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Research Article

A Note on Sequential Product of Quantum Effects

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The quantum effects for a physical system can be described by the set $\mathscr{E}(\mathscr{H})$ of positive operators on a complex Hilbert space $\mathscr{H}$ that are bounded above by the identity operator I. For $A, B \in \mathscr{E}(\mathscr{H})$, let $A \circ B = A^{1/2}BA^{1/2}$ be the sequential product and let $A * B = (AB + BA)/2$ be the Jordan product of $A, B \in \mathscr{E}(\mathscr{H})$. The main purpose of this note is to study some of the algebraic properties of effects. Many of our results show that algebraic conditions on $A \circ B$ and $A * B$ imply that A and B have 3×3 diagonal operator matrix forms with $I_{\overline{\mathscr{R}(A)} \cap \overline{\mathscr{R}(B)}}$ as an orthogonal projection on closed subspace $\overline{\mathscr{R}(A)} \cap \overline{\mathscr{R}(B)}$ being the common part of A and B. Moreover, some generalizations of results known in the literature and a number of new results for bounded operators are derived.

1. Introduction

Let $\mathscr{H}$, $\mathscr{B}(\mathscr{H})$, and $\mathscr{P}(\mathscr{H})$ be complex Hilbert space, the set of all bounded linear operators on $\mathscr{H}$, and the set of all orthogonal projections on $\mathscr{H}$, respectively. For $A \in \mathscr{B}(\mathscr{H})$, we will denote by $\mathscr{N}(A)$ and $\mathscr{R}(A)$ the null space and the range of A, respectively. An operator $A \in \mathscr{B}(\mathscr{H})$ is said to be injective if $\mathscr{N}(A) = \{0\}$. $\overline{\mathscr{R}(A)}$ is the closure of $\mathscr{R}(A)$. A is said to be positive if $(Ax, x) \geq 0$ for all $x \in \mathscr{H}$. A is said to be a contraction if $\|A\| \leq 1$. $P_{\mathscr{M}}$ is the orthogonal projection on a closed subspace $\mathscr{M} \subseteq \mathscr{H}$.

The elements of $\mathscr{E}(\mathscr{H}) = \{A \in \mathscr{B}(\mathscr{H}) : 0 \leq A \leq I\}$ are called quantum effects. The elements of $\mathscr{P}(\mathscr{H}) = \{P \in \mathscr{E}(\mathscr{H}) : P^2 = P\}$ are projections corresponding to quantum events and are called sharp effects. For $A, B \in \mathscr{E}(\mathscr{H})$, the sequential product of A and B is $A \circ B = A^{1/2}BA^{1/2}$. We interpret $A \circ B$ as the effect that occurs when A occurs first and B occurs second [1–9]. Let $A * B = (AB + BA)/2$ be the Jordan product of $A, B \in \mathscr{E}(\mathscr{H})$. If $AB = BA$, we say that A and B are compatible. We define the negation of $A \in \mathscr{E}(\mathscr{H})$ by $A' = I - A$.

In this note, we will study some properties of the sequential product or the Jordan product. Our results show that if one tries to impose classical conditions on $A \circ B = A^{1/2}BA^{1/2}$ and $A * B = (AB + BA)/2$, then this forces A and B to have closed relations with range relations. For example, let $T = A^nB^n$ for some $n \in \mathbb{Z}^+$. Then, $TT^* \in \mathscr{P}(\mathscr{H})$ (or $A * B \in \mathscr{P}(\mathscr{H})$) if and only if A and B have 3×3 diagonal operator matrix forms as follows:

$$\begin{aligned} A &= I_{\overline{\mathscr{R}(A)} \cap \overline{\mathscr{R}(B)}} \oplus A_{22} \oplus 0, \\ B &= I_{\overline{\mathscr{R}(A)} \cap \overline{\mathscr{R}(B)}} \oplus 0 \oplus B_{33}, \end{aligned} \tag{1}$$

where $I_{\overline{\mathscr{R}(A)} \cap \overline{\mathscr{R}(B)}}$ as an orthogonal projection on closed subspace $\overline{\mathscr{R}(A)} \cap \overline{\mathscr{R}(B)}$ is the common part of A and B. This results give us detailed information of matrix structures between two operators A and B. It is well known that if A or $B \in \mathscr{P}(\mathscr{H})$, then $A \circ B \leq B$ if and only if $AB = BA$ (see [2, Theorem 2.6(a)] and [10, Theorem 2.3]). We generate this result and show that, under some conditions, $A \circ B \leq B$ if and only if A and B have 3×3 operator matrix forms:

$$\begin{aligned} A &= I \oplus 0 \oplus A_{33}, \\ B &= B_{11} \oplus B_{22} \oplus 0. \end{aligned} \tag{2}$$

In [11, Lemma 3.4], the authors had gotten that if $A, B \in \mathscr{E}(\mathscr{H})$ and $\dim \mathscr{H} < \infty$, then $A \circ B + A' \circ B = B'$ if and only if $B = (1/2)I$. The authors said that they did not know if the condition $\dim \mathscr{H} < \infty$ can be relaxed. By some algebraic and spectral techniques, we extend some results in [11] to $\mathscr{B}(\mathscr{H})$. Some generalizations of results known in the literature and a number of new results for bounded operators are derived.

