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Research Article

Existence of Some Semilinear Nonlocal Functional Differential Equations of Neutral Type

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This paper is concerned with the existence of mild and strong solutions on the interval $[0, T]$ for some neutral partial differential equations with nonlocal conditions. The linear part of the equations is assumed to generate a compact analytic semigroup of bounded linear operators, whereas the nonlinear part satisfies the Carathëodory condition and is bounded by some suitable functions. We first employ the Schauder fixed-point theorem to prove the existence of solution on the interval $[\delta, T]$ for $\delta > 0$ that is small enough, and, then, by letting $\delta \to 0$ and using a diagonal argument, we have the existence results on the interval $[0, T]$. This approach allows one to drop the compactness assumption on a nonlocal condition, which generalizes recent conclusions on this topic. The obtained results will be applied to a class of functional partial differential equations with nonlocal conditions.

1. Introduction

The purpose of this paper is to study the existence of mild and strong solutions for the following neutral evolution problem with nonlocal initial conditions:

$$\frac{d}{dt}\left[u(t) + F(t, u(t))\right] = -Au(t) + G(t, u(t)), \quad t \in [0, T],$$
$$u(0) + g(u) = u_0, \tag{1}$$

in a Banach space $(X, \|\cdot\|)$, where $T > 0$ and $-A$ generates an analytic compact semigroup $T(\cdot)$ on X. The functions F, G, and g will be specified later. The Cauchy problem with the nonlocal condition $x(0) + g(x) = x_0$ was first considered by Byszewski [1], and since it reflects physical phenomena more precisely than the classical initial condition $x(0) = x_0$ does, this issue has gained enormous attention in the past several years. For more detailed information about the importance of nonlocal initial conditions in applications, we refer to the works of Byszewski [2], Byszewski and Lakshmikantham [3], and to many other authors [4–7] and the references therein.

Equation (1) has been studied by many authors under various assumptions on the linear part A, the nonlinear terms F, G, and the nonlocal condition g see, for example, [8–14]. A basic approach to this problem is to define the solution operator $\Phi : C([0, T], X) \to C([0, T], X)$ by

$$\begin{aligned}(\Phi u)(t) = {} & T(t)\left[u_0 + F(0, u(0)) - g(u)\right] \\ & - F(t, u(t)) + \int_0^t AT(t-s) F(s, u(s))\, ds \\ & + \int_0^t T(t-s) G(s, u(s))\, ds\end{aligned} \tag{2}$$

and to use various fixed-point theorems, including Schauder fixed-point theorem, Banach contraction principle, Leray-Schauder alternative, and Sadovskii fixed-point theorem, to show that Φ has a fixed point, which is the mild solution of (1). When using fixed-point theorems, it is necessary that the semigroup $T(t)$ generated by the linear part of (1) be compact; that is, $T(t)$ is a compact operator, for all $t > 0$, so that the norm continuity of $T(t)$, for $t > 0$, becomes a key point in the study of the existence of mild solutions. Thus, because of the absence of compactness of the solution operator at $t = 0$, most of the papers on the relevant topics (e.g., [8–10, 14]) assume complete continuity on the nonlocal term g. However, it is too restrictive in terms of applications.

Recently, Liang et al. [15] observed the nonlocal Cauchy problem [1, 3, 4, 6] that the nonlocal condition g is completely determined on $[\delta, T]$, for some $\delta > 0$; that is, such g ignores the fact that $t = 0$; for instance, in [4, 6], the function $g(u)$ is given by

$$g(u) = \sum_{i=0}^{p} c_i u(t_i), \tag{3}$$

where c_i's are given constants, and in this case, we have measurements at $t = 0 \le t_0 < t_1 < \cdots < t_p \le T$ rather than just at $t = 0$. Thus, by assuming that there is a $\delta \in (0, T)$ such that

$$\begin{aligned} g(\phi) = g(\psi), \quad \forall \phi, \varphi \in Y_r &:= \{\varphi \in C([0,T], X); \|\varphi(t)\| \\ &\le r, \quad \forall t \in [0,T]\}, \\ \text{with } \phi(s) = \psi(s), \quad & s \in [\delta, T], \end{aligned} \tag{4}$$

the authors utilize fixed-point theorem twice to deduce the existence results. More recently, Liu and Yuan [16] gave existence results using Schauder fixed-point theorem and a limiting process under the following hypothesis.

There is a $\delta \in (0, T)$ such that $F(\phi) = F(\psi)$ and $g(\phi) = g(\psi)$, for all $\phi, \varphi \in Y_r$, with $\phi(s) = \psi(s)$ and $s \in [\delta, T]$, with the nonlinear term F being bounded by an integrable function.

Motivated by the works in [15, 16], we drop the compactness assumption on the nonlocal condition g and discuss the existence of solutions for (1). The obtained results generalize recent conclusions on this topic.

The present work is organized as follows. Section 1 is devoted the introduction of the problem we studied. Section 2, we explain some known notations and results we will use. The basic hypotheses on (1) are also given in this section. In Section 3, we study the existence of mild solutions to (1) and in Section 4, we investigate some conditions for (1) to come up with strong solutions. In Section 5, an example is given to illustrate the existence results.

2. Preliminaries

Throughout this paper, $T > 0$ will be a fixed real number, X will be a Banach space with norm $\|\cdot\|$, and $-A : D(A) \subset X \to X$ is the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operators $T(\cdot)$ such that $0 \in \rho(A)$. Then, there exists a constant $M \ge 1$ such that $\|T(t)\| \le M$, for $t \ge 0$ and it is possible to define the fractional power A^{α}, for $0 < \alpha \le 1$, as a closed linear operator on its domain $D(A^{\alpha})$ with inverse $A^{-\alpha}$ (see [17]). The followings are the basic properties of A^{α}.

Theorem 1 (see [17], pages 69–75). *The following assertions hold:*

(i) *$D(A^{\alpha})$ is a Banach space with the norm $|x|_{\alpha} := \|A^{\alpha}x\|$, for $x \in D(A^{\alpha})$.*

(ii) *$T(t) : X \to D(A^{\alpha})$, for each $t > 0$.*

(iii) *$A^{\alpha}T(t)x = T(t)A^{\alpha}x$ for each $x \in D(A^{\alpha})$ and $t \ge 0$.*

(iv) *For every $t > 0$, $A^{\alpha}T(t)$ is bounded on X, and there exist $M_{\alpha} > 0$ and $\delta > 0$ such that*

$$\|A^{\alpha}T(t)\| \le \frac{M_{\alpha}}{t^{\alpha}} e^{-\delta t} \le \frac{M_{\alpha}}{t^{\alpha}}. \tag{5}$$

(v) *$A^{-\alpha}$ is a bounded linear operator in X with $D(A^{\alpha}) = \mathrm{Im}(A^{-\alpha})$.*

(vi) *If $0 < \alpha \le \beta$, then $D(A^{\beta}) \hookrightarrow D(A^{\alpha})$.*

Let X_{α} be the Banach space $D(A^{\alpha})$ endowed with the norm $\|\cdot\|_{\alpha}$. Then, we denote by C_{α} the operator norm of $A^{-\alpha}$, that is, $C_{\alpha} := \|A^{-\alpha}\|$, and let E be the Banach space $C([0,T], X)$ endowed with the supnorm given by

$$\|u\|_E := \sup_{0 \le t \le T} |u(t)|, \quad \text{for } u \in E, \tag{6}$$

and, for any $\delta \in (0, T)$, set $E_{\delta} := C([\delta, T], X)$. Moreover, let E_{α} be the Banach space $C([0,T], X_{\alpha})$ endowed with the supnorm given by

$$\|u\|_{E_{\alpha}} := \sup_{0 \le t \le T} \|x(t)\|_{\alpha} \quad \text{for } u \in E_{\alpha}. \tag{7}$$

The following hypotheses are the basic assumptions of this paper.

