y(t_0) e^{-\beta(t_2 - t_0)} - y(t_0) e^{-\beta(t_1 - t_0)} \right|$$

$$+ \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} \left| ((t_2 - s)^{\alpha-1} e^{-\beta(t_2 - s)} \right.$$

$$\left. - (t_1 - s)^{\alpha-1} e^{-\beta(t_1 - s)} \right) f(s, y_s) \left| ds \right.$$

$$+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left| (t_2 - s)^{\alpha-1} e^{-\beta(t_2 - s)} f(s, y_s) \right| ds$$

$$\le |y(t_0)| e^{\beta t_0} \left| e^{-\beta t_2} - e^{-\beta t_1} \right| + \frac{K_1 + K_2 \eta}{\Gamma(\alpha + 1)} (t_2 - t_1)^\alpha$$

$$+ \frac{K_1 + K_2 \eta}{\Gamma(\alpha)} \int_{t_0}^{t_1} \left((t_1 - s)^{\alpha-1} e^{-\beta(t_1 - s)} \right.$$

$$\left. - (t_2 - s)^{\alpha-1} e^{-\beta(t_2 - s)} \right) ds$$

$$\le |y(t_0)| e^{\beta t_0} \left| e^{-\beta t_2} - e^{-\beta t_1} \right| + \frac{K_1 + K_2 \eta}{\Gamma(\alpha + 1)} (t_2 - t_1)^\alpha$$

$$+ \frac{K_1 + K_2 \eta}{\Gamma(\alpha)} \int_{t_0}^{t_1} \left((t_1 - s)^{\alpha-1} e^{-\beta(t_1 - s)} \right.$$

$$\left. - (t_2 - s)^{\alpha-1} e^{-\beta(t_1 - s)} \right) ds$$

$$+ \frac{K_1 + K_2 \eta}{\Gamma(\alpha)} \int_{t_0}^{t_1} \left((t_2 - s)^{\alpha-1} e^{-\beta(t_1 - s)} \right.$$

$$\left. - (t_2 - s)^{\alpha-1} e^{-\beta(t_2 - s)} \right) ds. \tag{37}$$

Observing that

$$\frac{K_1 + K_2 \eta}{\Gamma(\alpha)}$$

$$\times \int_{t_0}^{t_1} \left((t_1 - s)^{\alpha-1} e^{-\beta(t_1 - s)} - (t_2 - s)^{\alpha-1} e^{-\beta(t_1 - s)} \right) ds$$

$$\le \frac{K_1 + K_2 \eta}{\Gamma(\alpha)} \int_{t_0}^{t_1} \left((t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1} \right) ds$$

$$\le \frac{K_1 + K_2 \eta}{\Gamma(\alpha + 1)} \left((t_1 - t_0)^\alpha - (t_2 - t_0)^\alpha + (t_2 - t_1)^\alpha \right)$$

$$\le \frac{K_1 + K_2 \eta}{\Gamma(\alpha + 1)} (t_2 - t_1)^\alpha, \tag{38}$$

from Taylor's theorem, we obtain

$$\frac{K_1 + K_2 \eta}{\Gamma(\alpha)}$$

$$\times \int_{t_0}^{t_1} \left((t_2 - s)^{\alpha-1} e^{-\beta(t_1 - s)} - (t_2 - s)^{\alpha-1} e^{-\beta(t_2 - s)} \right) ds$$

$$\le \frac{K_1 + K_2 \eta}{\Gamma(\alpha)} (t_2 - t_1)^{\alpha-1} \int_{t_0}^{t_1} \left(e^{-\beta(t_1 - s)} - e^{-\beta(t_2 - s)} \right) ds$$

$$\le \frac{K_1 + K_2 \eta}{\beta \Gamma(\alpha)} (t_2 - t_1)^{\alpha-1} \left(1 - e^{-\beta(t_2 - t_1)} \right)$$

$$= \frac{K_1 + K_2 \eta}{\Gamma(\alpha)} \left((t_2 - t_1)^\alpha + \frac{o(t_2 - t_1)}{t_2 - t_1} (t_2 - t_1)^\alpha \right), \tag{39}$$

where $\lim_{t_2 - t_1 \to 0} (o(t_2 - t_1)/(t_2 - t_1)) = 0$. By (37)–(39), we can conclude that

$$|(Py)(t_2) - (Py)(t_1)|$$

$$\le \eta e^{\beta t_0} \left| e^{-\beta t_2} - e^{-\beta t_1} \right|$$

$$+ \frac{2(K_1 + K_2 \eta)}{\Gamma(\alpha + 1)} (t_2 - t_1)^\alpha$$

$$+ \frac{K_1 + K_2 \eta}{\Gamma(\alpha)} \left((t_2 - t_1)^\alpha + \frac{o(t_2 - t_1)}{t_2 - t_1} (t_2 - t_1)^\alpha \right). \tag{40}$$

As $t_1 \rightarrow t_2$, the right-hand side of the above inequality tends to zero. The equicontinuity for the cases $t_1 < t_2 \leqslant t_0$ and $t_1 \leqslant t_0 \leqslant t_2$ is obvious.