2. Main Results

Our main interest is in sequential products of quantum effects. The next result gives some of the important properties of the sequential product.

Lemma 1 (see [2]). *Let $A, B \in \mathscr{E}(\mathscr{H})$ and $P, Q \in \mathscr{P}(\mathscr{H})$.*

(i) *$A \circ B = B \circ A$ if and only if $AB = BA$.*

(ii) *If $A \circ B \in \mathscr{P}(\mathscr{H})$, then $AB = BA$.*

(iii) *$P \circ Q \in \mathscr{P}(\mathscr{H})$ if and only if $PQ = QP$.*

Lemma 2 (see [12]). *Let $A \in \mathscr{B}(\mathscr{H})$ be a positive operator. If A has the operator matrix representation $A = (A_{ij})_{n\times n}$ with respect to the space decomposition $\mathscr{H} = \bigoplus_{i=1}^{n} \mathscr{H}_i$, then the following statements hold.*

(i) *A_{ii} as an operator on $\mathscr{H}_i$ is positive, $1 \le i \le n$.*

(ii) *$A_{ij} = A_{ii}^{1/2} D_{ij} A_{jj}^{1/2}$ for some contraction $D_{ij} \in \mathscr{B}(\mathscr{H}_j, \mathscr{H}_i)$, $1 \le i, j \le n$.*

(iii) *If $A_{i_0 i_0} = 0$ for some $1 \le i_0 \le n$, then $A_{i_0 j} = 0$ and $A_{k i_0} = 0$, $1 \le j, k \le n$.*

Lemma 3 (see [13, Lemma 2.2]). *Let $A \in \mathscr{B}(\mathscr{H})$ be a contraction and let A as an operator from $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$ into $\mathscr{H} = \mathscr{K}_1 \oplus \mathscr{K}_2$ have the operator matrix*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}. \tag{3}$$

If A_{11} is unitary from $\mathscr{H}_1$ onto $\mathscr{K}_1$, then $A_{12} = 0$ and $A_{21} = 0$.

In [11], Gudder had obtained that if $A, B \in \mathscr{E}(\mathscr{H})$ and $A + B = P \in \mathscr{P}(\mathscr{H})$, then A and B are compatible. Based on this result, we get the following interesting results.

Theorem 4. *Let $A, B \in \mathscr{E}(\mathscr{H})$ and $P, Q \in \mathscr{P}(\mathscr{H})$.*

(i) *$P \le A$ if and only if $PA = AP = P$; $A \le P$ if and only if $AP = PA = A$.*

(ii) *There exist $P, Q \in \mathscr{P}(\mathscr{H})$ such that $A = P + Q$ if and only if A is a projection.*

(iii) *If there exist $A, B \in \mathscr{E}(\mathscr{H})$ such that $P = A + B$, then $AB = BA = A - A^2$. In addition, if $\mathscr{R}(B) \subseteq \mathscr{R}(A)$, then $P = A + B$ if and only if $AB = BA = A - A^2$.*

Proof. Note that P and A, as operators on $\mathscr{H} = \mathscr{R}(P) \oplus \mathscr{N}(P)$, have the operator matrices

$$P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \qquad A = \begin{pmatrix} A_1 & A_3 \\ A_3^* & A_2 \end{pmatrix}, \tag{4}$$

respectively, where $0 \le A_1 \in \mathscr{B}(\mathscr{R}(P))$, $0 \le A_2 \in \mathscr{B}(\mathscr{N}(P))$, and $A_3 \in \mathscr{B}(\mathscr{N}(P), \mathscr{R}(P))$.

(i) By (4), it is clear that $P \le A$ if $PA = AP = P$. On the other hand, if $A - P = \begin{pmatrix} A_1 - I & A_3 \\ A_3^* & A_2 \end{pmatrix} \ge 0$, then $A_1 = I$ since $A_1 \oplus 0 = PAP \le P \in \mathscr{E}(\mathscr{H})$. From

$$A^2 = \begin{pmatrix} I + A_3 A_3^* & A_3 + A_3 A_2 \\ A_3^* + A_2 A_3^* & A_2^2 + A_3^* A_3 \end{pmatrix} \le I, \tag{5}$$

we get $A_3 A_3^* = 0$; that is $A_3 = 0$ and $AP = PA = A$. If $AP = PA = A$, then $A_2 = 0$ and $A_3 = 0$ in (4). We get that $A \le P$. On the other hand, since

$$P - A = \begin{pmatrix} I - A_1 & -A_3 \\ -A_3^* & -A_2 \end{pmatrix} \ge 0, \tag{6}$$

$A_2 = 0$ and $A_3 = 0$ by Lemma 2. Hence, $AP = PA = A$.

