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Research Article

A Study of (λ, μ)-Fuzzy Subgroups

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We deal with topics regarding (λ, μ)-fuzzy subgroups, mainly (λ, μ)-fuzzy cosets and (λ, μ)-fuzzy normal subgroups. We give basic properties of (λ, μ)-fuzzy subgroups and present some results related to (λ, μ)-fuzzy cosets and (λ, μ)-fuzzy normal subgroups.

1. Introduction

Since Zadeh [1] introduced the concept of a fuzzy set in 1965, various algebraic structures have been fuzzified. As a result, the theory of fuzzy group was developed. In 1971, Rosenfeld [2] introduced the notion of a fuzzy subgroup and thus initiated the study of fuzzy groups.

In recent years, some variants and extensions of fuzzy groups emerged. In 1996, Bhakat and Das proposed the concept of an $(\in, \in \vee q)$-fuzzy subgroup in [3] and investigated their fundamental properties. They showed that A is an $(\in, \in \vee q)$-fuzzy subgroup if and only if A_α is a crisp group for any $\alpha \in (0, 0.5]$ provided $A_\alpha \neq \emptyset$. A question arises naturally: can we define a type of fuzzy subgroups such that all of their nonempty α-level sets are crisp subgroups for any α in an interval $(\lambda, \mu]$? In 2003, Yuan et al. [4] answered this question by defining a so-called (λ, μ)-fuzzy subgroups, which is an extension of $(\in, \in \vee q)$-fuzzy subgroup. As in the case of fuzzy group, some counterparts of classic concepts can be found for (λ, μ)-fuzzy subgroups. For instance, (λ, μ)-fuzzy normal subgroups and (λ, μ)-fuzzy quotient groups are defined and their elementary properties are investigated, and an equivalent characterization of (λ, μ)-fuzzy normal subgroups was presented in [5]. However, there is much more research on (λ, μ)-fuzzy subgroups if we consider rich results both in the classic group theory and the fuzzy group theory in the sense of Rosenfeld.

In this paper, we conduct a detailed investigation on (λ, μ)-fuzzy subgroups. The research includes further properties of (λ, μ)-fuzzy subgroups, (λ, μ)-fuzzy normal subgroups, and (λ, μ)-fuzzy left and right cosets.

The rest of this paper is organized as follows. In Section 2, we give properties of (λ, μ)-fuzzy subgroups. In Section 3, we present properties of (λ, μ)-fuzzy normal subgroups. In Section 4, we define (λ, μ)-fuzzy left and right cosets and discuss their properties.

2. Properties of (λ, μ)-Fuzzy Subgroups

In this paper, G stands for a group with identity e, and $0 \leqslant \lambda < \mu \leqslant 1$. By a fuzzy subset of G, we mean a mapping from G to the closed unit interval $[0, 1]$. In this section, we present some basic properties of (λ, μ)-fuzzy subgroups.

Definition 1. Let A be a fuzzy subset of G. A is called a fuzzy subgroup of G if, for all $x, y \in G$,

(i) $A(xy) \geqslant A(x) \wedge A(y)$,

(ii) $A(x^{-1}) \geqslant A(x)$.

Definition 1 was introduced by Rosenfeld [2] in 1971.

Definition 2 (see [4]). Let A be a fuzzy subset of G. A is called a (λ, μ)-fuzzy subgroup of G if, for all $x, y \in G$,

(i) $A(xy) \vee \lambda \geqslant A(x) \wedge A(y) \wedge \mu$,

(ii) $A(x^{-1}) \vee \lambda \geqslant A(x) \wedge \mu$.

Clearly, a $(0, 1)$-fuzzy subgroup is just a fuzzy subgroup, and thus a (λ, μ)-fuzzy subgroup is a generalization of fuzzy subgroup.

Definition 3. For a fuzzy subset A of G and $\alpha \in [0,1]$, let $A_\alpha = \{x \in G \mid A(x) \geqslant \alpha\}$. Then A_α is called a level subset of A.