Step 4 (*P maps bounded sets into equiconvergent sets*). Let $y \in B_\eta$. Then

$$
\begin{aligned}
|(Py)(t)| \\
\leqslant |y(t_0)| e^{-\beta(t-t_0)} \\
+ \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} e^{-\beta(t-s)} |f(s, y_s)| \, ds \\
\leqslant \eta e^{-\beta(t-t_0)} \\
+ \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} e^{-\beta(t-s)} (k_1(s) + k_2(s)\eta) \, ds.
\end{aligned}
\tag{41}
$$

Therefore, (H_4) implies that $|(Py)(t)|$ uniformly (with respect to $y \in B(\eta)$) converges to 0 as $t \rightarrow \infty$. As a consequence of Steps 1–4, we can conclude that $P : BC \rightarrow BC$ is continuous and completely continuous.

Step 5 (*a priori bounds*). We now show that there exists an open set $U \subseteq BC$ with $y \neq \lambda P(y)$ for $\lambda \in (0,1)$ and $y \in \partial U$.

Let $y \in BC$ and $y = \lambda P(y)$ for some $0 < \lambda < 1$. Then, for each $t \in [t_0, \infty)$, we obtain

$$
y(t) = \lambda \left[y(t_0) e^{-\beta(t-t_0)} \right.
$$
$$
\left. + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} e^{-\beta(t-s)} f(s, y_s) \, ds \right].
\tag{42}
$$

By (H_3), we have that for all $\theta \in [-h, 0]$ and $t \geqslant t_0 + h$,

$$
\begin{aligned}
|y(t+\theta)| \\
\leqslant |y(t_0)| e^{-\beta(t+\theta-t_0)} \\
+ \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t+\theta} (t+\theta-s)^{\alpha-1} e^{-\beta(t+\theta-s)} |f(s, y_s)| \, ds \\
\leqslant |y(t_0)| e^{-\beta(t+\theta-t_0)} \\
+ \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t+\theta} (t+\theta-s)^{\alpha-1} e^{-\beta(t+\theta-s)} (K_1 + K_2 \|y_s\|) \, ds,
\end{aligned}
\tag{43}
$$

and thus

$$
\begin{aligned}
\|y_t\| \\
\leqslant |y(t_0)| e^{-\beta(t-h-t_0)} \\
+ \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} e^{-\beta(t-h-s)} (K_1 + K_2 \|y_s\|) \, ds.
\end{aligned}
\tag{44}
$$

From the arguments in (26)-(27), we can conclude that for each $t \in [t_0, \infty)$,

$$
\begin{aligned}
\frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} e^{-\beta(t-s)} \, ds \\
\leqslant \frac{(t_0+h)^{\alpha-1} e^{-\beta(t_0+h)}}{\beta \Gamma(\alpha)} + \frac{(t_0+h)^\alpha}{\Gamma(\alpha+1)} =: R_1.
\end{aligned}
\tag{45}
$$

Hence,

$$
\begin{aligned}
e^{\beta t} \|y_t\| \\
\leqslant \|\phi\| e^{\beta(h+t_0)} + e^{\beta(h+t)} K_1 R_1 \\
+ \frac{K_2 e^{\beta h}}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} e^{\beta s} \|y_s\| \, ds.
\end{aligned}
\tag{46}
$$

Let $R_2 = \|\phi\| e^{\beta(h+t_0)} + e^{\beta h} K_1 R_1$. Then, from **Lemma 2**, there exists K such that we have for all $t \geqslant t_0 + h$,

$$
\|y_t\| \leqslant R_2 + \frac{K K_2 R_2 e^{\beta h}}{\Gamma(\alpha+1)} (t-t_0)^\alpha e^{-\beta t}.
\tag{47}
$$

Since $\lim_{t \to \infty} (t-t_0)^\alpha e^{-\beta t} = 0$, there exists $R_3 > 0$ such that

$$
\|y\|_\infty \leqslant R_3.
\tag{48}
$$

Set

$$
U = \{ y \in BC : \|y\|_\infty < R_3 + 1 \}.
\tag{49}
$$

$P : U \rightarrow BC$ is continuous and completely continuous. From the choice of U, there is no $y \in \partial U$ such that $y = \lambda P(y)$, for $\lambda \in (0,1)$. As a consequence of Leray-Schauder fixed-point theorem, we deduce that P has a fixed point y in U.