(ii) If A is a projection, denote $P = A$ and $Q = 0$, then $A = P + Q$. Conversely, suppose that there exist two projections P and Q such that $A = P + Q$. If $x \in \mathscr{R}(P)$ is a unit vector, then $1 \ge (Ax, x) = (Px, x) + (Qx, x) = 1 + (Qx, x)$, so $(Qx \cdot x) = 0$. That is, $Qx = 0$ since Q is a positive operator. This shows that $QP = 0$. Similarly, $PQ = 0$. Hence, $PQ = QP$. The two projections P and Q are commutative; therefore, $P + Q = A$ is a projection.

(iii) Since $A \le P$, $AP = PA = A$ by item (i). So, $A(A+B) = AP = A = PA = (A + B)A$; that is, $AB = A - A^2 = BA$. Conversely, let $A, B \in \mathscr{E}(\mathscr{H})$. Then there exists $P \in \mathscr{P}(\mathscr{H})$ such that $\mathscr{R}(A) = \mathscr{R}(P)$. Since $\mathscr{R}(B) \subseteq \mathscr{R}(A)$, A and B can be written as operator matrices $A = A_1 \oplus 0$, $B = B_1 \oplus 0$ with respect to the space decomposition $\mathscr{H} = \mathscr{R}(P) \oplus \mathscr{N}(P)$, respectively, where A_1 is an injective positive operator. If $AB = A - A^2 = BA$, then $A_1 B_1 = A_1 - A_1^2$. It follows that $B_1 = I - A_1$ and $A + B = P$. $\square$

Let $P_{\overline{\mathscr{R}(A)}}$ denote the self-adjoint projection onto the closure of $\mathscr{R}(A)$. In general, that TT^* is a projection does not imply $T = T^*$. For example, if $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, then TT^* and T^*T are projections and $T \ne T^*$. But we have following result.

Theorem 5. *Let $A, B \in \mathscr{E}(\mathscr{H})$ and $T = A^n B^n$ for some $n \in \mathbb{Z}^+$. Then, $TT^* \in \mathscr{P}(\mathscr{H})$ if and only if $T^*T \in \mathscr{P}(\mathscr{H})$ if and only if A and B have 3×3 operator matrix forms as*

$$A = I \oplus A_{22} \oplus 0, \qquad B = I \oplus 0 \oplus B_{33} \tag{7}$$

with respect to the space decomposition $\mathscr{H} = [\overline{\mathscr{R}(A)} \cap \overline{\mathscr{R}(B)}] \oplus [\overline{\mathscr{R}(A)} \ominus (\overline{\mathscr{R}(A)} \cap \overline{\mathscr{R}(B)})] \oplus \mathscr{N}(A)$; that is, AB is a range projection on $\overline{\mathscr{R}(A)} \cap \overline{\mathscr{R}(B)}$.

Proof. As we know, $\sigma(TT^*) \setminus \{0\} = \sigma(T^*T) \setminus \{0\}$ (see [14, Section 1.2.1]). So, for arbitrary $T \in \mathscr{B}(\mathscr{H})$, TT^* is a projection if and only if T^*T is a projection. If A and B have the forms (7), then $T = T^* = P_{\overline{\mathscr{R}(A)} \cap \overline{\mathscr{R}(B)}}$ and $TT^* = T^*T \in \mathscr{P}(\mathscr{H})$.

Necessity. Let $S = TT^* = A^n B^{2n} A^n \in \mathscr{P}(\mathscr{H})$. Then, $S \le A^{2n} \le I$, and hence $S \le SA^{2n}S \le SAS \le S$. It follows that $S = SAS$. If we consider S as 2×2 matrix form $S = I \oplus 0$ with respective space decomposition $\mathscr{H} = \mathscr{R}(S) \oplus \mathscr{N}(S)$, then A has the corresponding matrix form $A = \begin{pmatrix} I & A_3 \\ A_3^* & A_2 \end{pmatrix}$. By Lemma 3, we that get $A_3 = 0$. Hence, $S = AS = SA$ and $\mathscr{R}(S) \subseteq \mathscr{R}(A)$. From

$$\begin{aligned} S &= SA^n B^{2n} A^n = SB^{2n} A^n \\ &= A^n B^{2n} A^n B^{2n} A^n = A^n B^{2n} S, \end{aligned} \tag{8}$$

we get $S = SA^n B^{2n} S = SB^{2n} S \le SBS \le S$. By similar proof that $S = SBS$ implies that $S = SB = BS$ and $\mathscr{R}(S) \subseteq \mathscr{R}(B)$. Now, from $A^nB^{2n}A^n = A^nB^{2n}A^nA^nB^{2n}A^n$ we derive that $A^nB^n(I - B^nA^{2n}B^n)B^nA^n = 0$; that is, $A^nB^n = A^nB^{2n}A^{2n}B^n = SA^nB^n = S$. We get that $S = A^nB^n = B^nA^n$. Hence, $AB = BA$. If we denote

$$\mathscr{H} = \left[\overline{\mathscr{R}(A)} \cap \overline{\mathscr{R}(B)}\right] \oplus \left[\overline{\mathscr{R}(A)} \ominus \left(\overline{\mathscr{R}(A)} \cap \overline{\mathscr{R}(B)}\right)\right] \oplus \mathscr{N}(A), \tag{9}$$