Proposition 4 (see [2]). *A fuzzy subset A of G is a fuzzy subgroup if and only if $A_\alpha \neq \emptyset$ is a crisp subgroup of G for every $\alpha \in (0,1]$.*

Proposition 5 (see [5]). *If A is a (λ,μ)-fuzzy subgroup of G, then $A(e) \vee \lambda \geqslant A(x) \wedge \mu$ for all $x \in G$.*

Corollary 6. *Let A be a (λ,μ)-fuzzy subgroup of G. Then*

(i) *if $A(x) \geqslant \mu$ for some $x \in G$, then $A(e) \geqslant \mu$.*

(ii) *If $A(x) < \mu$ for all $x \in G$ and $H = \{x \mid \lambda < A(x) < \mu\} \neq \emptyset$, then $A(e) = \max\{A(x) | x \in G\}$.*

(iii) *If $A(e) \leqslant \lambda$, then $A(x) \leqslant \lambda$ for all $x \in G$.*

Proof. (i) If $A(x) \geqslant \mu$ for some $x \in G$, then $A(e) \vee \lambda \geqslant A(x) \wedge \mu = \mu$ by Proposition 5; that is, $A(e) \geqslant \mu$.

(ii) When $x \in H$, by Proposition 5, $A(e) \vee \lambda \geqslant A(x) \wedge \mu = A(x)$; that is, $A(e) \geqslant A(x)$. Thus $A(e) > \lambda$.

When $x \notin H$, $A(x) \leqslant \lambda$ due to $A(x) < \mu$. $A(e) = A(e) \vee \lambda \geqslant A(x) \wedge \mu = A(x)$ by Proposition 5. Therefore, for all $x \in G$, $A(x) \leq A(e)$; that is, $A(e) = \max\{A(x) \mid x \in G\}$.

(iii) If $A(e) \leqslant \lambda$, then $\lambda = A(e) \vee \lambda \geqslant A(x) \wedge \mu$ for all $x \in G$, which implies $A(x) \leqslant \lambda$ due to $\lambda < \mu$. □

Corollary 7. *Let A be a (λ,μ)-fuzzy subgroup of G and $\lambda < A(e) < \mu$. Then $A(x) \leqslant A(e)$ holds for all $x \in G$.*

Proof. If $A(x) \geqslant \mu$ for some $x \in G$, then $A(e) \geqslant \mu$ by Corollary 6(i), which is a contradiction. Hence $A(x) < \mu$ for all $x \in G$. Since $e \in H$, $A(x) \leqslant A(e)$ holds for all $x \in G$ by Corollary 6(ii). □

Corollary 8. *Let A be a (λ,μ)-fuzzy subgroup of G and $\lambda < A(e) < \mu$. Then $A(x) = A(e)$ for all $x \in A_{A(e)}$.*

Proof. For every $x \in A_{A(e)}$, $A(x) \geqslant A(e)$. By Corollary 7, $A(x) \leqslant A(e)$. Hence $A(x) = A(e)$. □

Proposition 9 (see [4]). *Let A be a (λ,μ)-fuzzy subset of G. Then A is a (λ,μ)-fuzzy subgroup of G if and only if $A_\alpha \neq \emptyset$ is a subgroup of G for all $\alpha \in (\lambda,\mu]$.*

Proposition 10. *Let A be a (λ,μ)-fuzzy subgroup of G and $x \in G$. Assume*

(i) *if $A(x) \geqslant \mu$, then $A(x^{-1}) \geqslant \mu$.*

(ii) *If $\lambda < A(x) < \mu$, then $A(x^{-1}) = A(x)$.*

(iii) *If $A(x) \leqslant \lambda$, then $A(x^{-1}) \leqslant \lambda$.*

Proof. Since A is a (λ,μ)-fuzzy subgroup of G, $A_\alpha \neq \emptyset$ is a subgroup of G for all $\alpha \in (\lambda,\mu]$ by Proposition 9.

(i) If $A(x) \geqslant \mu$, then $x \in A_\mu$. Since A_μ is a subgroup of G, $x^{-1} \in A_\mu$; that is, $A(x^{-1}) \geqslant \mu$.

(ii) If $\lambda < A(x) < \mu$, then $A(x) = A(x) \vee \lambda \geqslant A(x^{-1}) \wedge \mu = A(x^{-1})$.

If $A(x^{-1}) \wedge \mu = \mu$, then $A(x^{-1}) \geqslant \mu$; that is, $A(x) \geqslant \mu$, which is contradictory to that $A(x) < \mu$. Hence $A(x) \geqslant A(x^{-1})$.