(H1) There exist $\beta \in (0, 1)$ and $L_1 > 0$ such that the function $F : [0,T] \times X \to X_{\beta}$ satisfies

$$\left\|A^{\beta}F(t,x) - A^{\beta}F(s,y)\right\| \le L_1 \left(|t-s| + \|x-y\|\right), \tag{8}$$

for all $t, s \in [0,T]$ and $x, y \in X$.

(H2) The function $G : [0,T] \times X \to X$ satisfies the following conditions.

(i) For each $t \in [0,T]$, the function $G(t, \cdot) : X \to X$ is continuous, and, for each $x \in X$, the function $G(\cdot, x) : [0,T] \to X$ is strongly measurable.

(ii) For each $k \in \mathbb{N}$ and $t \in [0,T]$, there exists a positive function $g_k \in L^1([0,T], \mathbb{R}^+)$ such that

$$\sup_{\|x\| \le k} \|G(t,x)\| \le g_k(t), \tag{9}$$

and there is a $\gamma \ge 0$ such that

$$\liminf_{k \to \infty} \frac{1}{k} \int_0^T g_k(s)\, ds = \gamma < \infty. \tag{10}$$

(H3) The function $g : L^1([0,T], X) \to X$ is continuous, and there exist constants $L_2, L_3 > 0$ such that

$$\|g(u)\| \le L_2 \|u\|_E + L_3, \tag{11}$$

for $u \in E$.

3. Mild Solutions

Definition 2. A continuous function $u : [0,T] \to X$ is called a mild solution of (1) on $[0,T]$ if, for each $t \in [0,T]$, the function $s \mapsto AT(t-s)F(s,u(s))$ is integrable on $[0,t)$, and the following equation is satisfied:

$$u(t) = T(t)[u_0 + F(0,u(0)) - g(u)] - F(t,u(t)) + \int_0^t AT(t-s)F(s,u(s))\,ds + \int_0^t T(t-s)G(s,u(s))\,ds, \quad (12)$$

for all $t \in [0,T]$.

To see the existence of mild solution of nonlocal problem (1), we will, in view of (12), locate the fixed point of a mapping Φ defined on E by

$$(\Phi u)(t) := T(t)[u_0 + F(0,u(0)) - g(u)] - F(t,u(t)) + \int_0^t AT(t-s)F(s,u(s))\,ds + \int_0^t T(t-s)G(s,u(s))\,ds, \quad 0 \le t \le T. \quad (13)$$

For this, we first observe the following result, where for all $k \in \mathbb{N}$, we let $B_k = \{u \in E : \|u(t)\| \le k, t \in [0,T]\}$.

Lemma 3. *Assume that hypotheses (H1)–(H3) are satisfied, and, in addition, there holds the following inequality:*
(H4)

$$M(L_2 + \gamma + C_\beta L_1) + L_1\left(C_\beta + M_{1-\beta}\frac{T^\beta}{\beta}\right) < 1. \quad (14)$$

Then, $\Phi B_k \subset B_k$, for some $k \in \mathbb{N}$.

Proof. Suppose, on the contrary, that, for each $k > 0$, there exist $u_k \in B_k$ and $t_k \in [0,T]$ such that $\|(\Phi u_k)(t_k)\| > k$. Then, we have

$$\begin{aligned} k &< \|(\Phi u_k)(t_k)\| \\ &\le \|T(t_k)[u_0 - g(u_k) + F(0,u_k(0))]\| \\ &\quad + \|F(t_k,u_k(t_k))\| \\ &\quad + \int_0^{t_k} \|AT(t_k-s)F(s,u_k(s))\|\,ds \\ &\quad + \int_0^{t_k} \|T(t_k-s)G(s,u_k(s))\|\,ds \\ &\le \|T(t_k)\| \\ &\quad \times \left(\|u_0\| + \|g(u_k)\| + \left\|A^{-\beta}A^\beta F(0,u_k(0))\right\|\right) \\ &\quad + \left\|A^{-\beta}A^\beta F(t_k,u_k(t_k))\right\| \\ &\quad + \int_0^{t_k} \left\|A^{1-\beta}T(t_k-s)A^\beta F(s,u_k(s))\right\|\,ds \\ &\quad + \int_0^{t_k} \|T(t_k-s)G(s,u_k(s))\|\,ds \\ &\le M(\|u_0\| + L_2\|u_k\|_E + L_3) \\ &\quad + \left(C_\beta(1+M) + M_{1-\beta}\frac{T^\beta}{\beta}\right) \\ &\quad \times \left(L_1(\|u_k\|_E + T) + \left\|A^\beta F(0,0)\right\|\right) \\ &\quad + M\int_0^{t_k} g_k(s)\,ds. \end{aligned} \quad (15)$$

Dividing the two sides by k and taking the lower limit as $k \to +\infty$, we have

$$M(L_2 + \gamma + C_\beta L_1) + L_1\left(C_\beta + M_{1-\beta}\frac{T^\beta}{\beta}\right) \ge 1, \quad (16)$$

which is a contradiction. This completes the proof. □

By Lemma 3, we see that the mapping $\Phi : E \to E$ defined by (13) maps B_k into itself. We will show that Φ has a fixed point in B_k. To see this, note first that Φ is continuous by the continuity of F, G and g. We decompose Φ as $\Phi = \Phi_1 + \Phi_2$, where

$$\begin{aligned} (\Phi_1 u)(t) &= -F(t,u(t)) + \int_0^t AT(t-s)F(s,u(s))\,ds, \\ (\Phi_2 u)(t) &= T(t)[u_0 + F(0,u(0)) - g(u)] \\ &\quad + \int_0^t T(t-s)G(s,u(s))\,ds. \end{aligned} \quad (17)$$

We show that Φ_1 is a contraction in B_k and Φ_2 is a compact operator in B_k.

Lemma 4. *Assume that hypotheses (H1)–(H4) are satisfied. If $u_0 \in X$, then Φ_1 is a contraction in B_k.*

Proof. Observe that, for $t \in [0,T]$ and $u, v \in B_k$, we have the assumption (H1) as follows:

$$\begin{aligned} &\|(\Phi_1 u)(t) - (\Phi_1 v)(t)\| \\ &\quad \le \|F(t,u(t)) - F(t,v(t))\| \\ &\qquad + \int_0^t \|AT(t-s)[F(s,u(s)) \\ &\qquad\qquad - F(s,v(s))]\|\,ds \end{aligned}$$

$$\leq C_\beta L_1 \|u(t) - v(t)\| + L_1 \left(\int_0^t \frac{M_{1-\beta}}{(t-s)^{1-\beta}} ds \right) \sup_{0\leq s\leq T} \|u(s) - v(s)\| \leq L_1 \left(C_\beta + M_{1-\beta} \frac{T^\beta}{\beta} \right) \sup_{0\leq s\leq T} \|u(s) - v(s)\| . \quad (18)$$

Hence,

$$\|\Phi_1 u - \Phi_1 v\|_E \leq \tilde{L} \|u - v\|_E, \quad (19)$$

where $\tilde{L} := L_1(C_\beta + M_{1-\beta}(T^\beta/\beta))$ which is, by (H4), less than 1. Thus, Φ_1 is a contraction. □

Lemma 5. *Assume that hypotheses (H1)–(H4) are satisfied, and, in addition, the following is given.*

(H5) *There exists a $\delta \in (0,T)$ such that $F(\cdot, u(\cdot)) = F(\cdot, v(\cdot))$, $G(\cdot, u(\cdot)) = G(\cdot, v(\cdot))$ and $g(u) = g(v)$, for any $u, v \in B_k$, with $u(s) = v(s)$ and $s \in [\delta, T]$.*

