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Research Article

On Exact Series Solution of Strongly Coupled Mixed Parabolic Problems

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This paper studies the construction of the exact solution for parabolic coupled systems of the type $u_t = Au_{xx}$, $A_1u(0,t)+B_1u_x(0,t) = 0$, $A_2u(l,t) + B_2u_x(l,t) = 0$, $0 < x < 1$, $t > 0$, and $u(x,0) = f(x)$, where A_1, A_2, B_1, and B_2 are arbitrary matrices for which the block matrix $\begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix}$ is nonsingular, and A is a positive stable matrix.

1. Introduction

Coupled partial differential systems with coupled boundary-value conditions are frequent in quantum mechanical scattering problems [1–3], chemical physics [4–6], thermoelastoplastic modelling [7], coupled diffusion problems [8–10], and other fields. In this paper, we consider systems of the type

$$u_t(x,t) - Au_{xx}(x,t) = 0, \quad 0 < x < 1, \ t > 0, \tag{1}$$

$$A_1u(0,t) + B_1u_x(0,t) = 0, \quad t > 0, \tag{2}$$

$$A_2u(1,t) + B_2u_x(1,t) = 0, \quad t > 0, \tag{3}$$

$$u(x,0) = f(x), \quad 0 \le x \le 1, \tag{4}$$

where the unknown $u = (u_1, u_2, \ldots, u_m)^T$ and the initial condition $f = (f_1, f_2, \ldots, f_m)^T$ are m-dimensional vectors, $A_i, B_i, i = 1, 2$, are $m\times m$ complex matrices, elements of $\mathbb{C}^{m\times m}$, and A is a matrix which satisfies the condition

$$\operatorname{Re}(z) > 0 \text{ for all eigenvalues } z \text{ of } A, \tag{5}$$

and we say that A is a positive stable matrix (where $\operatorname{Re}(z)$ denotes the real part of $z \in \mathbb{C}$). We assume that the block matrix

$$\begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix} \text{ is regular} \tag{6}$$

and also that

$$\text{the matrix pencil } A_1 + \rho B_1 \text{ is regular.} \tag{7}$$

Condition (7) is well known from the literature of singular systems of differential equations, and it involves the existence of some $\rho_0 \in \mathbb{C}$ such that matrix $A_1 + \rho_0 B_1$ is invertible [11].

Problem (1)–(4) with the less restrictive condition that (7) was solved in [12], but not with all of its blocks A_1, A_2, B_1, B_2, is singular (in particular $A_1 = I$). Mixed problems of the previously mentioned type, but with the Dirichlet conditions $u(0,t) = 0$, $u(1,t) = 0$ instead of (2) and (3), have been treated in [13, 14].

Throughout this paper, and as usual, matrix I denotes the identity matrix. The set of all the eigenvalues of a matrix C in $\mathbb{C}^{m\times m}$ is denoted by $\sigma(C)$, and its 2-norm $\|C\|$ is defined by [15, page 56]

$$\|C\| = \sup_{x\neq 0} \frac{\|Cx\|}{\|x\|}, \tag{8}$$

where for vector $y \in \mathbb{C}^m$, the Euclidean norm of y is $\|y\|$. By [15, page 556], it follows that

$$\left\| e^{At} \right\| \le e^{t\alpha(A)} \sum_{k=0}^{m-1} \frac{\left\| \sqrt{m} A \right\|^k t^k}{k!}, \quad t \ge 0, \tag{9}$$

where $\alpha(A) = \max\{\mathrm{Re}(w); w \in \sigma(A)\}$. We say that a subspace E of $\mathbb{C}^m$ is invariant by the matrix $A \in \mathbb{C}^{m\times m}$, if $A(E) \subset E$. If B is a matrix in $\mathbb{C}^{n\times m}$, we denote by $B^{\dagger}$ its Moore-Penrose pseudoinverse. A collection of examples, properties, and applications of this concept may be found in [11, 16], and $B^{\dagger}$ can be efficiently computed with the *MATLAB* and *Mathematica* computer algebra systems.

2. Preliminaries and Notation

In [17], eigenfunctions of problem (1)–(3) were constructed assuming other additional conditions besides (6) and (7). We recall in this section the notation and results needed. Let $\widetilde{A}_1$ and $\widetilde{B}_1$ be matrices defined by

$$\widetilde{A}_1 = \left(A_1 + \rho_0 B_1\right)^{-1} A_1, \qquad \widetilde{B}_1 = \left(A_1 + \rho_0 B_1\right)^{-1} B_1, \tag{10}$$

fulfilling the relation: $\widetilde{A}_1 + \rho_0 \widetilde{B}_1 = I$. Under hypothesis (6), matrix $B_2 - (A_2 + \rho_0 B_2)\widetilde{B}_1$ is regular; see [17, page 431], and let $\widetilde{A}_2$ and $\widetilde{B}_2$ be the matrices defined by

$$\begin{gathered} \widetilde{A}_2 = \left[B_2 - \left(A_2 + \rho_0 B_2\right) \widetilde{B}_1\right]^{-1} A_2, \\ \widetilde{B}_2 = \left[B_2 - \left(A_2 + \rho_0 B_2\right) \widetilde{B}_1\right]^{-1} B_2, \end{gathered} \tag{11}$$

so that they satisfy the relationships

$$\widetilde{B}_2 - \left(\widetilde{A}_2 + \rho_0 \widetilde{B}_2\right) \widetilde{B}_1 = I, \qquad \widetilde{B}_2 \widetilde{A}_1 - \widetilde{A}_2 \widetilde{B}_1 = I. \tag{12}$$