In addition, since $x \in A_{A(x)}$ and $A_{A(x)}$ is a subgroup of G, we have $x^{-1} \in A_{A(x)}$. Therefore, $A(x^{-1}) \geqslant A(x)$. In summary, $A(x^{-1}) = A(x)$.

(iii) Suppose $A(x^{-1}) > \lambda$. Let $\alpha_0 = \min\{A(x^{-1}), \mu\}$.

Then $\lambda < \alpha_0 \leqslant \mu$, and $A(x^{-1}) \geqslant \alpha_0$. Thus $x^{-1} \in A_{\alpha_0}$. By Proposition 9, A_{α_0} is a subgroup of G, and thus $x \in A_{\alpha_0}$. As a result, $A(x) \geqslant \alpha_0 > \lambda$, which is a contradiction to that $A(x) \leqslant \lambda$. Therefore, $A(x^{-1}) \leqslant \lambda$. □

Proposition 11. *Let A be a (λ,μ)-fuzzy subgroup of G and $x, y \in G$.*

(i) *If $A(x) \geqslant \mu$ and $A(y) \geqslant \mu$, then $A(xy) \geqslant \mu$.*

(ii) *If $\lambda < A(x) < \mu$, $A(x) < A(y)$, then $A(xy) = A(x) = A(yx)$.*

(iii) *If $A(x) \leqslant \lambda$, $A(y) > \lambda$, then $A(xy) \leqslant \lambda$ and $A(yx) \leqslant \lambda$.*

Proof. Since A is a (λ,μ)-fuzzy subgroup of G, $A_\alpha \neq \emptyset$ is a subgroup of G for all $\alpha \in (\lambda,\mu]$ by Proposition 9.

(i) From $A(x) \geqslant \mu$ and $A(y) \geqslant \mu$, it follows that $x, y \in A_\mu$. Since A_μ is a subgroup of G, we have $xy \in A_\mu$; that is, $A(xy) \geqslant \mu$.

(ii) Let $A(x) = \alpha$ and $A(y) = \beta$, $A(xy) = \gamma$. Then $\lambda < \alpha < \mu$ and $\beta > \alpha$.

Now we have the following implications successively:

$$\begin{aligned} \lambda < A(x) < A(y) &\Longrightarrow x, y \in A_\alpha \\ &\Longrightarrow xy \in A_\alpha \quad (\text{by Proposition 9}) \\ &\Longrightarrow A(xy) \geqslant \alpha \Longrightarrow \gamma \geqslant \alpha. \end{aligned} \tag{1}$$

If $\gamma > \alpha$, let $\alpha_0 = \min\{\gamma, \beta, \mu\}$. Then $\lambda < \alpha_0 \leqslant \mu$ and $xy, y \in A_{\alpha_0}$. By Proposition 9, A_{α_0} is a subgroup of G, thus $x = xyy^{-1} \in A_{\alpha_0}$. Hence $A(x) \geqslant \alpha_0 \geqslant \gamma > \alpha$, which is a contradiction. Consequently, $\gamma = \alpha$; that is, $A(xy) = A(x)$. Similarly, $A(yx) = A(x)$.

(iii) Suppose $A(xy) > \lambda$. Let $\alpha_0 = \min\{A(xy), A(y), \mu\}$.

Then $\lambda < \alpha_0 \leqslant \mu$ and $xy, y \in A_{\alpha_0}$. By Proposition 9, A_{α_0} is a subgroup of G, thus $x = xyy^{-1} \in A_{\alpha_0}$. It follows that $A(x) \geqslant \alpha_0 > \lambda$, which is a contradiction. Hence $A(xy) \leqslant \lambda$. Similarly, $A(yx) \leqslant \lambda$. □

Proposition 12. *Let $G = (a)$ be a cyclic group with generator a. If A is a (λ,μ)-fuzzy subgroup of G and $\lambda < A(a) < \mu$, then $A(x) \geqslant A(a)$ $\forall x \in G$.*

Proof. For any $x \in G$, there must be a positive integer m such that $x = a^m$. By the definition of (λ,μ)-fuzzy subgroup,

$$A(a^2) \vee \lambda \geqslant A(a) \wedge \mu = A(a), \tag{2}$$

which implies $A(a^2) \geqslant A(a)$. Similarly,

$$A\left(a^3\right) \vee \lambda \geqslant A\left(a^2\right) \wedge A(a) \wedge \mu \geqslant A(a) \wedge \mu = A(a), \quad (3)$$

and thus $A(a^3) \geqslant A(a)$. Continuing in this way, we have $A(a^m) \geqslant A(a)$; that is, $A(x) \geqslant A(a)$. □