Then the problem (1) *has at least one mild solution in B_k for some $k \in \mathbb{N}$.*

Proof. Let δ be given by (H5), and let

$$B_k(\delta) := \{u \in C([\delta, T], X) : \|u(t)\| \leq k, \ \forall t \in [\delta, T]\} . \quad (20)$$

For any $u \in B_k(\delta)$, let $\overline{u} \in B_k$ be defined by

$$\overline{u}(t) := \begin{cases} u(t), & t \in [\delta, T], \\ u(\delta), & t \in [0, \delta). \end{cases} \quad (21)$$

Now, we define Φ_δ on E_δ by

$$(\Phi_\delta u)(t) := T(t) [u_0 + F(0, \overline{u}(0)) - g(\overline{u})] - F(t, \overline{u}(t)) + \int_0^t AT(t-s) F(s, \overline{u}(s)) ds + \int_0^t T(t-s) G(s, \overline{u}(s)) ds, \quad \delta \leq t \leq T. \quad (22)$$

Then, by Lemma 3, we see that $\Phi_\delta B_k(\delta) \subset B_k(\delta)$. Consider Φ_δ as the sum $\Phi_\delta = \Phi_{1,\delta} + \Phi_{2,\delta}$, where

$$(\Phi_{1,\delta} u)(t) = -F(t, \overline{u}(t)) + \int_0^t AT(t-s) F(s, \overline{u}(s)) ds, \quad \forall u \in B_k(\delta), \ t \in [\delta, T], \quad (23)$$

and $\Phi_{2,\delta}$ is defined on $B_k(\delta)$ by

$$(\Phi_{2,\delta} u)(t) := T(t) (u_0 + F(0, \overline{u}(0)) - g(\overline{u})) + \int_0^t T(t-s) G(s, \overline{u}(s)) ds, \quad t \in [\delta, T]. \quad (24)$$

With a similar argument as in the proof of Lemma 4, one sees that $\Phi_{1,\delta}$ is a contraction on $B_k(\delta)$.

For the compactness of $\Phi_{2,\delta}$, note first that $\Phi_{2,\delta}$ is continuous by the continuity of G and g. Now, to show that the set $\{\Phi_{2,\delta} u : u \in B_k(\delta)\}$ is relatively compact in $C([\delta, T], X)$, we will prove that, for each $t \in [\delta, T]$, the two sets

$$\{T(t) (u_0 + F(0, \overline{u}(0)) - g(\overline{u})) : u \in B_k(\delta)\}, \quad V(t) := \left\{ \int_0^t T(t-s) G(s, \overline{u}(s)) ds : u \in B_k(\delta) \right\} \quad (25)$$

are relatively compact in X and that

$$\{T(\cdot) (u_0 + F(0, \overline{u}(0)) - g(\overline{u})) : u \in B_k(\delta)\}, \quad \left\{ \int_0^\cdot T(\cdot - s) G(s, \overline{u}(s)) ds : u \in B_k(\delta) \right\} \quad (26)$$

are equicontinuous families of functions on $[\delta, T]$. In fact, it follows from (H3) and the compactness of $T(t)$, for $t \in [\delta, T]$ that, for each $t \in [\delta, T]$,

$$\{T(t) (u_0 + F(0, \overline{u}(0)) - g(\overline{u})) : u \in B_k(\delta)\} \quad \text{is relatively compact in } X. \quad (27)$$

Moreover, since for each $t > 0$, and $\epsilon \in (0, t)$, the set

$$\left\{ T(\varepsilon) \int_0^{t-\varepsilon} T(t - \varepsilon - s) G(s, \overline{u}(s)) ds : u \in B_k(\delta) \right\} \quad (28)$$

is relatively compact, then, in view of

$$\left\| \int_0^t T(t-s) G(s, \overline{u}(s)) ds - T(\varepsilon) \int_0^{t-\varepsilon} T(t - \varepsilon - s) G(s, \overline{u}(s)) ds \right\| \leq \int_{t-\epsilon}^t \|T(t-s)\|_{L(X)} \|G(s, \overline{u}(s))\| ds \leq M \int_{t-\epsilon}^t g_k(s) ds, \quad (29)$$

we see by (H2) that there are relative compact sets arbitrarily close to $V(t)$, and, hence, $V(t)$ is also relatively compact in X. Now, by the norm continuity of $T(t)$, for $t > 0$, we see that

$$\|[T(t+h) - T(t)] (u_0 + F(0, \overline{u}(0)) - g(\overline{u}))\| \longrightarrow 0 \quad \text{as } h \longrightarrow 0 \quad (30)$$

independently of $u \in B_k$, and, hence, $\{T(\cdot)(u_0 + F(0,\overline{u}(0) - g(\overline{u})) : u \in B_k(\delta)\}$ is an equicontinuous family of functions on $[\delta, T]$. Finally, let $\epsilon > 0$ be arbitrarily small, and we see that

$$\begin{aligned}
&\left\| \int_0^{t+h} T(t+h-s)\, G(s,\overline{u}(s))\, ds - \int_0^{t} T(t-s)\, G(s,\overline{u}(s))\, ds \right\| \\
&\leq \int_0^{t-\epsilon} \| [T(t+h-s) - T(t-s)]\, G(s,\overline{u}(s)) \| \, ds \\
&\quad + \int_{t-\epsilon}^{t} \| [T(t+h-s) - T(t-s)] \times G(s,\overline{u}(s)) \| \, ds \\
&\quad + \int_t^{t+h} \| T(t+h-s)\, G(s,\overline{u}(s)) \| \, ds \\
&\leq \int_0^{t-\epsilon} \| T(t-s+h) - T(t-s) \|_{L(X)} \times g_k(s)\, ds + 2M \int_{t-\epsilon}^{t} g_k(s)\, ds \\
&\quad + M \int_t^{t+h} g_k(s)\, ds,
\end{aligned} \tag{31}$$

which is, by the norm continuity of $T(t)$, for $t > 0$, arbitrarily small and independent of $u \in B_k$ as $h \to 0$. Therefore, $\{\int_0^{\cdot} T(\cdot - s)G(s,u(s))ds : u \in B_k(\delta)\}$ is an equicontinuous family of functions on $[\delta, T]$ and so is $\{\Phi_{2,\delta}u : u \in B_k(\delta)\}$. It follows from Arzela-Ascoli's theorem that $\{\Phi_{2,\delta}u : u \in B_k(\delta)\}$ is relatively compact on E_δ. Thus, the mapping $\Phi_{2,\delta}$ defined by (24) is compact.

By the fixed-point theorem of Sadovskiĭ [18], this shows that Φ_δ has a fixed point in $B_k(\delta)$; that is, there is a $\varphi \in B_k(\delta)$ such that

$$\begin{aligned}
\varphi(t) := {} & T(t)\left[u_0 + F\left(0,\overline{\varphi}(0)\right) - g\left(\overline{\varphi}\right)\right] \\
& - F\left(t,\overline{\varphi}(t)\right) + \int_0^t AT(t-s)\, F\left(s,\overline{\varphi}(s)\right) ds \\
& + \int_0^t T(t-s)\, G\left(s,\overline{\varphi}(s)\right) ds, \quad \delta \leq t \leq T.
\end{aligned} \tag{32}$$

Now, define a function ψ on $[0, T]$ by

$$\begin{aligned}
\psi(t) := {} & T(t)\left[u_0 + F\left(0,\overline{\varphi}(0)\right) - g\left(\overline{\varphi}\right)\right] \\
& - F\left(t,\overline{\varphi}(t)\right) + \int_0^t AT(t-s)\, F\left(s,\overline{\varphi}(s)\right) ds \\
& + \int_0^t T(t-s)\, G\left(s,\overline{\varphi}(s)\right) ds, \quad 0 \leq t \leq T.
\end{aligned} \tag{33}$$