Assuming that the following condition

$$\begin{gathered} \text{exists } b_1 \in \sigma\left(\widetilde{B}_1\right) - \{0\}, \qquad b_2 \in \sigma\left(\widetilde{B}_2\right), \\ v \in \mathbb{C}^m - \{0\}, \\ \text{such that} \left(\widetilde{B}_1 - b_1 I\right) v = \left(\widetilde{B}_2 - b_2 I\right) v = 0, \end{gathered} \tag{13}$$

and that values b_1, b_2 of condition (13) satisfy

$$b_1 b_2 \in \mathbb{R}, \quad \text{where } b_1 \in \mathbb{R} \text{ or } 2 b_1 b_2 \left(\mathrm{Re}\left(b_1^{-1}\right) - \rho_0\right) = 1 \quad \text{if } b_1 \notin \mathbb{R}, \tag{14}$$

we can define the function

$$\alpha\left(\rho_0, b_1, b_2, \lambda\right) = \frac{\left(1 - b_2 + \rho_0 b_1 b_2\right)\left(1 - \rho_0 b_1\right)}{b_1} - b_1 b_2 \lambda^2, \quad \lambda > 0. \tag{15}$$

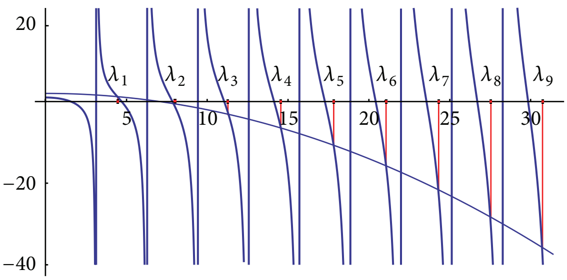

Figure 1: Graphical representation of $y = \lambda \cot(\lambda)$ and determination of the eigenvalues λ_n.

Note that under hypothesis (14) we have guaranteed the existence of the solutions for

$$\lambda \cot(\lambda) = \frac{\left(1 - b_2 + \rho_0 b_1 b_2\right)\left(1 - \rho_0 b_1\right)}{b_1} - b_1 b_2 \lambda^2. \tag{16}$$

Equation (16) has a unique solution λ_k in each interval $(k\pi, (k+1)\pi)$ for $k \ge 1$, as seen in Figure 1. Also, it is straightforward to prove the following lemma.

Lemma 1. *Under hypothesis* (14)*, the roots* λ_k *of* (16) *satisfy* $\lim_{n\to\infty} \lambda_n = +\infty$*. Also, if* $b_1 b_2 \neq 0$*, then*

$$\lim_{n\to\infty} \sin\left(\lambda_n\right) = 0, \qquad \lim_{n\to\infty} \left|\cos\left(\lambda_n\right)\right| = 1. \tag{17}$$

Otherwise, if $b_1 b_2 = 0$*, then*

$$\lim_{n\to\infty} \left|\sin\left(\lambda_n\right)\right| = 1, \qquad \lim_{n\to\infty} \cos\left(\lambda_n\right) = 0. \tag{18}$$

However, in all cases it is

$$\lim_{n\to\infty} \left(\lambda_{n+1} - \lambda_n\right) = \pi. \tag{19}$$

Proof. Function $f(\lambda) = \lambda \cot(\lambda)$ has vertical asymptotes at the points $\lambda = k\pi$, $k \in \mathbb{N}$, and $f(\lambda)$ has zeros at the points $\lambda = (\pi/2) + k\pi$, $k \in \mathbb{N}$. Thus, as we have stated, the real coefficient function $((1 - b_2 + \rho_0 b_1 b_2)(1 - \rho_0 b_1)/b_1) - b_1 b_2 \lambda^2$ intersects the graph of the function $f(\lambda)$ in each interval $(k\pi, (k+1)\pi)$, where $\lambda_k \in (k\pi, (k+1)\pi)$ is the point of intersection. Thus, the sequence $\{\lambda_k\}_{k\ge 1}$ is monotonicaly increasing with $\lim_{k\to\infty} \lambda_k = \infty$. We have to consider two possibilities.

(i) $b_1 b_2 > 0$. Function $((1 - b_2 + \rho_0 b_1 b_2)(1 - \rho_0 b_1)/b_1) - b_1 b_2 \lambda^2$ is therefore decreasing, and as seen in Figure 1, for large enough k, then $\lambda_k \in ((\pi/2) + k\pi, (k+1)\pi)$.

(ii) $b_1 b_2 < 0$. Function $((1 - b_2 + \rho_0 b_1 b_2)(1 - \rho_0 b_1)/b_1) - b_1 b_2 \lambda^2$ is therefore increasing, and as seen in Figure 1, for large enough k, then $\lambda_k \in (k\pi, (\pi/2) + k\pi)$.

Thus, observe that if $b_1 b_2 \neq 0$, then $(\pi/2) < \lambda_{k+1} - \lambda_k < (3\pi/2)$ for large sufficiently k. For λ_k, reemploying in (16), one gets

$$\lambda_k \cot\left(\lambda_k\right) = \frac{\left(1 - b_2 + \rho_0 b_1 b_2\right)\left(1 - \rho_0 b_1\right)}{b_1} - b_1 b_2 \lambda_k^2, \tag{20}$$

dividing by λ_k^2 and taking limits where $k \to \infty$:

$$\lim_{k\to\infty} \frac{\cot(\lambda_k)}{\lambda_k} = -b_1 b_2 \neq 0. \tag{21}$$

This demonstrates that sequences $\{\lambda_k\}_{k\geq 1}$ and $\{\cot(\lambda_k)\}_{k\geq 1}$ are infinite equivalents and

$$\lim_{k\to\infty} \cot(\lambda_k) = \infty, \tag{22}$$

where $\lim_{k\to\infty} \tan(\lambda_k) = 0$. Moreover, as $\{\cos(\lambda_k)\}_{k\geq 1}$ is bounded, one gets that $\lim_{k\to\infty} \sin(\lambda_k) = 0$ and $\lim_{k\to\infty} |\cos(\lambda_k)| = 1$. Taking into account that

$$\tan(\lambda_{k+1} - \lambda_k) = \frac{\tan(\lambda_{k+1}) - \tan(\lambda_k)}{1 + \tan(\lambda_{k+1})\tan(\lambda_k)}, \tag{23}$$

considering limits where $k \to \infty$, one gets $\lim_{k\to\infty} \tan(\lambda_{k+1} - \lambda_k) = 0$, and with $(\pi/2) < \lambda_{k+1} - \lambda_k < (3\pi/2)$, then $\lim_{k\to\infty}(\lambda_{k+1} - \lambda_k) = \pi$.