then A and B can be rewritten as $A = A_{11} \oplus A_{22} \oplus 0$ and $B = B_{11} \oplus 0 \oplus B_{33}$, where A_{11}, A_{22}, and B_{11} are injective, densely defined operators and $A_{11}B_{11} = B_{11}A_{11}$. Since $S = A^nB^{2n}A^n = (AB)^{2n} = (A_{11}B_{11})^{2n} \oplus 0 \oplus 0$ is projection, this implies that $A_{11}B_{11} = B_{11}A_{11} = I$. So, $A_{11}^{-1} = B_{11} \in \mathscr{E}(\mathscr{H})$. Hence, $A_{11} = B_{11} = I$; A and B have the matrix forms as in (7). $\square$

In Theorem 5, $T = A^nB^n = B^nA^n = AB = BA = I \oplus 0 \oplus 0 = P_{\overline{\mathscr{R}(A)} \cap \overline{\mathscr{R}(B)}}$.

Theorem 6. *Let $A, B \in \mathscr{E}(\mathscr{H})$. Then, $A * B \in \mathscr{P}(\mathscr{H})$ if and only if A and B have 3×3 operator matrix forms as*

$$A = I \oplus A_{22} \oplus 0, \qquad B = I \oplus 0 \oplus B_{33} \tag{10}$$

with respect to the space decomposition $\mathscr{H} = [\overline{\mathscr{R}(A)} \cap \overline{\mathscr{R}(B)}] \oplus [\overline{\mathscr{R}(A)} \ominus (\overline{\mathscr{R}(A)} \cap \overline{\mathscr{R}(B)})] \oplus \mathscr{N}(A)$. In particular, $A * B = 0$ if and only if $AB = 0$.

Proof. By (10), if $AB = BA = P_{\overline{\mathscr{R}(A)} \cap \overline{\mathscr{R}(B)}}$, then clearly $A * B \in \mathscr{P}(\mathscr{H})$.

Necessity. Observing that A and B as operators on $\mathscr{H} = \overline{R(A)} \oplus \mathscr{N}(A)$ have the forms as $A = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} B_1 & B_3 \\ B_3^* & B_{33} \end{pmatrix}$, where A_1 is injective, densely defined. Then

$$A * B = \begin{pmatrix} \dfrac{(A_1B_1 + B_1A_1)}{2} & \dfrac{A_1B_3}{2} \\ \dfrac{B_3^*A_1}{2} & 0 \end{pmatrix} \tag{11}$$

is a projection implies that $A_1B_3 = 0$ by Lemma 2. So, $B_3 = 0$ because A_1 is injective, densely defined. B_1 can be further written as $B_1 = B_{11} \oplus 0$ with respect to space decomposition $\overline{\mathscr{R}(A)} = (\overline{\mathscr{R}(A)} \cap \overline{\mathscr{R}(B)}) \oplus (\overline{\mathscr{R}(A)} \ominus (\overline{\mathscr{R}(A)} \cap \overline{\mathscr{R}(B)}))$, where B_{11} is injective, densely defined. Similarly, A_1 has corresponding form as $A_1 = A_{11} \oplus A_{22}$ with A_{11} and A_{22} being injective, densely defined. So

$$A * B = \frac{(A_{11}B_{11} + B_{11}A_{11})}{2} \oplus 0 \oplus 0. \tag{12}$$

We say that $(A_{11}B_{11} + B_{11}A_{11})/2$ is injective. In fact, if $\mathscr{N}(A_{11}B_{11} + B_{11}A_{11}) \neq \{0\}$, then $A_{11}B_{11} = -B_{11}A_{11}$ on $\mathscr{N}(A_{11}B_{11} + B_{11}A_{11})$ and hence $A_{11}^2B_{11}^2 = B_{11}^2A_{11}^2$ on $\mathscr{N}(A_{11}B_{11} + B_{11}A_{11})$. Therefore, $A_{11}B_{11} = B_{11}A_{11}$ on $\mathscr{N}(A_{11}B_{11} + B_{11}A_{11})$. Hence, for every $0 \neq x \in \mathscr{N}(A_{11}B_{11} + B_{11}A_{11})$,

$$\frac{A_{11}B_{11} + B_{11}A_{11}}{2}x = A_{11}B_{11}x = 0. \tag{13}$$

Since A_{11} and B_{11} are injective, we get $x = 0$, which contradicts the assumption. Now, $A * B \in \mathscr{P}(\mathscr{H})$ implies that $A_{11}B_{11} + B_{11}A_{11} = 2I$. For every unit vector $x \in \overline{\mathscr{R}(A)} \cap \overline{\mathscr{R}(B)}$,

$$1 = \langle x, x \rangle = \frac{1}{2}\langle A_{11}B_{11}x, x \rangle + \frac{1}{2}\langle B_{11}A_{11}x, x \rangle. \tag{14}$$

Since $A_{11}B_{11}$ is contraction, we derive that $\langle A_{11}B_{11}x, x \rangle = 1$ and $\langle B_{11}A_{11}x, x \rangle = 1$ for every unit vector $x \in \overline{\mathscr{R}(A)} \cap \overline{\mathscr{R}(B)}$. This concludes that $A_{11}B_{11} = B_{11}A_{11} = I$. So, $A_{11}^{-1} = B_{11} \in \mathscr{E}(\mathscr{H})$. Hence, $A_{11} = B_{11} = I$, A and B have the matrix forms as in (7).