Proposition 13. *If $G = (a)$ is a cyclic group and $A(a) \geqslant \mu$, then $G = A_\mu$.*

Proof. For any $x \in G$, there must be a positive integer m such that $x = a^m$. By the definition of (λ, μ)-fuzzy subgroup,

$$A\left(a^2\right) \vee \lambda \geqslant A(a) \wedge \mu = \mu, \quad (4)$$

which implies $A(a^2) \geqslant \mu$. Similarly,

$$A\left(a^3\right) \vee \lambda \geqslant A\left(a^2\right) \wedge A(a) \wedge \mu = \mu, \quad (5)$$

which implies $A(a^3) \geqslant \mu$. Continuing in this way, we have $A(a^m) \geqslant \mu$. Hence $A(x) \geqslant \mu$; that is, $x \in A_\mu$. Therefore $G = A_\mu$. □

Corollary 14. *Let G be a cyclic group with generators a and b, A a (λ, μ)-fuzzy subgroup of G. If $\lambda < A(a) < \mu$, then $A(a) = A(b)$.*

Proof. By Proposition 12, $A(b) \geqslant A(a)$. If $A(b) \geqslant \mu$, then $G = A_\mu$ by Proposition 13. Particularly, $A(a) \geqslant \mu$, which is a contradiction to that $A(a) < \mu$. Hence $A(b) < \mu$; thus $\lambda < A(b) < \mu$. By Proposition 12, $A(a) \geqslant A(b)$. Consequently, $A(a) = A(b)$. □

Corollary 15. *Let G be a cyclic group of a prime order with generator a, A a (λ, μ)-fuzzy subgroup of G. If $\lambda < A(a) < \mu$, then $A(x) = A(a)$ for $x \neq e$.*

Proof. When the order of G is a prime, every $x \neq e$ in G is a generator of G. By Corollary 14, $A(x) = A(a)$ for $x \neq e$. □

3. Properties of (λ, μ)-Fuzzy Normal Subgroups

The notion of fuzzy normal subgroup was first proposed and investigated by Wu [6] in 1981.

Definition 16 (see [6]). Let A be a fuzzy subgroup of G. A is called a fuzzy normal subgroup of G if for all $x, y \in G$,

$$A\left(xyx^{-1}\right) \geqslant A(y). \quad (6)$$

Proposition 17 (see [6]). *Let A be a fuzzy subgroup of G. A is a fuzzy normal subgroup if and only if for all $x, y \in G$,*

$$A(xy) \geqslant A(yx). \quad (7)$$

Proposition 18 (see [6]). *Let A be a fuzzy subset of G. Then A is a fuzzy normal subgroup of G if and only if $A_\alpha \neq \emptyset$ is a normal subgroup of G for all $\alpha \in (0, 1]$.*

In 2005, the following notion of (λ, μ)-fuzzy normal subgroup was put forward by Yao [5].

Definition 19. Let A be a (λ, μ)-fuzzy subgroup of G. A is called a (λ, μ)-fuzzy normal subgroup of G if, for all $x, y \in G$,

$$A\left(xyx^{-1}\right) \vee \lambda \geqslant A(y) \wedge \mu. \quad (8)$$

Clearly, a $(0, 1)$-fuzzy normal subgroup is just a fuzzy normal subgroup, and thus a (λ, μ)-fuzzy normal subgroup is a generalization of fuzzy normal subgroup.

Proposition 20 (see [5]). *Let A be a (λ, μ)-fuzzy subgroup of G. A is a (λ, μ)-fuzzy normal subgroup if and only if, for all $x, y \in G$,*

$$A(xy) \vee \lambda \geqslant A(yx) \wedge \mu. \quad (9)$$

Proposition 21 (see [5]). *Let A be a (λ, μ)-fuzzy subset of G. Then A is a (λ, μ)-fuzzy normal subgroup of G if and only if $A_\alpha \neq \emptyset$ is a normal subgroup of G for all $\alpha \in (\lambda, \mu]$.*

Proposition 22. *Let A be a (λ, μ)-fuzzy normal subgroup of G and $x \in G$.*

(i) *If $A(x) \geqslant \mu$, then $A(yxy^{-1}) \geqslant \mu$ for all $y \in G$.*