If $b_1 b_2 = 0$, then one obtains two possibilities.

(i) If $((1 - b_2 + \rho_0 b_1 b_2)(1 - \rho_0 b_1)/b_1) > 0$, as one can see in Figure 1, for large enough k, $\lambda_k \in (k\pi, (\pi/2) + k\pi)$.

(ii) If $((1 - b_2 + \rho_0 b_1 b_2)(1 - \rho_0 b_1)/b_1) < 0$, as one can see in Figure 1, for large enough k, $\lambda_k \in ((\pi/2) + k\pi, (k+1)\pi)$.

Thus, observe that if $b_1 b_2 = 0$, then also $(\pi/2) < \lambda_{k+1} - \lambda_k < (3\pi/2)$ for k sufficiently large. For λ_k, reemploying in (16), one gets

$$\lambda_k \cot(\lambda_k) = \frac{(1 - b_2 + \rho_0 b_1 b_2)(1 - \rho_0 b_1)}{b_1}, \tag{24}$$

dividing by λ_k and taking limits where $k \to \infty$, one gets that $\lim_{k\to\infty}\cot(\lambda_k) = 0$, and as the sequence $\{\sin(\lambda_k)\}_{k\geq 1}$ is bounded, one gets that $\lim_{k\to\infty} \cos(\lambda_k) = 0$ and $\lim_{k\to\infty} |\sin(\lambda_k)| = 1$. Moreover, one gets that

$$\cot(\lambda_{k+1} - \lambda_k) = \frac{\cot(\lambda_{k+1})\cot(\lambda_k) + 1}{\cot(\lambda_k) - \cot(\lambda_{k+1})}, \tag{25}$$

considering limits where $k \to \infty$, one gets

$$\lim_{k\to\infty} \cot(\lambda_{k+1} - \lambda_k) = \infty, \tag{26}$$

as the sequence $\{\cos(\lambda_{k+1} - \lambda_k)\}_{k\geq 1}$ is bounded, we have that $\lim_{k\to\infty} \sin(\lambda_{k+1} - \lambda_k) = 0$, and with $(\pi/2) < \lambda_{k+1} - \lambda_k < (3\pi/2)$, one gets that $\lim_{k\to\infty}(\lambda_{k+1} - \lambda_k) = \pi$.

If $b_1 b_2 = 0$ and $((1 - b_2 + \rho_0 b_1 b_2)(1 - \rho_0 b_1)/b_1) = 0$, (16) reduces to $\lambda\cot(\lambda) = 0$, whose roots are $\lambda_k = (\pi/2) + k\pi$, $k \in \mathbb{N}$, and trivially $\lambda_{k+1} - \lambda_k = \pi$. Then $\lim_{k\to\infty}(\lambda_{k+1} - \lambda_k) = \pi$. □

Under hypothesis $\alpha(\rho_0, b_1, b_2, \lambda_0) < 1$ there is a root $\lambda_0 \in (0, \pi)$, and we can define the set of eigenvalues of the problem (1)–(3) as

$$\begin{aligned}\mathcal{F} = \{&\lambda_k \in (k\pi, (k+1)\pi);\\ &\lambda_k \cot(\lambda_k) = \alpha(\rho_0, b_1, b_2, \lambda_k), k \geq 1\} \cup \mathcal{F}_0,\end{aligned} \tag{27}$$

where

$$\mathcal{F}_0 = \begin{cases} \emptyset, & \text{if } \alpha(\rho_0, b_1, b_2, \lambda_0) \geq 1 \\ \lambda_0 \in (0,\pi), & \text{if } \alpha(\rho_0, b_1, b_2, \lambda_0) < 1. \end{cases} \tag{28}$$

Thus, by [17, page 433] a set of solutions of problem (1) is given by

$$\begin{aligned} u(x,t,\lambda_k) &= e^{-\lambda_k A t}\{\sin(\lambda_k x)\widetilde{A}_1 - \lambda_k \cos(\lambda_k x)\widetilde{B}_1\} C(\lambda_k), \\ &\qquad \lambda_k \in \mathcal{F}, \end{aligned} \tag{29}$$

where $C(\lambda_k)$ satisfies

$$G(\rho_0, b_1, b_2, \lambda_k) C(\lambda_k) = 0. \tag{30}$$

Observe that if p is the degree of minimal polynomial of A, the matrix $G(\rho_0, b_1, b_2, \lambda_k)$ is defined by

$$G(\rho_0, b_1, b_2, \lambda_k) = \begin{pmatrix} \widetilde{B}_1 A - A\widetilde{B}_1 \\ \vdots \\ \widetilde{B}_1 A^{p-1} - A^{p-1}\widetilde{B}_1 \\ (\widetilde{A}_2\widetilde{A}_1 + \lambda_k^2 \widetilde{B}_2\widetilde{B}_1) + \alpha(\rho_0, b_1, b_2, \lambda_k) I \\ \{(\widetilde{A}_2\widetilde{A}_1 + \lambda_k^2 \widetilde{B}_2\widetilde{B}_1) + \alpha(\rho_0, b_1, b_2, \lambda_k) I\} A \\ \vdots \\ \{(\widetilde{A}_2\widetilde{A}_1 + \lambda_k^2 \widetilde{B}_2\widetilde{B}_1) + \alpha(\rho_0, b_1, b_2, \lambda_k) I\} A^{p-1} \end{pmatrix}. \tag{31}$$

In order to ensure that $C(\lambda_k) \neq 0$ satisfies (30) we have

$$\operatorname{rank} G(\rho_0, b_1, b_2, \lambda_k) < m, \tag{32}$$

and under condition (32), the solution of (30) is given by

$$C(\lambda_k) = \left(I - G(\rho_0, b_1, b_2, \lambda_k)^{\dagger} G(\rho_0, b_1, b_2, \lambda_k)\right) S, \quad S \in \mathbb{C}^m. \tag{33}$$