Then, $\psi = \varphi$ on $[\delta, T]$, and $\psi \in B_k$. Consequently, (H5) guarantees that

$$\begin{aligned}
\psi(t) := {} & T(t)\left[u_0 + F\left(0,\psi(0)\right) - g\left(\psi\right)\right] \\
& - F\left(t,\psi(t)\right) + \int_0^t AT(t-s)\, F\left(s,\psi(s)\right) ds \\
& + \int_0^t T(t-s)\, G\left(s,\psi(s)\right) ds, \quad 0 \leq t \leq T.
\end{aligned} \tag{34}$$

That is, ψ is a mild solution of (1). □

For the main results in this section, we introduce a family of nonlocal neutral problems as follows. Firstly, we define, for each $\delta \in (0, T)$, an operator $\mathscr{B}_\delta$ on E by

$$(\mathscr{B}_\delta u)(t) := \begin{cases} u(\delta), & 0 \leq t \leq \delta, \\ u(t), & \delta < t \leq T, \end{cases} \tag{35}$$

for all $u \in E$. It is clear that $\mathscr{B}_\delta$ is bounded on E and $\|\mathscr{B}_\delta\|_{L(E)} \leq 1$, and, hence, $\mathscr{B}_\delta B_k \subset B_k$. Now, for each $\delta \in [0, T]$, we define $g_\delta : E \to X$ by

$$g_\delta(u) = g\left(\mathscr{B}_\delta u\right), \quad \forall u \in E, \tag{36}$$

$F_\delta : [0, T] \times X \to X$ by

$$F_\delta(t, u(t)) = F\left(t, \mathscr{B}_\delta u(t)\right), \quad \forall u \in E, \tag{37}$$

and $G_\delta : [0, T] \times X \to X$ by

$$G_\delta(t, u(t)) = G\left(t, \mathscr{B}_\delta u(t)\right), \quad \forall u \in E. \tag{38}$$

Consider the following nonlocal neutral problem:

$$\begin{gathered}
u'(t) + F_\delta(t, u(t)) = Au(t) + G_\delta(t, u(t)), \quad t \in [0, T], \\
u(0) + g(u) = u_0 \in X.
\end{gathered} \tag{NNP_δ}$$

In view of (35)–(38), the following result is an immediate corollary of Lemmas 4 and 5.

Lemma 6. *Suppose that (H1)–(H4) are satisfied. Then, for any $\delta \in (0, T]$, the problem (NNP_δ) has at least one mild solution in B_k.*

Theorem 7. *Suppose that, hypotheses (H1)–(H4) are satisfied. Then, problem (1) has at least one mild solution in B_k for some $k \in \mathbb{N}$.*

Proof. Choose a decreasing sequence $\{\delta_n\}_{n\in\mathbb{N}} \subset (0, T)$ so that $\lim_{n\to\infty}\delta_n = 0$, and, then, by Lemma 6, we see that for each $n \in \mathbb{N}$, there is an u_n such that

$$\begin{aligned}
u_n(t) := {} & T(t)\Big[u_0 + F\Big(0, \big(\mathscr{B}_{\delta_n} u_n\big)(0)\Big) - g\big(\mathscr{B}_{\delta_n} u_n\big)\Big] - F\Big(t, \big(\mathscr{B}_{\delta_n} u_n\big)(t)\Big) \\
& + \int_0^t AT(t-s)\, F\Big(s, \big(\mathscr{B}_{\delta_n} u_n\big)(s)\Big) ds \\
& + \int_0^t T(t-s)\, G\Big(s, \big(\mathscr{B}_{\delta_n} u_n\big)(s)\Big) ds, \\
& 0 \leq t \leq T.
\end{aligned} \tag{39}$$

Now, for each $n \in \mathbb{N}$, we define Φ_{δ_n} on B_k by

$$\begin{aligned}\left(\Phi_{\delta_n} u\right)(t) := T(t)\left[u_0 + F\left(0, \left(\mathscr{B}_{\delta_n} u\right)(0)\right) - g_{\delta_n}(u)\right] \\ - F\left(t, \left(\mathscr{B}_{\delta_n} u\right)(t)\right) \\ + \int_0^t AT(t-s) F\left(s, \left(\mathscr{B}_{\delta_n} u\right)(s)\right) ds \\ + \int_0^t T(t-s) G\left(s, \left(\mathscr{B}_{\delta_n} u\right)(s)\right) ds, \\ 0 \le t \le T.\end{aligned} \tag{40}$$

Then, (39) implies that Φ_{δ_n} has a fixed point in B_k which is a mild solution for the nonlocal Cauchy problem (NCP_{δ_n}). Decompose Φ_{δ_n} as $\Phi_{\delta_n} = \Phi_{1,\delta_n} + \Phi_{2,\delta_n}$, where

$$\begin{aligned}\left(\Phi_{1,\delta_n} u\right)(t) = -F\left(t, \left(\mathscr{B}_{\delta_n} u\right)(t)\right) \\ + \int_0^t AT(t-s) F\left(s, \left(\mathscr{B}_{\delta_n} u\right)(s)\right) ds, \\ 0 \le t \le T, \\ \Phi_{2,\delta_n} u(t) = T(t)\left[u_0 + F\left(0, \left(\mathscr{B}_{\delta_n} u\right)(0)\right) - g_{\delta_n}(u)\right] \\ + \int_0^t T(t-s) G\left(s, \left(\mathscr{B}_{\delta_n} u\right)(s)\right) ds, \\ 0 \le t \le T.\end{aligned} \tag{41}$$

With the same argument as in the proof of Lemma 4, we have that Φ_{1,δ_n} is a contraction. Furthermore, since the sequence $\{\mathscr{B}_{\delta_n} u_n\}_{n\in\mathbb{N}}$ lies in B_k, then a similar argument as in the proof of Lemma 5 (see (27)–(31)) shows that, for each $t \in [0, T]$, the sets

$$\begin{gathered}\left\{T(t)\left(u_0 + F(0, u(0)) - g\left(\mathscr{B}_{\delta_n} u_n\right)\right)\right\}_{n\in\mathbb{N}}, \\ \left\{\int_0^t T(t-s) G\left(s, \left(\mathscr{B}_{\delta_n} u_n\right)(s)\right) ds\right\}_{n\in\mathbb{N}}\end{gathered} \tag{42}$$

are both relatively compact in X and that the sequence of functions

$$\left\{\int_0^{\cdot} T(\cdot - s) G\left(s, \left(\mathscr{B}_{\delta_n} u_n\right)(s)\right) ds\right\}_{n\in\mathbb{N}} \tag{43}$$

is equicontinuous on $[0, T]$. Hence, it follows from Ascoli-Arzela theorem that

$$\begin{gathered}\left\{\int_0^{\cdot} T(\cdot - s) G\left(s, \left(\mathscr{B}_{\delta_n} u_n\right)(s)\right) ds\right\}_{n\in\mathbb{N}} \\ \text{is relatively compact on } E.\end{gathered} \tag{44}$$

Now, let $\{\epsilon_n\}_{n\in\mathbb{N}} \subset (0, T)$ be a decreasing sequence such that $\lim_{n\to\infty} \epsilon_n = 0$, and let $\{u_{n_0}\}_{n_0\in\mathbb{N}}$ be a subsequence of $\{u_n\}_{n\in\mathbb{N}}$. Then, a similar argument as in the proof of Lemma 5 insures that $\{T(\cdot)(u_0 + F(0, u(0)) - g(\mathscr{B}_{\delta_{n_0}} u_{n_0}))\}_{n_0\in\mathbb{N}}$ is an equicontinuous sequence of functions on $[\epsilon_1, T]$. Thus, Ascoli-Arzela theorem guarantees that the sequence

$$\begin{gathered}\left\{T(\cdot)\left(u_0 + F(0, u(0)) - g\left(\mathscr{B}_{\delta_{n_0}} u_{n_0}\right)\right)\right\}_{n_0\in\mathbb{N}} \\ \text{is relatively compact in } C([\epsilon_1, T], X).\end{gathered} \tag{45}$$