(ii) *If $\lambda < A(x) < \mu$, then $A(yxy^{-1}) = A(x)$ for all $y \in G$.*

(iii) *If $y \in G$ and $\lambda < A(xy) < \mu$, then $A(xy) = A(yx)$.*

(iv) *If $y \in G$ and $A(xy) \geqslant \mu$, then $A(yx) \geqslant \mu$.*

(v) *If $y \in G$ and $A(xy) \leqslant \lambda$, then $A(yx) \leqslant \lambda$.*

Proof. (i) If $A(x) \geqslant \mu$, then $x \in A_\mu$. By Proposition 21, A_μ is a normal subgroup of G and thus $yxy^{-1} \in A_\mu$. Hence $A(yxy^{-1}) \geqslant \mu$.

(ii) Let $A(x) = \alpha$. Then $\lambda < \alpha < \mu$. By Proposition 21, A_α is a normal subgroup of G. Hence $yxy^{-1} \in A_\alpha$; that is, $A(yxy^{-1}) \geqslant \alpha = A(x)$.

Suppose $A(yxy^{-1}) > \alpha$. Set $\alpha_0 = \min\{A(yxy^{-1}), \mu\}$. Then $\lambda < \alpha_0 \leqslant \mu$. By Proposition 21, A_{α_0} is a normal subgroup of G, and thus $yxy^{-1} \in A_{\alpha_0}$. Therefore, $x = y^{-1}(yxy^{-1})y \in A_{\alpha_0}$; that is, $A(x) \geqslant \alpha_0 > \alpha$, which is a contradiction to that $A(x) = \alpha$. Consequently, $A(yxy^{-1}) = A(x)$.

(iii) If $\lambda < A(xy) < \mu$, then $A(yx) = A(x^{-1}(xy)x) = A(xy)$ by (ii); that is, $A(xy) = A(yx)$.

(iv) If $A(xy) \geqslant \mu$, then $xy \in A_\mu$. Since A_μ is a normal subgroup of G by Proposition 21, $yx = x^{-1}(xy)x \in A_\mu$; that is, $A(yx) \geqslant \mu$.

(v) Suppose $A(yx) > \lambda$ on the contrary.

If $A(yx) \geqslant \mu$, then, by (i), $A(xy) \geqslant \mu$, which is contradictory to that $A(xy) < \mu$. If $A(yx) < \mu$, then, by (iii), $A(xy) = A(yx) > \lambda$, which is contradictory to that $A(xy) \leqslant \lambda$. Hence $A(yx) \leqslant \lambda$. □

Proposition 23. *Let A be a (λ, μ)-fuzzy subgroup of G. Then A is a (λ, μ)-fuzzy normal subgroup of G if and only if $A([x, y]) \vee \lambda \geqslant A(x) \wedge \mu$ for all $x, y \in G$, where $[x, y] = x^{-1}y^{-1}xy$ is a commutator in G.*

Proof. For any $x, y \in G$,

$$\begin{aligned}
A([x,y]) \vee \lambda &= A\left(x^{-1}y^{-1}xy\right) \vee \lambda \\
&= A\left(x^{-1}\left(y^{-1}xy\right)\right) \vee \lambda \vee \lambda \\
&\geqslant \left(A\left(x^{-1}\right) \wedge A\left(y^{-1}xy\right) \wedge \mu\right) \vee \lambda \\
&= \left(A\left(x^{-1}\right) \vee \lambda\right) \wedge \left(A\left(y^{-1}xy\right) \vee \lambda\right) \wedge \mu.
\end{aligned} \tag{10}$$

Since A is a (λ,μ)-fuzzy normal subgroup of G, $A(x^{-1}) \vee \lambda \geqslant A(x) \wedge \mu$ and $A(y^{-1}xy) \vee \lambda \geqslant A(x) \wedge \mu$. Therefore,

$$A([x,y]) \vee \lambda \geqslant A(x) \wedge \mu. \tag{11}$$

Conversely, if $A([x,y]) \vee \lambda \geqslant A(x) \wedge \mu$, then

$$\begin{aligned}
A\left(y^{-1}xy\right) \vee \lambda &= A\left(xx^{-1}y^{-1}xy\right) \vee \lambda \\
&= A(x[x,y]) \vee \lambda \vee \lambda \\
&\geqslant (A(x) \wedge A([x,y]) \wedge \mu) \vee \lambda \\
&= (A(x) \vee \lambda) \wedge (A([x,y]) \vee \lambda) \wedge \mu \\
&\geqslant (A(x) \vee \lambda) \wedge (A(x) \wedge \mu) \wedge \mu \\
&\geqslant A(x) \wedge A(x) \wedge \mu \\
&= A(x) \wedge \mu.
\end{aligned} \tag{12}$$