The eigenfunctions associated to the problem (1) are then given by

$$\begin{aligned} u(x,t,\lambda_k) &= e^{-\lambda_k A t}\{\sin(\lambda_k x)\widetilde{A}_1 - \lambda_k \cos(\lambda_k x)\widetilde{B}_1\} C(\lambda_k), \\ &\qquad \lambda_k \in \mathcal{F}. \end{aligned} \tag{34}$$

Also $\lambda = 0$ is an eigenvalue of problem (1), if

$$1 \in \sigma\left(-\widetilde{A}_2\widetilde{A}_1\right). \tag{35}$$

Under hypothesis (35), if $G(\rho_0, 0) = \widetilde{A}_2\widetilde{A}_1 + I$, then, if we denote by

$$C(0) = \left(I - G(\rho_0, 0)^{\dagger} G(\rho_0, 0)\right) S, \quad S \in \mathbb{C}^m, \tag{36}$$

one gets that function

$$u(x,0) = \left(x\widetilde{A}_1 - \widetilde{B}_1\right) C(0) \tag{37}$$

is an eigenfunction of problem (1) associated to eigenvalue $\lambda = 0$.

All these results are summarized in Theorem 2.1 of [17, page 434]. Our goal is to find the exact solution of the problem (1)–(4). We provide conditions for the function $f(x)$ and the matrix coefficients in order to ensure the existence of a series solution of the problem. The paper is organized as follows. In Section 3 a series solution for the problem is presented. In Section 4 we proceed with an algorithm and give an illustrative example.

3. A Series Solution

By the superposition principle, a possible candidate to the series solution of problem (1)–(4) is given by

$$u(x,t) = \begin{cases} u(x,0) + \sum_{\lambda_n \in \mathcal{F}} u(x,t,\lambda_k), & 0 \in \mathcal{F}, \\ \sum_{\lambda_n \in \mathcal{F}} u(x,t,\lambda_k), & 0 \notin \mathcal{F}, \end{cases} \tag{38}$$

where $u(x,t,\lambda_k)$ and $u(x,0)$ are defined by (34) and (37), respectively, for suitable vectors $C(\lambda_n)$ and $C(0)$.

Assuming that series (38) and the corresponding derivatives $u_x(x,t)$, $u_{xx}(x,t)$, and $u_t(x,t)$ are convergent (we will demonstrate this later), (38) will be a solution of (1)–(3). Now, we need to determine vectors $C(\lambda)$ and $C(0)$ so that (38) satisfies (4).

Note that, taking v to satisfy (13), from (12) one gets

$$\widetilde{A}_2 v = \left(\frac{b_2 - \rho_0 b_1 b_2}{b_1}\right) v, \qquad \widetilde{A}_1 v = \left(1 - \rho_0 b_1\right) v. \tag{39}$$

Under condition (39), we will consider the scalar Sturm-Liouville problem:

$$\begin{gathered} X''(x) + \lambda^2 X(x) = 0, \\ \left(1 - \rho_0 b_1\right) X(0) + b_1 X'(0) = 0, \\ -\left(\frac{1 - b_2 + \rho_0 b_1 b_2}{b_1}\right) X(1) + b_2 X'(1) = 0, \end{gathered} \tag{40}$$

which provides a family of eigenvalues $\mathcal{F}$ given in (27). Then, the associated eigenfunctions are

$$\begin{gathered} X_{\lambda_n}(x) = \left(1 - \rho_0 b_1\right) \sin\left(\lambda_n x\right) - b_1 \lambda_n \cos\left(\lambda_n x\right), \quad \lambda_n > 0, \\ X_0(x) = \left(1 - \rho_0 b_1\right) x - b_1, \quad \text{if } \lambda_0 = 0. \end{gathered} \tag{41}$$

By the theorem of convergence of the Sturm-Liouville for functional series [18, chapter 11], with the initial condition $f(x) = (f_1(x), \ldots, f_m(x))^t$ given in (4) satisfying the following properties:

$$\begin{gathered} f \in \mathcal{C}^2([0,1]), \\ \left(1 - \rho_0 b_1\right) f(0) + b_1 f'(0) = 0, \\ -\left(\frac{1 - b_2 + \rho_0 b_1 b_2}{b_1}\right) f(1) + b_2 f'(1) = 0, \end{gathered} \tag{42}$$

each component f_i of f, for $1 \le i \le m$, has a series expansion which converges absolutely and uniformly on the interval $[0,1]$; namely,

$$\begin{aligned} f_i(x) = {} & \alpha\left(\left(1 - \rho_0 b_1\right) x - b_1\right) e_{0i} \\ & + \sum_{\lambda_n \in \mathcal{F}} \left(\left(1 - \rho_0 b_1\right) \sin\left(\lambda_n x\right) - b_1 \lambda_n \cos\left(\lambda_n x\right)\right) e_{\lambda_n i}, \end{aligned} \tag{43}$$

where

$$\alpha = \begin{cases} 1 & \text{if } \dfrac{\left(1 - b_2 + \rho_0 b_1 b_2\right)\left(1 - \rho_0 b_1\right)}{b_1} = 1 \\ 0 & \text{if } \dfrac{\left(1 - b_2 + \rho_0 b_1 b_2\right)\left(1 - \rho_0 b_1\right)}{b_1} \neq 1, \end{cases}$$

$$e_{0i} = b_1 \frac{\int_0^1 \left(\left(1 - \rho_0 b_1\right) x - b_1\right) f_i(x)\, dx}{\int_0^1 \left(\left(1 - \rho_0 b_1\right) x - b_1\right)^2 dx} \quad \text{if } \lambda_0 = 0,$$

$$\begin{aligned} & e_{\lambda_n i} \\ & = b_1 \lambda_n \frac{\int_0^1 \left(\left(1 - \rho_0 b_1\right) \sin\left(\lambda_n x\right) - b_1 \lambda_n \cos\left(\lambda_n x\right)\right) f_i(x)\, dx}{\int_0^1 \left(\left(1 - \rho_0 b_1\right) \sin\left(\lambda_n x\right) - b_1 \lambda_n \cos\left(\lambda_n x\right)\right)^2 dx} \\ & \qquad \text{if } \lambda_n > 0. \end{aligned} \tag{44}$$