Thus, by (44) and (45), we see that $\{u_{n_0}\}_{n_0\in\mathbb{N}}$ is relatively compact in $C([\epsilon_1, T], X)$, and, hence, we can select a subsequence of $\{u_{n_0}\}_{n_0\in\mathbb{N}}$ denoted by $\{u_{n_1}\}_{n_1\in\mathbb{N}}$, which is a Cauchy sequence in $C([\epsilon_1, T], X)$. By a similar process, we can select a subsequence of $\{u_{n_1}\}_{n_1\in\mathbb{N}}$ denoted by $\{u_{n_2}\}_{n_2\in\mathbb{N}}$, which is a Cauchy sequence in $C([\epsilon_2, T], X)$. Repeat the above argument, and use a diagonal argument to obtain a subsequence of $\{u_{n_0}\}_{n_0\in\mathbb{N}}$ denoted by $\{v_n\}_{n\in\mathbb{N}}$. Then, for every $t \in (0, T]$, $\{v_n(t)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X, and thus, we can define the function v_∞ by

$$v_\infty(t) = \begin{cases} 0, & t = 0, \\ \lim_{n\to\infty} v_n(t), & 0 < t \le T. \end{cases} \tag{46}$$

It is clear that v_∞ is strongly measurable, $v_\infty \in B_k$, and

$$\int_0^T \left\| v_n(t) - v_\infty(t) \right\| dt \le 2Tk. \tag{47}$$

It therefore follows from Lebesgue's dominated convergence theorem that there is a subsequence $\{\tau_n\}_{n\in\mathbb{N}}$ of $\{\delta_{n_0}\}_{n_0\in\mathbb{N}}$ such that

$$\begin{aligned}\int_0^T \left\| \left(\mathscr{B}_{\tau_n} v_n\right)(t) - v_\infty(t) \right\| dt \\ \le \int_0^T \left\| \left(\mathscr{B}_{\tau_n} v_n\right)(t) - v_n(t) \right\| dt \\ + \int_0^T \left\| v_n(t) - v_\infty(t) \right\| dt \longrightarrow 0 \\ \text{as } n \longrightarrow \infty.\end{aligned} \tag{48}$$

This shows that the sequence $\{\mathscr{B}_{\delta_n} u_n\}_{n\in\mathbb{N}}$ is relatively compact on E, and, hence, by the continuity of g, it follows that

$$\begin{gathered}\left\{T(\cdot)\left(u_0 + F(0, u(0)) - g\left(\mathscr{B}_{\delta_n} u_n\right)\right)\right\}_{n\in\mathbb{N}} \\ \text{is relatively compact on } E.\end{gathered} \tag{49}$$

By (44) and (49), we see the relative compactness of $\{u_n\}_{n\in\mathbb{N}}$ on E. Thus, there is a subsequence of $\{u_n\}_{n\in\mathbb{N}}$ denoted by $\{u_\nu\}_{\nu\in\mathbb{N}}$ and a function $u_\infty \in E$ such that

$$\lim_{\nu\to\infty} \left\| u_\nu - u_\infty \right\|_E = 0. \tag{50}$$

It is clear that $u_\infty \in B_k$. Since

$$\begin{aligned}\left\|\mathcal{B}_{\delta_\nu} u_\nu - u_\infty\right\|_E &= \sup_{0\le t\le T}\left\|\mathcal{B}_{\delta_\nu} u_\nu(t) - u_\infty(t)\right\| \\ &\le \sup_{0\le t\le \delta_\nu}\left\|u_\nu(\delta_\nu) - u_\infty(t)\right\| \\ &\quad + \sup_{\delta_\nu\le t\le T}\left\|u_\nu(t) - u_\infty(t)\right\| \\ &\le \left\|u_\nu(\delta_\nu) - u_\infty(\delta_\nu)\right\| \\ &\quad + \sup_{0\le t\le \delta_\nu}\left\|u_\infty(\delta_\nu) - u_\infty(t)\right\| \\ &\quad + \left\|u_\nu - u_\infty\right\|_E \\ &\le 2\left\|u_\nu - u_\infty\right\|_E \\ &\quad + \sup_{0\le t\le \delta_\nu}\left\|u_\infty(\delta_\nu) - u_\infty(t)\right\|,\end{aligned} \tag{51}$$

then (50) and the uniform continuity of u_∞ imply that $\lim_{\nu\to\infty}\|\mathcal{B}_{\delta_\nu} u_\nu - u_\infty\|_E = 0$. By taking limits in (39), we see that u_∞ is a mild solution of (1) and this completes the proof. □

We will consider the case more generally; that is, the nonlocal condition g is defined on E rather than $L^1([0,T],X)$.

Theorem 8. *Suppose that, hypotheses (H1) and (H2) are satisfied, and, in addition, there hold the following hypotheses.*

(H6) *The function $g : E \to X$ is continuous, and inequality* (11) *also holds.*

(H7) $\lim_{\epsilon\to 0}\sup_{\phi\in B_k}\|g(\phi) - g(\phi^\epsilon)\|_E = 0$, *where*

$$\phi^\epsilon(t) = \begin{cases}\phi(\epsilon), & 0\le t\le \epsilon,\\ \phi(t), & \epsilon < t\le T.\end{cases} \tag{52}$$

If inequality (H4) holds, then, problem (1) *has at least one mild solution in B_k, for some $k\in\mathbb{N}$.*

Proof. Let $\{u_n\}_{n\in\mathbb{N}}$ and $\{\epsilon_n\}_{n\in\mathbb{N}}$ be the sequences defined as in the proof of Theorem 7. With the same arguments as in the proof of Lemma 4, we see that $\Phi B_k \subset B_k$, for some $k\in\mathbb{N}$. Moreover, it follows from the same arguments as in the proof of Theorem 7 that (44), and (45) also hold, and, for every subsequence $\{u_{n_0}\}_{n_0\in\mathbb{N}}$ of $\{u_n\}_{n\in\mathbb{N}}$, there exist a subsequence $\{v_n\}_{n=1}^\infty$ and a function $v_\infty : (0,T] \to X$ such that v_∞ is continuous on $(0,T]$ and, for every ϵ_k,

$$\begin{gathered}\lim_{n\to\infty}\max_{t\in[\epsilon_k,T]}\left\|v_n(t) - v_\infty(t)\right\| = 0,\\ \text{or}\quad \lim_{n\to\infty}\left\|\mathcal{B}_{\epsilon_k}\left[v_n - v_\infty\right]\right\|_E = 0.\end{gathered} \tag{53}$$

Let $\varepsilon > 0$ be given. It follows from (H7) and (53) that there is a $\delta > 0$ such that

$$\left\|g(\phi) - g(\psi)\right\| < \frac{\varepsilon}{4} \quad \text{whenever } \phi = \psi \text{ on } [\delta, T] \tag{54}$$

and that, for every $k\in\mathbb{N}$, there is an N_k such that $n > N_k$ implies that

$$\left\|g\left(\mathcal{B}_{\epsilon_k}\left[v_n - v_\infty\right]\right)\right\| < \frac{\varepsilon}{4}. \tag{55}$$

Choose K that is large enough so that $\epsilon_K < \delta$, and define $\varphi : [0,T] \to X$ by

$$\varphi(t) = \begin{cases} v_\infty(\epsilon_K), & 0\le t\le \epsilon_K,\\ v_\infty(t), & \epsilon_K\le t\le T.\end{cases} \tag{56}$$