In particular, if $AB = 0$, then $BA = 0$ and $A * B = (AB + BA)/2 = 0$. On the other hand, if $A * B = 0$, then $B_3 = 0$ and $A_1B_1 = -B_1A_1$ in (11). We have $A_1^2B_1 = -A_1B_1A_1 = B_1A_1^2$. Therefore, $A_1B_1 = B_1A_1$; that is, $AB = 0$. $\square$

Next, we are now interested in the question of when $A \circ B \geq B$ or $A \circ B \leq B$. In Theorem 2.6 of [2] it is proved that, if $\mathscr{H}$ is finite dimensional and $A \circ B \geq B$, then $AB = BA = B$, and it is asked whether this holds for infinite-dimensional spaces $\mathscr{H}$. In [5, Theorem 2.6], the authors answer this question positively. Here, we include a different proof because it is very short.

Theorem 7. *Let $A, B \in \mathscr{E}(\mathscr{H})$ such that $A \circ B \geq B$ if and only if*

$$A = I \oplus A_1, \qquad B = B_1 \oplus 0,$$
$$\text{where} \quad A_1 \in \mathscr{B}\left(\mathscr{N}(I - A)^{\perp}\right), \; B_1 \in \mathscr{B}\left(\mathscr{N}(I - A)\right). \tag{15}$$

Proof. If $AB = BA = B$, then clearly $A \circ B \geq B$. On the other hand, for arbitrary $0 < \delta < 1$, let $\Delta_1 = [1 - \delta, 1] \cap \sigma(A)$ and $\Delta_2 = [0, 1 - \delta) \cap \sigma(A)$. Let $A = \int_0^{\|A\|} \lambda dE_\lambda$ be the spectral representation of A. Thus, A has the operator matrix form $A = A_1 \oplus A_2$ with respect to the space decomposition $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$, where $\mathscr{H}_1 = E(\Delta_1)\mathscr{H}$ and $\mathscr{H}_2 = E(\Delta_2)\mathscr{H}$. It is clear that $A_2 \leq (1 - \delta)I_{\mathscr{H}_2}$. Let B have corresponding matrix form. Since $B \leq A^{1/2}BA^{1/2} \leq ABA \leq A^{3/2}BA^{3/2} \leq \cdots \leq A^nBA^n \leq \cdots$, $n \in \mathbb{N}$. Hence

$$\begin{pmatrix} A_1^n & 0 \\ 0 & A_2^n \end{pmatrix}\begin{pmatrix} B_1 & B_3 \\ B_3^* & B_2 \end{pmatrix}\begin{pmatrix} A_1^n & 0 \\ 0 & A_2^n \end{pmatrix} = \begin{pmatrix} A_1^nB_1A_1^n & A_1^nB_3A_2^n \\ A_2^nB_3^*A_1^n & A_2^nB_2A_2^n \end{pmatrix} \geq \begin{pmatrix} B_1 & B_3 \\ B_3^* & B_2 \end{pmatrix}. \tag{16}$$

It follows $A_2^nB_2A_2^n \geq B_2 \geq 0$ for all $n \in \mathbb{N}$. Since A_2^n is convergence by strong operator topology to zero, we get that $B_2 = 0$. By Lemma 2, we know that $B_3 = 0$. Hence, $A_1^nB_1A_1^n \geq B_1$ for arbitrary $0 < \delta < 1$. Note that $\bigcap_{0<\delta<1}[[1-\delta, 1] \cap \sigma(A)] \subseteq \{1\}$. Hence. $A_1 = I_{\mathscr{H}_1}$ and A, B have the form (15). $\square$

Note that if $A, B \in \mathscr{E}(\mathscr{H})$, then (i) $AB = BA = B \Leftrightarrow B(I-A) = 0 \Leftrightarrow \mathscr{R}(I-A) \subseteq \mathscr{N}(B)$; (ii) $B \circ A = B \Leftrightarrow B^{1/2}(I-A)B^{1/2} = 0 \Leftrightarrow B(I-A) = 0$; (iii) $B \circ A \geq B \Leftrightarrow -B^{1/2}(I-A)B^{1/2} \geq 0 \Leftrightarrow B(I-A) = 0$. By Theorem 7, it is easy to get the following results.