Thus,

$$\begin{aligned} f(x) = {} & \alpha\left(\left(1 - \rho_0 b_1\right) x - b_1\right) E(0) \\ & + \sum_{\lambda_n \in \mathcal{F}} \left(\left(1 - \rho_0 b_1\right) \sin\left(\lambda_n x\right) \right. \\ & \left. - b_1 \lambda_n \cos\left(\lambda_n x\right)\right) E\left(\lambda_n\right), \end{aligned} \tag{45}$$

where $E(0) = \begin{pmatrix} e_{01} \\ \vdots \\ e_{0m} \end{pmatrix}$ and $E(\lambda_n) = \begin{pmatrix} e_{\lambda_n 1} \\ \vdots \\ e_{\lambda_n m} \end{pmatrix}$. On the other hand, from (38) and taking into account (34) and (37), one gets

$$\begin{aligned} f(x) = u(x,0) = {} & \alpha\left(x\widetilde{A}_1 - \widetilde{B}_1\right) C(0) \\ & + \sum_{\lambda_n \in \mathcal{F}} \left(\sin\left(\lambda_n x\right) \widetilde{A}_1 - \lambda_n \cos\left(\lambda_n x\right) \widetilde{B}_1\right) C\left(\lambda_n\right). \end{aligned} \tag{46}$$

We can equate the two expressions; if $C(0)$ and $C(\lambda_n)$, apart from conditions (33) and (36), satisfy $\{C(0), C(\lambda)\} \subset \operatorname{Ker}(\widetilde{B}_1 - b_1 I)$. Then, we have

$$\begin{aligned} C(\lambda_n) &= E(\lambda_n) \\ &= \frac{\int_0^1 ((1-\rho_0 b_1)\sin(\lambda_n x) - b_1\lambda_n\cos(\lambda_n x)) f(x)\,dx}{\int_0^1 ((1-\rho_0 b_1)\sin(\lambda_n x) - b_1\lambda_n\cos(\lambda_n x))^2 dx}, \quad \text{if } \lambda_n > 0, \\ C(0) &= E(0) = \frac{\int_0^1 ((1-\rho_0 b_1)x - b_1) f(x)\,dx}{\int_0^1 ((1-\rho_0 b_1)x - b_1)^2 dx} \quad \text{if } \lambda_0 = 0. \end{aligned} \tag{47}$$

Note that $C(0)$ and $C(\lambda) \in \operatorname{Ker}(\widetilde{B}_1 - b_1 I)$, if

$$f(x) \in \operatorname{Ker}(\widetilde{B}_1 - b_1 I). \tag{48}$$

Then $u(x,t)$ defined by

$$\begin{aligned} u(x,t) = \alpha((1-\rho_0 b_1)x - b_1) C(0) &+ \sum_{\lambda_n \in \mathcal{F}} e^{-\lambda_n^2 A t}((1-\rho_0 b_1)\sin(\lambda_n x) \\ &- b_1\lambda_n\cos(\lambda_n x)) C(\lambda_n), \end{aligned} \tag{49}$$

where α and $C(\lambda_n)$ are defined by (44) and (47), satisfies the initial condition (4). Note that conditions (30)–(32) hold if

$$G(\rho_0, b_1, b_2, \lambda_k) f(x) = 0, \tag{50}$$

and then

$$\begin{gathered} (\widetilde{B}_1 - b_1 I) A^j f(x) = 0, \quad 0 \le j < p, \\ (\widetilde{A}_2\widetilde{A}_1 + \lambda_n^2 \widetilde{B}_2\widetilde{B}_1 + \alpha(\rho_0, b_1, b_2, \lambda) I) A^j f(x) = 0, \quad 0 \le j < p. \end{gathered} \tag{51}$$

It is easy to check that conditions (48), (51) are equivalent to the condition

$$A^j f(x) \in \operatorname{Ker}(\widetilde{B}_1 - b_1 I) \cap \operatorname{Ker}(\widetilde{B}_2 - b_2 I), \quad 0 \le j < p. \tag{52}$$

Condition (52) holds if

$$\begin{gathered} f(x) \in \operatorname{Ker}(\widetilde{B}_1 - b_1 I) \cap \operatorname{Ker}(\widetilde{B}_2 - b_2 I), \quad 0 \le x \le 1, \\ \operatorname{Ker}(\widetilde{B}_1 - b_1 I) \cap \operatorname{Ker}(\widetilde{B}_2 - b_2 I), \end{gathered} \tag{53}$$

is an invariant subspace with respect to matrix A.

Now we study the convergence of the solution given by (49) with α defined by (44) and $C(\lambda_n)$ by (47). Using Parseval's identity for scalar Sturm-Liouville problems [19], there exists a positive constant $M_1 > 0$ so that $\|C(\lambda_n)\| \le M_1$. Taking formal derivatives in (49), one gets

$$\begin{aligned} u_t(x,t) &= \sum_{\lambda_n \in \mathcal{F}} (-\lambda_n^2) e^{-\lambda_n^2 A t} A (\sin(\lambda_n x)(1-\rho_0 b_1) - \lambda_n\cos(\lambda_n x) b_1) C(\lambda_n), \\ u_x(x,t) &= \sum_{\lambda_n \in \mathcal{F}} \lambda_n e^{-\lambda_n^2 A t} (\cos(\lambda_n x)(1-\rho_0 b_1) + \lambda_n \sin(\lambda_n x) b_1) C(\lambda_n) + \alpha(1-\rho_0 b_1) C(0), \\ u_{xx}(x,t) &= \sum_{\lambda_n \in \mathcal{F}} \lambda_n^2 e^{-\lambda_n^2 A t} (-\sin(\lambda_n x)(1-\rho_0 b_1) + \lambda_n\cos(\lambda_n x) b_1) C(\lambda_n). \end{aligned} \tag{54}$$