Step 6 (*uniform asymptotic stability of solutions*). Let $B \subset C([-h, 0], \mathbb{R})$ be bounded; that is, there exists $d \geqslant 0$ such that

$$
\|\psi\| = \sup_{\theta \in [-h, 0]} |\psi(\theta)| \leqslant d \quad \forall \psi \in B.
\tag{50}
$$

From the similar arguments in Step 4, we can deduce that there exists $R_4 > 0$ such that for all solutions $y(t, t_0, \phi)$ of IVP (1) with initial data $\phi \in B$, we have

$$
\|y\|_\infty \leqslant R_4, \quad \forall \phi \in B.
\tag{51}
$$

Now we consider two solutions $x = x(t)$ and $y = y(t)$ of IVP (1) corresponding to the initial values ψ and ϕ. Note that for all $t \geqslant t_0$,

$$
\begin{aligned}
|x(t) - y(t)| \\
\leqslant |x(t_0) - y(t_0)| e^{-\beta(t-t_0)} \\
+ \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} e^{-\beta(t-s)} (|f(s, x_s)| + |f(s, y_s)|) \, ds \\
\leqslant 2d e^{-\beta(t-t_0)} \\
+ \frac{2}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} e^{-\beta(t-s)} (k_1(s) + k_2(s) R_4) \, ds.
\end{aligned}
\tag{52}
$$

Then, the proof of uniform asymptotic stability of solutions can be done by making use of (H_4) and (52).

The proof of Theorem 5 is completed. □

4. Examples

Example 1. Consider the fractional functional differential equation

$$D^{1/2}\left[y\left(t\right)e^{t}\right] = \frac{e^{2t}}{8\left(e^{t}+e^{-t}\right)}\sin^{4}\left(y\left(t-1\right)\right)+e^{t}, \quad t \geqslant 0,$$

$$y\left(t\right) = \phi\left(t\right), \quad -1 \leqslant t \leqslant 0,$$

$$(53)$$

where $f(t, y_t) = (e^t/8(e^t + e^{-t}))\sin^4(y(t-1)) + 1$. It is clear that condition (H_0) holds. Let $x_t, y_t \in C([-1, 0], \mathbb{R})$. Then for all $t \in [0, \infty)$, we have

$$\left| f\left(t, x_t\right) - f\left(t, y_t\right) \right|$$

$$= \frac{e^{t}}{8\left(e^{t}+e^{-t}\right)}\left|\sin^{4}\left(x\left(t-1\right)\right)-\sin^{4}\left(y\left(t-1\right)\right)\right|$$

$$\leqslant \frac{e^{t}}{2\left(e^{t}+e^{-t}\right)}\left|x\left(t-1\right)-y\left(t-1\right)\right|$$

$$\leqslant \frac{1}{2}\left|x\left(t-1\right)-y\left(t-1\right)\right|.$$

$$(54)$$

On the other hand, note that $f(t, 0) = 1$ for each $t \in [0, \infty)$ and $(1/2)(e^{-1}/\Gamma(1/2) + 1/\Gamma(3/2)) < 1$. Hence, conditions (H_1) and (10) hold. By Theorem 4, we conclude that IVP (53) has a unique solution in the space $BC([-1, \infty), \mathbb{R})$, and the solution of IVP (53) is uniformly asymptotically stable.