Corollary 8. *Consider $A, B \in \mathscr{E}(\mathscr{H})$.*

$$\begin{aligned} B = P_{\mathscr{N}(I-A)}B &\Longleftrightarrow A \circ B \geq B \Longleftrightarrow A \circ B = B \\ &\Longleftrightarrow B \circ A \geq B \Longleftrightarrow B \circ A = B \\ &\Longleftrightarrow A = I \oplus A_1, \\ B = B_1 \oplus 0, A_1, B_1 &\text{ are defined in (15)} \\ &\Longleftrightarrow AB = BA = B. \end{aligned} \tag{17}$$

From Corollary 10, we know that $AP_{\overline{\mathscr{R}(B)}} = P_{\overline{\mathscr{R}(B)}}A = P_{\overline{\mathscr{R}(A)} \cap \overline{\mathscr{R}(B)}}$. However, $A \circ B \leq B$ does not imply $AB = BA$. One can check this fact by choices $A = \begin{pmatrix} 1/4 & 1/4 \\ 1/4 & 1/4 \end{pmatrix}$ and $B = \begin{pmatrix} 3/4 & 0 \\ 0 & 1/4 \end{pmatrix}$ in $\mathbb{C}^2$ (see [2]). However, we obtain the following result.

Theorem 9. *Let $A, B \in \mathscr{E}(\mathscr{H})$ and $P \in \mathscr{P}(\mathscr{H})$ such that $\mathscr{R}(P) = \overline{\mathscr{R}(A)} \cap \overline{\mathscr{R}(B)}$.*

(i) *If $AP = P$, then $A \circ B \leq B$ if and only if A and B have 3×3 operator matrix forms*

$$A = I \oplus 0 \oplus A_{33}, \qquad B = B_{11} \oplus B_{22} \oplus 0 \tag{18}$$

with respect to the space decomposition $\mathscr{H} = \mathscr{R}(P) \oplus [\overline{\mathscr{R}(B)} \ominus \mathscr{R}(P)] \oplus \mathscr{N}(B)$.

(ii) *If $BP = P$, then $A \circ B \leq B$ if and only if A and B have 3×3 operator matrix forms*

$$A = A_{11} \oplus A_{22} \oplus 0, \qquad B = I \oplus 0 \oplus B_{33} \tag{19}$$

with respect to the space decomposition $\mathscr{H} = \mathscr{R}(P) \oplus [\overline{\mathscr{R}(A)} \ominus \mathscr{R}(P)] \oplus \mathscr{N}(A)$.

Proof. By (18) and (19), it is clear that $AB = BA$ and $A \circ B = A^{1/2}BA^{1/2} = B^{1/2}AB^{1/2} \leq B$.

Necessity. (i) If $AP = P$, by Lemma 3, A and B as operators on $\mathscr{R}(P) \oplus [\overline{\mathscr{R}(B)} \ominus \mathscr{R}(P)] \oplus \mathscr{N}(B)$ have the operator matrix forms

$$\begin{gathered} A = \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & A_{33} \end{pmatrix}, \\ B = \begin{pmatrix} B_{11} & B_{12} & 0 \\ B_{12}^* & B_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{gathered} \tag{20}$$

If $A \circ B \leq B$, then $0 \leq ABA \leq A^{1/2}BA^{1/2} \leq B$. So

$$B - ABA = \begin{pmatrix} 0 & B_{12} & 0 \\ B_{12}^* & B_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \geq 0. \tag{21}$$

By Lemma 2, we have $B_{12} = 0$. So, (18) holds.

(ii) If $BP = P$, then A and B as operators on $\mathscr{R}(P) \oplus [\overline{\mathscr{R}(A)} \ominus \mathscr{R}(P)] \oplus \mathscr{N}(A)$ can be denoted as

$$\begin{gathered} A = \begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{12}^* & A_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ B = \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & B_{33} \end{pmatrix}. \end{gathered} \tag{22}$$

We have

$$B - ABA = \begin{pmatrix} I - A_{11}^2 & -A_{11}A_{12} & 0 \\ -A_{12}^*A_{11} & -A_{12}^*A_{12} & 0 \\ 0 & 0 & B_{33} \end{pmatrix} \geq 0. \tag{23}$$

By Lemma 2, we have $A_{12}^*A_{12} = 0$; that is, $A_{12} = 0$ and (18) holds. □

Let $A, B \in \mathscr{E}(\mathscr{H})$ and $P = P_{\overline{\mathscr{R}(A)} \cap \overline{\mathscr{R}(B)}}$. Theorem 9 implies that if $AP = P$ or $BP = P$, then $A \circ B \leq B \Leftrightarrow AB = BA$. In particular, if A or $B \in \mathscr{P}(\mathscr{H})$, then $AP_{\overline{\mathscr{R}(A)} \cap \overline{\mathscr{R}(B)}} = P_{\overline{\mathscr{R}(A)} \cap \overline{\mathscr{R}(B)}}$ or $BP_{\overline{\mathscr{R}(A)} \cap \overline{\mathscr{R}(B)}} = P_{\overline{\mathscr{R}(A)} \cap \overline{\mathscr{R}(B)}}$ hold automatically. We get the following corollary.