These series are all bounded in their respective norms:

$$\begin{aligned} \|u(x,t)\| &\le \sum_{\lambda_n \in \mathcal{F}} \left[\left\| e^{-\lambda_n^2 A t} \right\| |1-\rho_0 b_1| M_1 + \left\| \lambda_n e^{-\lambda_n^2 A t} \right\| |b_1| M_1 \right] + \alpha(|1-\rho_0 b_1| x + |b_1|) \|C(0)\|, \\ \|u_t(x,t)\| &\le \sum_{\lambda_n \in \mathcal{F}} \left[\left\| \lambda_n^2 e^{-\lambda_n^2 A t} A \right\| |1-\rho_0 b_1| M_1 + \left\| \lambda_n^2 e^{-\lambda_n^2 A t} \right\| |b_1| M_1 \right], \\ \|u_x(x,t)\| &\le \sum_{\lambda_n \in \mathcal{F}} \left[\left\| \lambda_n^2 e^{-\lambda_n^2 A t} \right\| |1-\rho_0 b_1| M_1 + \left\| \lambda_n^2 e^{-\lambda_n^2 A t} \right\| |b_1| M_1 \right] + \alpha|1-\rho_0 b_1| \|C(0)\|, \\ \|u_{xx}(x,t)\| &\le \sum_{\lambda_n \in \mathcal{F}} \left[\left\| \lambda_n^2 e^{-\lambda_n^2 A t} \right\| |1-\rho_0 b_1| M_1 + \left\| \lambda_n^3 e^{-\lambda_n^2 A t} \right\| |b_1| M_1 \right]. \end{aligned} \tag{55}$$

To check that the series is uniformly convergent in each domain $[0,1] \times [c,d]$, it is sufficient to verify that the series

$$\sum_{\lambda_n \in \mathcal{F}} \lambda_n^3 e^{-\lambda_n^2 A t} \tag{56}$$

is uniformly convergent in this domain. This is trivial because, using (9), one gets

$$\left\| \lambda_n^3 e^{-\lambda_n^2 A t} \right\| \le e^{-\lambda_n^2 \alpha(A) t} \sum_{k=0}^{m-1} \frac{(\sqrt{m}\|A\| t)^k \lambda_n^{2k+3}}{k!}, \tag{57}$$

and from the d'Alembert test series applied to each summand, taking into account (5) and the relation (19), $\lim_{n\to\infty}(\lambda_{n+1}-\lambda_n)=\pi$, given in Lemma 1, one gets for $3 \le r \le 2\ (m-1)+3$ that

$$\begin{aligned}\lim_{n\to\infty} e^{(\lambda_n^2-\lambda_{n+1}^2)\alpha(A)t}\left(\frac{\lambda_{n+1}}{\lambda_n}\right)^r &\le \lim_{n\to\infty} e^{(\lambda_n^2-\lambda_{n+1}^2)\alpha(A)t}\left(\frac{n+2}{n}\right)^r \\ &= e^{-\alpha(A)t\pi\lim_{n\to\infty}(\lambda_n+\lambda_{n+1})} = 0 < 1.\end{aligned} \tag{58}$$

Thus, the series (56) is convergent.

Independence of the series solution (49) with respect to the chosen $\rho_0 \in \mathbb{R}$ can be demonstrated using the same technique as given in [20].

We can summarize the results in the following theorem.

Theorem 2. *Consider the homogeneous problem with homogeneous conditions* (1)–(4) *under hypotheses* (5), (6), *and* (7) *verifying conditions* (13) *and* (14). *Let $f(x)$ be a vectorial function satisfying* (42). *Let $\mathcal{F}$ be the set defined by* (27), *and let $G(\rho_0, b_1, b_2, \lambda_k)$ be the matrix defined by* (31), *taking as eigenvalues of problems $\lambda \in \mathcal{F}$ satisfying*

$$\operatorname{rank}\left(G\left(\rho_0, b_1, b_2, \lambda_k\right)\right) < m, \tag{59}$$

including the eigenvalue $\lambda = 0$ if $1 \in \sigma(-\widetilde{A}_2\widetilde{A}_1)$, and taking as eigenfunctions $u(x,t,\lambda_k)$ defined by (34). *Let α be given by* (44) *and vectors $C(\lambda_n)$ defined by* (47). *Then, $u(x,t)$, as defined in* (49), *is a series solution of problem* (1)–(4).

4. Algorithm and Example

We can summarize the process to calculate the solution of the homogeneous problem with homogeneous conditions (1)–(4) in Algorithm 1.

Example 1. We will consider the homogeneous parabolic problem with homogeneous conditions (1)–(4), where the matrix $A \in \mathbb{C}^{4\times4}$ is chosen as

$$A = \begin{pmatrix} 2 & 0 & 0 & -1 \\ 1 & 2 & 1 & -2 \\ -1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{60}$$

and the 4×4 matrices A_i, B_i, $i \in \{1,2\}$, are

$$\begin{aligned} A_1 &= \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}, & A_2 &= \begin{pmatrix} 0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&0&0 \end{pmatrix}, \\ B_1 &= \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, & B_2 &= \begin{pmatrix} 1&0&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}. \end{aligned} \tag{61}$$

Also, the vectorial valued function $f(x)$ will be defined as

$$f(x) = \begin{pmatrix} 0 \\ x^2-1 \\ 0 \\ 0 \end{pmatrix}. \tag{62}$$

Observe that the method proposed in [12] cannot be applied to solve this problem.

We will follow Algorithm 1 step to step.

(1) Matrix A satisfies the condition (5), because $\sigma(A) = \{1,2\}$. That is, A is positive stable.

(2) Each of the matrices A_i, B_i, $i \in \{1,2\}$, is singular, and the block matrix

$$\begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix} = \left(\begin{array}{cccc|cccc} 0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\ \hline 0&1&0&0&1&0&0&0\\1&0&0&0&1&0&0&0\\0&0&0&1&0&0&1&0\\0&0&0&0&0&0&0&1 \end{array}\right) \tag{63}$$

is regular.

(3) Note that although A_1 is singular, taking $\rho_0 = 1 \in \mathbb{R}$, the matrix pencil

$$A_1 + \rho_0 B_1 = I_{4\times4} \tag{64}$$

is regular. Therefore, we take $\rho_0 = 1$.