Thus, (H7), (54), and (55) insure that

$$\begin{aligned}\left\|g(v_n) - g(v_m)\right\| &\le \left\|g(v_n) - g\left(\mathcal{B}_{\epsilon_K} v_n\right)\right\| \\ &\quad + \left\|g\left(\mathcal{B}_{\epsilon_K} v_n\right) - g(\varphi)\right\| \\ &\quad + \left\|g(\varphi) - g\left(B_{\epsilon_K} v_m\right)\right\| \\ &\quad + \left\|g\left(\mathcal{B}_{\epsilon_K} v_m\right) - g(v_m)\right\| < \varepsilon.\end{aligned} \tag{57}$$

And, hence, by the continuity of g and the compactness of $T(t)$, for $t > 0$, (49) is also valid in this case. Therefore, a similar argument as in the last paragraph of the proof of Theorem 7 shows the existence of a mild solution for (1). □

4. Strong Solutions

Definition 9. A mild solution u is called a strong solution if u is continuously differentiable on $(0,T]$ with $u' \in L^1([0,T],X)$ and satisfies (1).

In the following, we establish a result of a strong solution for (1).

Theorem 10. *Let X be a reflexive Banach space. Suppose that there hold the following hypotheses.*

(H8) *The function $F : [0,T]\times X \to X_1$ is a continuous function and there exists L_4 such that*

$$\left\|AF(t,x) - AF(s,y)\right\| \le L_4\left(|t-s| + \|x-y\|\right), \tag{58}$$

for all $t,s\in[0,T]$ and $x,y\in X$.

(H9) *$G(\cdot,\cdot)$ is Lipschitz continuous; that is, there exists a constant $L_0 > 0$ such that*

$$\left\|G(t,x) - G(s,y)\right\| \le L_0\left(|t-s| + \|x-y\|\right), \tag{59}$$

for all $(t,x),(s,y)\in[0,T]\times X$.

(H10) *The function $g : L^1([0,T],X) \to X$ is continuous, $g(u)\in X_1$, for all $u\in E$, and*

$$\left\|g(u)\right\|_1 \le L_5\|u\|_E + L_6, \tag{60}$$

for some $L_5, L_6 > 0$.

(H11) *There holds the following inequality:*

$$L_4\left(C_1 + MT\right) + 2MTL_0 < 1. \tag{61}$$

If $u_0 \in X_1$ and inequality (H4) also holds with L_1, L_2; C_β is replaced by L_4, L_5; and $C_1 := \|A^{-1}\|$, respectively, then (1) *has a strong solution on $[0,T]$.*

Proof. Let Φ be the operator defined by (13). By (H8), (H9) and (H10), one can use a similar argument as in the proof of Lemma 3 to deduce that there is a $k \in \mathbb{N}$ such that $\Phi B_k \subset B_k$. For this k, consider the set

$$B = \{u \in E : \|u\|_E \le k, \|u(t) - u(s)\| \le L|t-s|,\ \forall t, s \in [0,T]\}, \tag{62}$$

for some k and L that are large enough. It is clear that B is nonempty, convex, and closed. We will prove that Φ has a fixed point on B. Obviously, from the proofs of Lemmas 4 and 5 and Theorem 7, it is sufficient to show that, for any $x \in B$,

$$\|(\Phi u)(t_2) - (\Phi u)(t_1)\| \le L|t_2 - t_1|, \quad \forall t_2, t_1 \in [0,T]. \tag{63}$$

We first fix an element $w \in B$ and observe that, for any $s \in [0,T]$,

$$\begin{aligned}
&\|AF(s,u(s))\| \\
&\quad\le \|A[F(s,u(s)) - F(0,(0))]\| \\
&\qquad + \|AF(0,u(0))\| \\
&\quad\le L_4(s + \|u(s) - u(0)\|) + \|AF(0,u(0))\| \\
&\quad\le L_4(T+2k) + k_0, \\
&\|G(s,u(s))\| \\
&\quad\le \|G(s,u(s)) - G(s,w(s))\| \\
&\qquad + \|G(s,w(s)) - G(0,w(0))\| \\
&\qquad + \|G(0,w(0))\| \le L_0\|u(s) - w(s)\| \\
&\qquad + L_0(1+L)s + \|G(0,w(0))\| \\
&\quad\le L_0[2k + (1+L)T] + k_1,
\end{aligned} \tag{64}$$

where $k_0 := \|AF(0,w(0))\|$ and $k_1 := \|G(0,w(0))\|$. Now,

$$\begin{aligned}
&\|(\Phi u)(t_2) - (\Phi u)(t_1)\| \\
&\quad\le \left\|\int_{t_1}^{t_2} AT(s)[u_0 + F(0,u(0)) - g(u)]\,ds\right\| \\
&\qquad + \|F(t_2,u(t_2)) - F(t_1,u(t_1))\| \\
&\qquad + \left\|\int_0^{t_2} AT(t_2 - s)F(s,u(s))\,ds - \int_0^{t_1} AT(t_1 - s)F(s,u(s))\,ds\right\| \\
&\qquad + \left\|\int_0^{t_2} T(t_2 - s)G(s,u(s))\,ds - \int_0^{t_1} T(t_1 - s)G(s,u(s))\,ds\right\| \\
&\quad\le \left\|\int_{t_1}^{t_2} AT(s)u_0\,ds + \int_{t_1}^{t_2} T(s)AF(0,u(0))\,ds - \int_{t_1}^{t_2} T(s)Ag(u)\,ds\right\| \\
&\qquad + C_1\|A[F(t_2,u(t_2)) - F(t_1,u(t_1))]\| \\
&\qquad + \left\|\int_0^{t_1} T(s)A[F(t_2 - s,u(t_2 - s)) - F(t_1 - s,u(t_1 - s))]\,ds + \int_{t_1}^{t_2} T(s)AF(t_2 - s,u(t_2 - s))\,ds\right\| \\
&\qquad + \left\|\int_0^{t_1} T(s)[G(t_2 - s,u(t_2 - s)) - G(t_1 - s,u(t_1 - s))]\,ds + \int_{t_1}^{t_2} T(t_2 - s)G(s,u(s))\,ds\right\|.
\end{aligned} \tag{65}$$

Thus, from (H8), (H9), and (H10), it follows that

$$\begin{aligned}
&\|(\Phi u)(t_2) - (\Phi u)(t_1)\| \\
&\quad\le M\|Au_0\||t_2 - t_1| \\
&\qquad + M\|AF(0,u(0))\||t_2 - t_1| \\
&\qquad + M(kL_5 + L_6)|t_2 - t_1| \\
&\qquad + C_1L_4(1+L)|t_2 - t_1| \\
&\qquad + MTL_4(1+L)|t_2 - t_1| \\
&\qquad + M[L_4(T+2k) + k_0]|t_2 - t_1| \\
&\qquad + MTL_0(1+L)|t_2 - t_1| \\
&\qquad + M\{L_0[2k + (1+L)T] + k_1\}|t_2 - t_1| \\
&\quad\le \{K_0 + L[(C_1 + MT)L_4 + 2MTL_0]\}|t_2 - t_1|,
\end{aligned} \tag{66}$$

where K_0 is a constant independent of L. Since (H11) implies that

$$K^* := L_4(C_1 + MT) + 2MTL_0 < 1, \tag{67}$$

then

$$\|(\Phi u)(t_2) - (\Phi u)(t_1)\| \le L|t_2 - t_1|, \quad \forall t_1, t_2 \in [0,T], \tag{68}$$