Corollary 10 (see [2, Theorem 2.6(a)] and [10, Theorem 2.3]). *Let $A, B \in \mathscr{E}(\mathscr{H})$. If A or $B \in \mathscr{P}(\mathscr{H})$, then $A \circ B \leq B$ if and only if $AB = BA$.*

In [11, Lemma 3.4], the authors had gotten that if $A, B \in \mathscr{E}(\mathscr{H})$ and $\dim \mathscr{H} < \infty$, then $A \circ B + A' \circ B = B'$ if and only if $B = (1/2)I$. The authors said they did not know if the condition $\dim \mathscr{H} < \infty$ can be relaxed. In the following, we show that the condition $\dim \mathscr{H} < \infty$ in [11, Lemma 3.4] can be relaxed.

Theorem 11. *Consider $A, B \in \mathscr{E}(\mathscr{H})$. $A \circ B + A' \circ B = B'$ if and only if $B = (1/2)I$.*

Proof. If $A \circ B + A' \circ B = A^{1/2}BA^{1/2} + (I-A)^{1/2}B(I-A)^{1/2} = B'$, then

$$A^{1/2}BA^{1/2} = I - B - (I-A)^{1/2}B(I-A)^{1/2}. \tag{24}$$

So

$$\begin{aligned} ABA &= A^{1/2}(I-B)A^{1/2} \\ &\quad - (I-A)^{1/2}A^{1/2}BA^{1/2}(I-A)^{1/2} \\ &= A - A^{1/2}BA^{1/2} \\ &\quad - (I-A)^{1/2}A^{1/2}BA^{1/2}(I-A)^{1/2} \\ &= A - \left[I - B - (I-A)^{1/2}B(I-A)^{1/2}\right] \\ &\quad - (I-A)^{1/2}\left[I - B - (I-A)^{1/2}B(I-A)^{1/2}\right] \\ &\quad \times (I-A)^{1/2} \\ &= 2A - 2I + 2B - AB - BA + ABA \\ &\quad + 2(I-A)^{1/2}B(I-A)^{1/2} \\ &= 2A - 2I + 2B - AB - BA + ABA \\ &\quad + 2\left[I - B - A^{1/2}BA^{1/2}\right] \\ &= 2A - AB - BA + ABA - 2A^{1/2}BA^{1/2}. \end{aligned} \tag{25}$$

We get

$$2A = AB + BA + 2A^{1/2}BA^{1/2}, \tag{26}$$

which is equal to $A - A^{1/2}BA^{1/2} - AB = -[A - BA - A^{1/2}BA^{1/2}]$. Put $T = A^{1/2} - BA^{1/2} - A^{1/2}B$. Then, $T = T^*$ and $A^{1/2}T = -TA^{1/2}$. Product T from right, we get

$$A^{1/2}T^2 = -TA^{1/2}T = T^2A^{1/2}. \tag{27}$$

Since $A \geq 0$ and $T = T^*$, we derive that T^2 is positive, and hence $A^{1/2}T^2 = -TA^{1/2}T = T^2A^{1/2} \geq 0$. Note that $-TA^{1/2}T \leq 0$. We get that $TA^{1/2}T = TA^{1/4}[TA^{1/4}]^* = 0$; that is, $TA^{1/4} = 0$. Therefore, $A^{1/2}T = TA^{1/2} = 0$. Since $A^{1/2}T = A - A^{1/2}BA^{1/2} - AB$ and $TA^{1/2} = A - A^{1/2}BA^{1/2} - BA$, we obtain that $AB = BA$. In this case, A, B, as operators on $\overline{\mathscr{R}(A)} \oplus \mathscr{N}(A)$, have 2×2 operator matrix form

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}, \qquad \text{where } A_1, B_1 \in \mathscr{B}\left(\overline{\mathscr{R}(A)}\right),\ B_2 \in \mathscr{B}(\mathscr{N}(A)) \tag{28}$$

and A_1 is injective, densely defined. By (26), we get that $A = 2AB$. By (28), we get, $B_1 = (1/2)I_{\overline{\mathscr{R}(A)}}$. By (24), we get that $B_2 = (1/2)I_{\mathscr{N}(A)}$. Hence, $B = (1/2)I$. Conversely, by (28), it is clear that $B = (1/2)I$ implies that $A \circ B + A' \circ B = B'$. □

For $\mathscr{A}, \mathscr{B} \subseteq \mathscr{E}(\mathscr{H})$ with $\mathscr{A} = \{A_i\}$ and $\mathscr{B} = \{B_j\}$, the sequential product of $\mathscr{A}$ and $\mathscr{B}$ is defined by $\mathscr{A} \circ \mathscr{B} = \{A_i \circ B_j\}$. We interpret $\mathscr{A} \circ \mathscr{B}$ to be the measurement obtained when $\mathscr{A}$ is performed first and $\mathscr{B}$ is performed second. The sequential product is noncommutative and nonassociative in general. We write $\mathscr{A} \approx \mathscr{B}$ if the nonzero elements of $\mathscr{A}$ are a permutation of the nonzero elements of $\mathscr{B}$. "$\approx$" is an equivalence relation, and when $\mathscr{A} \approx \mathscr{B}$ we say that $\mathscr{A}$ and $\mathscr{B}$ are equivalent. In this case, the two submeasurements are identical up to an ordering of their outcomes [11].