(4) By (10) we have

$$\begin{aligned} \widetilde{A_1} &= \left(A_1 + \rho_0 B_1\right)^{-1} A_1 = A_1, \\ \widetilde{B_1} &= \left(A_1 + \rho_0 B_1\right)^{-1} B_1 = B_1. \end{aligned} \tag{65}$$

(5) By (11) we have

$$\begin{aligned} \widetilde{A_2} &= \left(B_2 - \left(A_2 + \rho_0 B_2\right)\widetilde{B_1}\right)^{-1} A_2 = \begin{pmatrix} -1&0&0&0\\0&-1&0&0\\0&0&0&1\\0&0&0&0 \end{pmatrix}, \\ \widetilde{B_2} &= \left(B_2 - \left(A_2 + \rho_0 B_2\right)\widetilde{B_1}\right)^{-1} B_2 = \begin{pmatrix} -1&0&0&0\\-1&0&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}. \end{aligned} \tag{66}$$

(6) We have $\sigma(\widetilde{B_1}) = \{0,1\}$ and $\sigma(\widetilde{B_2}) = \{0,1,-1\}$. Note that in this case the condition (13) holds because with $b_1 = 1$ and $b_2 = 0 \in \sigma(\widetilde{B}_2)$ there exists a common eigenvector $v \in \mathbb{C}^4$, $v = (0,1,0,0)^t$, and thus $\operatorname{Ker}(\widetilde{B}_1 - I) \cap \operatorname{Ker}(\widetilde{B}_2) \ne (0,0,0,0)^t$. We are therefore in Case 1 of Algorithm 1.

Input data: $A, A_1, A_2, B_1, B_2 \in \mathbb{C}^{m\times m}$, $f(x) \in \mathbb{C}^m$.
Result: $u(x,t)$.
(1) Check that matrix A satisfies (5).
(2) Check that matrices $A_i, B_i \in \mathbb{C}^{m\times m}, i \in \{1,2\}$ are singular, and check that the block matrix $\begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix}$ is regular.
(3) Determine a number $\rho_0 \in \mathbb{R}$ so that the matrix pencil $A_1 + \rho_0 B_1$ is regular.
(4) Determine matrices $\widetilde{A_1}$ and $\widetilde{B_1}$ defined by (10).
(5) Determine matrices $\widetilde{A_2}$ and $\widetilde{B_2}$ defined by (11).
(6) Consider the following cases:
  (i) *Case 1.* Condition (13) holds, that is, matrices $\widetilde{B}_1$ and $\widetilde{B}_2$ have a common eigenvector $v \neq 0$ associated with eigenvalues $b_1 \in \sigma(\widetilde{B}_1) - \{0\}$ and $b_2 \in \sigma(\widetilde{B}_2)$. In this case continue with *step (7)*.
  (ii) *Case 2.* Condition (13) does not hold. In this case the algorithm stops because it is not possible to find the solution of (1)–(4) for the given data.
(7) Determine $b_1 \in \sigma(\widetilde{B}_1)$, $b_1 \neq 0$, $b_2 \in \sigma(\widetilde{B}_2)$ and vector $v \neq 0$ verifying $v \in \operatorname{Ker}(\widetilde{B}_1 - b_1 I) \cap \operatorname{Ker}(\widetilde{B}_2 - b_2 I)$ such that:
  (i) Conditions (53) hold, that is:
  1.1: $\operatorname{Ker}(\widetilde{B}_1 - b_1 I) \cap \operatorname{Ker}(\widetilde{B}_2 - b_2 I)$ is an invariant subspace respect matrix A.
  1.2: $f(x) \in \operatorname{Ker}(\widetilde{B}_1 - b_1 I) \cap \operatorname{Ker}(\widetilde{B}_2 - b_2 I)$, $\forall x \in [0,1]$.
  (ii) Conditions (14) hold, that is:
  1.3: $\dfrac{(1 - b_2 + \rho_0 b_1 b_2)(1 - \rho_0 b_1)}{b_1} \in \mathbb{R}$, $b_1 b_2 \in \mathbb{R}$.
  (iii) The vectorial function $f(x)$ satisfies (42), that is:
  1.4: $f \in \mathscr{C}^2([0,1])$.
  1.5: $(1 - \rho_0 b_1) f(0) + b_1 f'(0) = 0$.
  1.6: $-\left(\dfrac{1 - b_2 + \rho_0 b_1 b_2}{b_1}\right) f(1) + b_2 f'(1) = 0$.
If these conditions are not satisfied, return to step (6) of Algorithm 1 discarding the values taken for b_1 and b_2.
(8) Determine the positive solutions of (16) and determine $\mathscr{F}$ defined by (27).
(9) Determine degree p of minimal polynomial of matrix A.
(10) Building block matrix $G(\rho_0, b_1, b_2, \lambda_k)$ defined by (31).
(11) Determine $\lambda \in \mathscr{F}$ so that rank $G(\rho_0, b_1, b_2, \lambda_k) < m$.
(12) Include the eigenvalue $\lambda = 0$ if $1 \in \sigma(-\widetilde{A}_2 \widetilde{A}_1)$.
(13) Determine α given by (44).
(14) Determine vectors $C(\lambda_n)$ defined by (47).
(15) Determine functions $u(x,t,\lambda_n)$ defined by (34).
(16) Determine the series solution $u(x,t)$ of problem (1)–(4) defined by (49).

Algorithm 1: Solution of the homogeneous problem with homogeneous conditions (1)–(4).

(7) We take the values $b_1 = 1$ and $b_2 = 0$ and will check the conditions given in step 7 of the algorithm.

(1.1) One gets that

$$\operatorname{Ker}(\widetilde{B}_1 - I) \cap \operatorname{Ker}(\widetilde{B}_2) = \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle. \quad (67)$$

Let $x \in \operatorname{Ker}(\widetilde{B}_1 - I) \cap \operatorname{Ker}(\widetilde{B}_2)$. Then $x = \left(\begin{smallmatrix} 0 \\ \lambda \\ 0 \\ 0 \end{smallmatrix}\right)$, $\lambda \in \mathbb{C}$. In this case one gets

$$Ax = \begin{pmatrix} 0 \\ 2\lambda \\ 0 \\ 0 \end{pmatrix} \in \operatorname{Ker}(\widetilde{B}_1 - I) \cap \operatorname{Ker}(\widetilde{B}_2), \quad (68)$$

and then the subspace $\operatorname{Ker}(\widetilde{B}_1 - I) \cap \operatorname{Ker}(\widetilde{B}_2)$ is invariant by matrix A.