whenever

$$L \ge \frac{K_0}{1 - K^*}. \tag{69}$$

Therefore, Φ has a fixed point u which is a mild solution of (1). By the above calculation, we see that, for this $u(\cdot)$, all of the functions

$$\begin{aligned} p(t) &= F(t, u(t)), \\ q(t) &= T(t)[u_0 + F(0, u(0)) - g(u)], \\ x(t) &= \int_0^t AT(t-s) F(s, u(s)) \, ds, \\ y(t) &= \int_0^t T(t-s) G(s, u(s)) \, ds \end{aligned} \tag{70}$$

are Lipschitz continuous, respectively. Since u is Lipschitz continuous on $[0, T]$ and the space X is reflexive, then a result of [19] asserts that $u(\cdot)$ is a.e. differentiable on $(0, T)$ and $u'(\cdot) \in L^1([0, T], X)$. A similar argument shows that $p(\cdot)$, $q(\cdot)$, $x(\cdot)$, and $y(\cdot)$ also have this property. Furthermore, with a standard argument as in [17] (Theorem 4.2.4), we have

$$\begin{aligned} x'(t) &= AF(t, u(t)) - A \int_0^t AT(t-s) F(s, u(s)) \, ds, \\ y'(t) &= G(t, u(t)) - A \int_0^t T(t-s) G(s, u(s)) \, ds. \end{aligned} \tag{71}$$

So the following holds, for almost all $t \in [0, T]$:

$$\begin{aligned} &\frac{d}{dt}[u(t) + F(t, u(t))] \\ &\quad = \frac{d}{dt}[T(t)(u_0 + F(0, u(0)) - g(u))] \\ &\quad\quad + x'(t) + y'(t) \\ &\quad = -AT(t)(x_0 + F(0, u(0)) - g(u)) \\ &\quad\quad + AF(t, u(t)) - Ax(t) + G(t, u(t)) - Ay(t) \\ &\quad = -A[T(t)(u_0 + F(0, u(0)) - g(u)) \\ &\quad\quad\quad -F(t, u(t)) + x(t) + y(t)] + G(t, u(t)) \\ &\quad = -Au(t) + G(t, u(t)). \end{aligned} \tag{72}$$

This shows that $u(\cdot)$ is also a strong solution to the nonlocal Cauchy problem (1), and the proof is completed. □

The following result is an immediate corollary of Theorems 8 and 10.

Corollary 11. *Suppose that the hypotheses (H7)–(H9), and (H11) are satisfied, and in addition, there holds the following hypotheses.*

(H12) *The function $g : E \to X$ is continuous, $g(u) \in X_1$, for all $u \in E$, and inequality* (60) *sustains.*

If $u_0 \in X_1$ and inequality (H4) also holds with L_1, L_2; and C_β is replaced by L_4, L_5; $C_1 := \|A^{-1}\|$, respectively, then (1) *has a strong solution on* $[0, T]$.

5. An Example

In the last section, our existence results will be applied to solve the following system:

$$\begin{aligned} &\frac{\partial}{\partial t}\left[u(t, x) + \int_0^1 b(x, s) u(t, s) \, ds\right] \\ &\quad = \frac{\partial^2}{\partial x^2} u(t, x) + h\left(t, x, \int_0^1 u(t, s) \, ds\right), \\ &\quad\quad\quad 0 \le t \le T, \\ &u(t, 0) = u(t, 1) = 0, \quad 0 \le t \le T, \\ &u(0, x) + \int_0^T k_1(t, x) \int_0^1 k_2(r, u(t, r)) \, dr \, dt = u_0(x), \\ &\quad\quad\quad 0 \le x \le 1, \end{aligned} \tag{73}$$

where $0 < T \le 1$ and $u_0 \in X := L^2([0, 1], \mathbb{R})$ equipped with L^2 norm $\|\cdot\|$.

The operator $A : D(A) \subset X \to X$ defined by

$$\begin{aligned} D(A) &= \{f \in X : f, f'' \in X, f(0) = f(1) = 0\}, \\ Af &= -f''. \end{aligned} \tag{74}$$

Then, $-A$ generates a compact, analytic semigroup $T(\cdot)$ of uniformly bounded linear operators. It is well known that $0 \in \rho(A)$, and, thus, the fractional powers of A are well-defined where the eigenvalues of A are $n^2\pi^2$ and the corresponding normalized eigenvectors are $e_n(x) = \sqrt{2}\sin(n\pi x)$, $n = 1, 2, \ldots$. Moreover,

$$\begin{aligned} Az &= \sum_{n=1}^{\infty} n^2\pi^2 \langle z, e_n \rangle e_n, \quad \forall z \in D(A), \\ A^{-1/2} z &= \sum_{n=1}^{\infty} \frac{1}{n} \langle z, e_n \rangle e_n, \quad \forall z \in X, \end{aligned} \tag{75}$$

with $\|A^{-1/2}\| = 1$, and the operator $A^{1/2}$ is given by

$$A^{1/2} z = \sum_{n=1}^{\infty} n \langle z, e_n \rangle e_n, \tag{76}$$

with domain $D(A^{1/2}) := \{f \in X : \sum_{n=1}^{\infty} n \langle f, e_n \rangle e_n \in X\}$.

We need the following assumptions to solve (73) with our results.

(A1) The function $b : [0, 1] \times [0, 1] \to \mathbb{R}$ satisfies the following conditions.

(a) $(x, y) \mapsto (\partial/\partial x) b(x, y)$ is welldefined and measurable with

$$C := \int_0^1 \int_0^1 \left(\frac{\partial}{\partial x} b(x, y)\right)^2 dy \, dx < \infty. \tag{77}$$

(b) $b(0, x) = b(1, x) = 0$, for each $x \in [0, 1]$.

(A2) The function $h : [0,T] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ satisfies the following conditions.

(a) For each $(x, y) \in [0,1] \times \mathbb{R}$, the function $h(\cdot, x, y)$ is measurable.

(b) For each $(t, y) \in [0,T] \times \mathbb{R}$, the function $h(t, \cdot, y)$ is continuous.

(c) There is an $l \in \mathbb{R}^+$ such that

$$\left|h(t_1, x, y_1) - h(t_2, x, y_2)\right| \le l\left(\left|t_1 - t_2\right| + \left|y_1 - y_2\right|\right), \quad \forall x \in [0,1]. \tag{78}$$

(A3) The functions k_1 and k_2 satisfy the following conditions, respectively:

(a) $k_1 \in L^2([0,1] \times [0,T])$,

(b) $k_2 \in L^2([0,T] \times \mathbb{R})$ and there is an l_0 such that

$$\left|k_2(t, y) - k_2(s, z)\right| \le l_0\left(|t - s| + |y - z|\right), \quad \forall t, s \in [0,T],\ y, z \in \mathbb{R}. \tag{79}$$

Let E be the Banach space $C([0,T], X)$ equipped with supnorm, let $F : [0,T] \times X \to X$ be defined by

$$(F(t, \varphi))(x) = \int_0^1 b(x, s)\varphi(s)\,ds \quad (t, \varphi) \in [0,T] \times X,\ x \in [0,1], \tag{80}$$

and let $G : [0,T] \times X \to X$ be defined by

$$(G(t, \phi))(x) = h\left(t, x, \int_0^1 \phi(s)\,ds\right) \quad (t, \phi) \in [0,T] \times X,\ x \in [0,1]. \tag{81}$$

Moreover, if $u : [0,T]\times[0,1] \to \mathbb{R}$, we define $u : [0,T] \to X$ by

$$u(t)(\cdot) = u(t, \cdot). \tag{82}$$

Assumptions (A1) and (A2) imply the following conclusions.