The results in [11, Theorem 3.1] could be modified as the following. Note that, in [2, Theorem 4.4], it had proved that $A \circ B + A' \circ B = B$ if and only if $AB = BA$.

Theorem 12. *Suppose, $A, B \in \mathscr{E}(\mathscr{H})$, $\mathscr{A} = \{A, A'\}$, and $\mathscr{B} = \{B, B'\}$. If $\mathscr{A} \circ \mathscr{B} \approx \mathscr{B} \circ \mathscr{A}$, then $AB = BA$.*

Proof. Denote

$$\begin{aligned} T &=: \left(A \circ B,\ A \circ B',\ A' \circ B,\ A' \circ B'\right) \\ &=: \left(X_{00},\ X_{01},\ X_{10},\ X_{11}\right), \\ &\left(B \circ A,\ B' \circ A,\ B \circ A',\ B^2 \circ A'\right) \\ &\quad =: \left(X_{00}^T,\ X_{01}^T,\ X_{10}^T,\ X_{11}^T\right), \end{aligned} \tag{29}$$

respectively. If there exists one corresponding term $X_{ij} = X_{ij}^T$, $0 \leq i, j \leq 1$, then $AB = BA$ by Lemma 1. Next, we consider equality for noncorresponding terms.

Case I. If $T = (X_{01}^T, X_{00}^T, X_{11}^T, X_{10}^T)$, then by comparing the third and the fourth components in two sides, we get that $X_{10} + X_{11} = X_{10}^T + X_{11}^T$; that is, $B \circ A' + B' \circ A' = A'$. So, $AB = BA$.

Case II. If $T = (X_{01}^T, X_{10}^T, X_{11}^T, X_{00}^T)$ or $T = (X_{11}^T, X_{00}^T, X_{01}^T, X_{10}^T)$, then by comparing the first and the third components in two sides, we get that $X_{00} + X_{10} = X_{01}^T + X_{11}^T$, that is, $A \circ B + A' \circ B = B'$. By Theorem 11, we get $AB = BA$.

Case III. If $T = (X_{01}^T, X_{11}^T, X_{00}^T, X_{10}^T)$, then by comparing the first and the second components in two sides, we get that $X_{00} + X_{01} = X_{01}^T + X_{11}^T$; that is, $A = B'$, and hence $AB = BA$.

Case IV. If $T = (X_{10}^T, X_{00}^T, X_{11}^T, X_{01}^T)$, then by comparing the first and the second components in two sides, we get that $X_{00} + X_{01} = X_{00}^T + X_{10}^T$; that is, $A = B$. So, $AB = BA$.

Case V. If $T = (X_{10}^T, X_{11}^T, X_{00}^T, X_{01}^T)$, then by comparing the first and the third components in two sides, we get that $X_{00} + X_{10} = X_{00}^T + X_{10}^T$; that is, $A \circ B + A' \circ B = B$. So $AB = BA$.

Case VI. If $T = (X_{10}^T, X_{11}^T, X_{01}^T, X_{00}^T)$, then by comparing the third and the fourth components in two sides we get $X_{10} + X_{11} = X_{00}^T + X_{01}^T$, that is, $B \circ A + B' \circ A = A'$. By Theorem 11, we get that $A = (1/2)I$ and $AB = BA$.

Case VII. If $T = (X_{11}^T, X_{10}^T, X_{00}^T, X_{01}^T)$ or $T = (X_{11}^T, X_{10}^T, X_{01}^T, X_{00}^T)$, then by comparing the first and the second components in two sides, we get $X_{00} + X_{01} = X_{10}^T + X_{11}^T$; that is, $B \circ A' + B' \circ A' = A$. By Theorem 11, we get that $A = (1/2)I$ and $AB = BA$. □

The converse does not hold. Indeed, $\mathscr{A} \circ \mathscr{A} \approx \mathscr{A} \circ \mathscr{A}$ and yet the elements in $\mathscr{A}$ need not be commutative. In the following, we give a characterization of the two submeasurements that are identical up to an arbitrary ordering of their outcomes.

Corollary 13. *Suppose that $A, B \in \mathscr{E}(\mathscr{H})$, $\mathscr{A} = \{A, A'\}$, and $\mathscr{B} = \{B, B'\}$. An arbitrary permutation of the elements in $\mathscr{B} \circ \mathscr{A}$ is equivalent to $\mathscr{A} \circ \mathscr{B}$ if and only if $A = B = (1/2)I$.*

Proof. If $A = B = (1/2)I$, then $A = A' = B = B'$, and clearly an arbitrary permutation of the elements in $\mathscr{B} \circ \mathscr{A}$ is equivalent to $\mathscr{A} \circ \mathscr{B}$.

Conversely, by Cases IV and VII in the proof of Theorem 12, we have $A = B = (1/2)I$. □

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