(1.2) It is trivial to check that

$$f(x) \in \operatorname{Ker}(\widetilde{B}_1 - I) \cap \operatorname{Ker}(\widetilde{B}_2), \quad \forall x \in [0,1]. \quad (69)$$

(1.3) With these values ρ_0, b_1, and b_2, one gets that

$$\frac{(1 - b_2 + \rho_0 b_1 b_2)(1 - \rho_0 b_1)}{b_1} = 0 \in \mathbb{R}. \quad (70)$$

With these values b_1 and b_2, one gets

$$b_1 b_2 = 0 \in \mathbb{R}. \quad (71)$$

(1.4) It is trivial to check that $f(x) \in \mathscr{C}^2([0,1])$.

(1.5) It is trivial to check that $(1-\rho_0 b_1) f(0) + b_1 f'(0) = (0,0,0,0)^t$.

(1.6) It is trivial to check that $-((1 - b_2 + \rho_0 b_1 b_2)/b_1) f(1) + b_2 f'(1) = (0,0,0,0)^t$.

(8) Equation (16) is of the form

$$\lambda \cot(\lambda) = 0 \tag{72}$$

We can solve (72) exactly, $\lambda_k = (\pi/2) + k\pi$, with an additional solution $\lambda_0 \in]0, \pi[$, because

$$\frac{(1 - b_2 + \rho_0 b_1 b_2)(1 - \rho_0 b_1)}{b_1} = 0 < 1, \tag{73}$$

and then $\lambda_0 = (\pi/2)$. Thus, we have a numerable family of solutions of (72) which we denote by $\mathcal{F}$, given by.

$$\mathcal{F} = \left\{\lambda_k = \frac{\pi}{2} + k\pi;\ \lambda_k \in (k\pi, (k+1)\pi),\ k \geq 1\right\} \cup \mathcal{F}_0, \quad \mathcal{F}_0 = \left\{\lambda_0 = \frac{\pi}{2}\right\}. \tag{74}$$

(9) The minimal polynomial of matrix A is given by $p(x) = (x-2)^3(x-1)$. Then $p = 4$.

(10) If λ_k is a positive solution of (72), the matrix $G(\rho_0, b_1, b_2, \lambda_k)$ given by (31) takes the form

$$G(1,1,0,\lambda_k) = \left(\begin{array}{cccc}
0 & 0 & 0 & -1 \\
0 & 0 & 1 & -2 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & -3 \\
0 & 0 & 4 & -6 \\
4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & -7 \\
0 & 0 & 12 & -13 \\
12 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\ \hline
-\lambda_k^2 & 0 & 0 & 0 \\
-\lambda_k^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\ \hline
-2\lambda_k^2 & 0 & 0 & \lambda_k^2 \\
-2\lambda_k^2 & 0 & 0 & \lambda_k^2 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\ \hline
-4\lambda_k^2 & 0 & 0 & 3\lambda_k^2 \\
-4\lambda_k^2 & 0 & 0 & 3\lambda_k^2 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\ \hline
-8\lambda_k^2 & 0 & 0 & 7\lambda_k^2 \\
-8\lambda_k^2 & 0 & 0 & 7\lambda_k^2 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{array}\right). \tag{75}$$

(11) Since the second column $G(1,1,0,\lambda_k)$ is zero, we have that $\mathrm{rank}(G(1,1,0,\lambda_k)) < 4$. Thus, each one of the positive solutions given by (74) is an eigenvalue.

(12) It is trivial to check that $1 \notin \sigma(-\widetilde{A}_2\widetilde{A}_1)$, because

$$-\widetilde{A}_2\widetilde{A}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \sigma\left(-\widetilde{A}_2\widetilde{A}_1\right) = \{0\}. \tag{76}$$

Then we do not include 0 as an eigenvalue.

(13) Taking into account that $((1 - b_2 + \rho_0 b_1 b_2)(1 - \rho_0 b_1)/b_1) = 0 < 1$, one gets $\alpha = 0$.

(14) Vectors $C(\lambda_n)$ defined by (47) take the values

$$C(\lambda_n) = \frac{64(-1)^n}{\pi^4(2n+1)^4}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}. \tag{77}$$

(15) Using the minimal theorem [21, page 571], one gets that

$$e^{Au} = \begin{pmatrix} e^{2u} & 0 & 0 & -e^u(e^u - 1) \\ -\frac{1}{2}e^{2u}(u-2)u & e^{2u} & e^{2u}u & \frac{1}{2}e^u(2 + e^u(-2 + (-2+u)u)) \\ -e^{2u}u & 0 & e^{2u} & e^{2u}u \\ 0 & 0 & 0 & e^u \end{pmatrix}. \tag{78}$$

Next, by considering (78) with $u = -((\pi/2) + n\pi)^2 t$ and simplifying, we obtain the value of $e^{-((\pi/2)+n\pi)^2 At}$. Taking into account that all eigenvalues λ_n are positive, the associated eigenfunctions are

$$u(x,t,\lambda_n) = e^{-\lambda_n^2 At}\left((1-\rho_0 b_1)\sin(\lambda_n x) - b_1\lambda_n\cos(\lambda_n x)\right) C(\lambda_n). \tag{79}$$

(16) We replace the values of $C(\lambda_n)$ given by (77) in (79) and take into account the value of the matrix $e^{-((\pi/2)+n\pi)^2 At}$. After simplification, we finally obtain the solution of (1)–(4) given by

$$u(x,t) = \left(\sum_{n\geq 0} -\frac{32(-1)^n e^{-(1/2)(\pi+2n\pi)^2 t}\cos((1/2)(\pi+2n\pi)x)}{\pi^3(2n+1)^3}\right) \times \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}. \tag{80}$$

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