Theorem 12. *The functions F and G have the following inequalities.*

(a) *F satisfies hypothesis (H1) with $\beta = 1/2$ and $L_1 = C$ that is,*

$$\left\|A^{1/2}F(t_1, \varphi_1) - A^{1/2}F(t_2, \varphi_2)\right\| \le C\left\|\phi_1 - \phi_2\right\|. \tag{83}$$

(b) *G satisfies the hypothesis (H2) with $g_k = l(T + k) + \|G(0,0)\|$ and $\gamma = l$; that is,*

$$\left\|G(t_1, \varphi_1) - G(t_2, \varphi_2)\right\| \le l\left(\left|t_1 - t_2\right| + \left\|\varphi_1 - \varphi_2\right\|\right), \quad \sup_{\|\phi\|\le k}\left\|G(t, \phi)\right\| \le l(T + k) + \|G(0,0)\|. \tag{84}$$

Proof. (a) By the definition of F and assumption (A1), we see that $F(\cdot,\cdot) \in D(A)$ and

$$\begin{aligned} &\left\|A^{1/2}F(t_1, \varphi_1) - A^{1/2}F(t_2, \varphi_2)\right\|^2 \\ &\quad = \int_0^1 \int_0^1 \left(\frac{\partial}{\partial x}b(x, s)\left(\varphi_1(y) - \varphi_2(y)\right)\right)^2 dydx \\ &\quad \le \int_0^1 \int_0^1 \left(\frac{\partial}{\partial x}b(x, y)\right)^2 dydx \cdot \left\|\phi_1 - \phi_2\right\|^2 \\ &\quad \le C\left\|\phi_1 - \phi_2\right\|^2, \end{aligned} \tag{85}$$

for all $(t_1, \phi_1), (t_2, \phi_2) \in [0,T] \times X$. Hence, F satisfies hypothesis (H1).

(b) By the part of (c) of assumption (A2) and Hölder's inequality, we have

$$\begin{aligned} &\left\|G(t_1, \varphi_1) - G(t_2, \varphi_2)\right\| \\ &\quad = \left(\int_0^1 \left|h\left(t_1, x, \int_0^1 \varphi_1(s)\,ds\right) - h\left(t_2, x, \int_0^1 \varphi_2(s)\,ds\right)\right|^2 dx\right)^{1/2} \\ &\quad \le \left(\int_0^1 \left(l^2\left|t_1 - t_2\right| + \left|\int_0^1 \varphi_1(s)\,ds - \int_0^1 \varphi_2(s)\,ds\right|\right)^2 dx\right)^{1/2} \\ &\quad \le l\left(\left|t_1 - t_2\right| + \left(\int_0^1 \left|\varphi_1(s) - \varphi_2(s)\right| ds\right)\right) \\ &\quad \le l\left(\left|t_1 - t_2\right| + \left\|\varphi_1 - \varphi_2\right\|\right). \end{aligned} \tag{86}$$

So $G(t, \cdot)$ is a continuous function from X into X, for each $t \in [0,T]$. Moreover, let $k \in \mathbb{N}$ be arbitrary, and it follows that

$$\begin{aligned} &\sup_{\|\phi\|\le k}\left\|G(t, \phi)\right\| \\ &\quad \le \sup_{\|\phi\|\le k}\left(\left\|G(t, \phi) - G(0,0)\right\| + \|G(0,0)\|\right) \\ &\quad \le \sup_{\|\phi\|\le k}\left(l\left(t + \|\phi\|\right) + \|G(0,0)\|\right) \\ &\quad \le l(T + k) + \|G(0,0)\|. \end{aligned} \tag{87}$$

So $G(\cdot,\cdot)$ satisfies (H2). □

Now, we define $g : E \to X$ by

$$(g(u))(x) = \int_0^T \int_0^1 k_1(t, x)\,k_2(r, u(t, r))\,drdt, \quad \forall u \in E,\ 0 \le x \le 1. \tag{88}$$

Theorem 13. *g satisfies the following properties.*

(a) *g is a continuous function from E into X.*

(b) $\|g(u)\| \leq l_0 \|k_1\|_{L^2([0,1]\times[0,T])} \|u\|_E + \|g(0)\|$.

(c) *If $\phi \in E$, then $\lim_{\epsilon\to 0} \sup_{\phi\in B_k} \|g(\phi) - g(\phi^\epsilon)\| = 0$, where*

$$\phi^\epsilon(t) = \begin{cases} \phi(\epsilon), & 0 \leq t \leq \epsilon, \\ \phi(t), & \epsilon < t \leq T. \end{cases} \quad (89)$$

Proof. (a) This follows since

$$\begin{aligned} &\|g(u) - g(v)\| \\ &= \left(\int_0^1 \left(\int_0^T \int_0^1 k_1(t,x) \times [k_2(r, u(t,r)) - k_2(r, v(t,r))] \, dr \, dt \right)^2 dx \right)^{1/2} \\ &\leq \left(\int_0^1 \left(\int_0^T \int_0^1 k_1(t,x) \times (l_0 |u(t,r) - v(t,r)|) \, dr \, dt \right)^2 dx \right)^{1/2} \\ &\leq \left(\int_0^1 \left(\int_0^T k_1(t,x) \times \left(l_0 \sup_{t\in[0,T]} \int_0^1 |u(t,r) - v(t,r)| \, dr \right) \times dt \right)^2 dx \right)^{1/2} \\ &\leq l_0 \|u - v\|_E \left(\int_0^1 \left(\int_0^T k_1(t,x) dt \right)^2 dx \right)^{1/2} \\ &\leq l_0 \|k_1\|_{L^2([0,1]\times[0,T])} \|u - v\|_E. \end{aligned} \quad (90)$$

(b) This is clear from the proof of part (a).

(c) Let $k \in \mathbb{N}$ and $\eta > 0$ be arbitrary. Since $\phi \in E$, then ϕ is uniformly continuous on $[0, T]$, and, hence, there is a $\delta > 0$ such that $\|\phi(t) - \phi(\epsilon)\| < \eta$, for all $t \in [0, \epsilon]$, whenever $0 < \epsilon < \delta$. Thus, $0 < \epsilon < \delta$ implies that $\|\phi - \phi^\epsilon\|_E < \eta$. Since g is continuous from E into X by the part of (a), the assertion follows. □

Theorem 13 show that g satisfies the hypotheses (H6) and (H7) with $L_4 = l_0\|k_1\|_{L^2([0,1]\times[0,T])}$ and $L_5 = \|g(0)\|$ respectively. Consequently, since (73) is transformed into

$$\begin{gathered} \frac{d}{dt}(u(t) + F(t, u(t))) = Au(t) + G(t, u(t)), \\ t \in [0, T], \\ u(0) + g(u) = z_0, \end{gathered} \quad (91)$$

the following result is deduced by Theorem 8.

Theorem 14. *If*

$$\left(l_0 \|k_1\|_{L^2([0,1]\times[0,T])} + l \right) + 2C\left(1 + M_{1/2}\right) < 1, \quad (92)$$

then (73) *has a mild solution.*

Theorem 12 also shows that G satisfies (H9) with $L_0 = l$. If k_1 also satisfies.

(A4) $k_1(x, y)$ is twice differentiable with respect to x, $(\partial^2/\partial^2 x)k_1(x, y) \in L^2([0,1] \times [0,T])$, and

$$\overline{C} := \int_0^1 \int_0^1 \left(\frac{\partial^2}{\partial^2 x} b(x,y) \right)^2 dy \, dx < \infty. \quad (93)$$

then Corollary 11 indicates the following result.

Theorem 15. *Assume that assumptions (A2)–(A4) are satisfied and the function*

$$u_0(x) := u(0, x) \in W^{2,2}([0, T]). \quad (94)$$

If inequalities (92) *and* $\overline{C}(1 + M) + 2l < 1$ *hold, then* (73) *has a strong solution.*

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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