Example 2. Consider the fractional functional differential equation

$$D^{1/2}\left[y\left(t\right)e^{t}\right] = 10e^{t}(t+1)^{-3/4}\frac{y\left(t-1\right)}{1+\left|y\left(t-1\right)\right|}, \quad t \geqslant 0,$$

$$y\left(t\right) = \phi\left(t\right), \quad -1 \leqslant t \leqslant 0,$$

$$(55)$$

where $f(t, y_t) = 10(t + 1)^{-3/4}(y(t-1)/(1 + |y(t-1)|))$. It is easy to see that condition (H_2) holds. Let $y_t \in C([-1, 0], \mathbb{R})$. Then, for all $t \in [0, \infty)$, we find that

$$\left| f\left(t, y_t\right) \right|$$

$$= \left|10(t+1)^{-3/4}\frac{y\left(t-1\right)}{1+\left|y\left(t-1\right)\right|}\right| \leqslant 10(t+1)^{-3/4}\left|y\left(t-1\right)\right|,$$
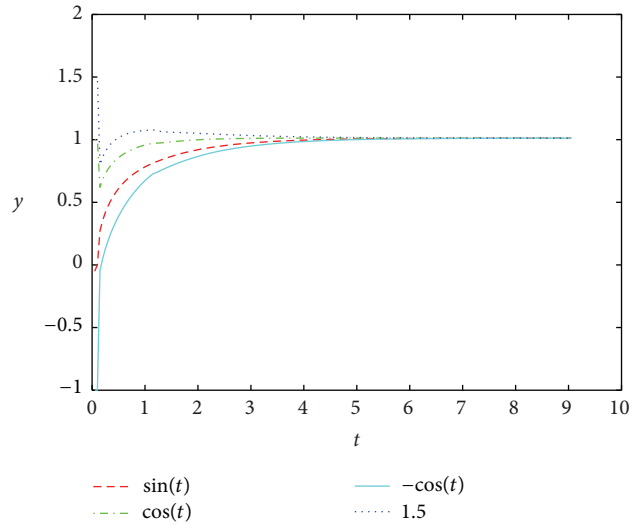
$$(56)$$

FIGURE 1: The numerical solutions of **Example 1** with the initial conditions $\phi(t) = \sin(t), \cos(t), -\cos(t)$, and 1.5, respectively.
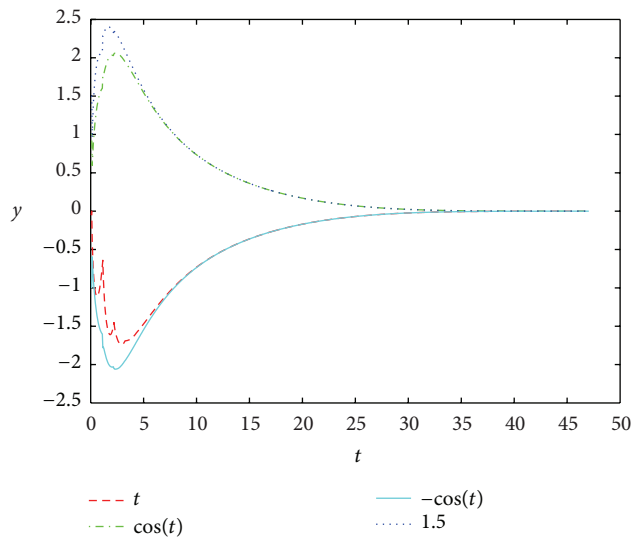

FIGURE 2: The numerical solutions of **Example 2** with the initial conditions $\phi(t) = t, \cos(t), -\cos(t)$, and 1.5, respectively.

where $10(t + 1)^{-3/4} \in BC([0, \infty), \mathbb{R}_+)$ with $\sup_{t \geqslant 0} 10(t + 1)^{-3/4} = 10$ and

$$\frac{1}{\Gamma\left(1/2\right)}\int_{0}^{t}(t-s)^{-1/2}e^{-(t-s)}10(s+1)^{-3/4}ds$$

$$\leqslant \frac{10}{\Gamma\left(1/2\right)}\int_{0}^{t}(t-s)^{-1/2}s^{-3/4}ds$$

$$= \frac{10\Gamma\left(1/4\right)}{\Gamma\left(3/4\right)}t^{-1/4} \longrightarrow 0 \quad \text{as} \quad t \longrightarrow \infty.$$

$$(57)$$

Thus, conditions (H_3) and (H_4) hold, and the global existence and the uniform asymptotic stability of solutions of IVP (55) can be obtained by applying Theorem 5.

By using the algorithm given in [19], we numerically simulate Example 1 with the initial conditions $\phi(t) = \sin(t)$, $\cos(t), -\cos(t), 1.5$, and Example 2 with $\phi(t) = t, \cos(t), -\cos(t), 1.5$; see Figures 1 and 2. From the numerical results, it can be noted that both of the solutions of Examples 1 and 2 converge uniformly, and the solutions of Example 1 converge faster than the ones of Example 2. The numerical results confirm the theoretical analysis.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

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