Hence A is a (λ,μ)-fuzzy normal subgroup of G. □

Proposition 24. *If G is an abelian group and A is a (λ,μ)-fuzzy subgroup of G, then A is a (λ,μ)-fuzzy normal subgroup of G.*

Proof. Since G is an abelian group, we have $[xy] = e$; hence $A([xy]) \vee \lambda = A(e) \vee \lambda \geqslant A(x) \wedge \mu$ for all $x, y \in G$ by Proposition 5. By Proposition 23, A is a (λ,μ)-fuzzy normal subgroup of G. □

Since a cyclic group is an abelian group, the following result is immediate by Proposition 24.

Corollary 25. *If G is a cyclic group and A is a (λ,μ)-fuzzy subgroup of G, then A is a (λ,μ)-fuzzy normal subgroup of G.*

4. Properties of Left Cosets and Right Cosets of A

Definition 26 (see [5]). Let A be a (λ,μ)-fuzzy subgroup of G and $a \in G$. Define fuzzy subsets $a \circ A$ and $A \circ a$ of G respectively by

$$\begin{aligned}
(a \circ A)(x) &= \left(A\left(a^{-1}x\right) \vee \lambda\right) \wedge \mu, \quad \forall x \in G, \\
(A \circ a)(x) &= \left(A\left(xa^{-1}\right) \vee \lambda\right) \wedge \mu, \quad \forall x \in G.
\end{aligned} \tag{13}$$

$a \circ A$ and $A \circ a$ will be called a left coset and a right coset of A, respectively.

Clearly, we have $e \circ A = A \circ e$, $\lambda \leqslant (A \circ a)(x) \leqslant \mu$ and $\lambda \leqslant (a \circ A)(x) \leqslant \mu$ which are valid for all $x \in G$.

The following conclusions can be found in [5].

Proposition 27 (see [5]). *Let A and B be (λ,μ)-fuzzy subgroups of G and $a, b \in G$. Then*

(i) $a \circ (b \circ A) = (ab) \circ A$.

(ii) $(A \circ a) \circ b = A \circ (ab)$.

(iii) $a \circ A = b \circ B \Leftrightarrow e \circ A = (a^{-1}b) \circ B \Leftrightarrow (b^{-1}a) \circ A = e \circ B$.

(iv) $A \circ a = B \circ b \Leftrightarrow A \circ e = B \circ (ba^{-1}) \Leftrightarrow A \circ (ab^{-1}) = B \circ e$.

In particular, we have the following corollary when $A = B$.

Corollary 28. *Let A be a (λ,μ)-fuzzy subgroup of G and $a, b \in G$. Then*

(i) $a \circ A = b \circ A \Leftrightarrow e \circ A = (a^{-1}b) \circ A \Leftrightarrow (b^{-1}a) \circ A = e \circ A$;

(ii) $A \circ a = A \circ b \Leftrightarrow A \circ e = A \circ (ba^{-1}) \Leftrightarrow A \circ (ab^{-1}) = A \circ e$.

Firstly, we present some basic properties of $e \circ A$.

Proposition 29. *Let A be a (λ,μ)-fuzzy subgroup of G and $x \in G$.*

(i) *If $\lambda < A(x) < \mu$, then $(e \circ A)(x) = A(x)$.*

(ii) *If $A(x) \geqslant \mu$, then $(e \circ A)(x) = \mu$.*

(iii) *If $A(x) \leqslant \lambda$, then $(e \circ A)(x) = \lambda$.*

Proof. (i) If $\lambda < A(x) < \mu$, then $(e \circ A)(x) = (A(x) \vee \lambda) \wedge \mu = A(x)$. (ii) and (iii) can be similarly proved. □

Proposition 30. *Let A be a (λ,μ)-fuzzy subgroup of G. Then $e \circ A$ is a (λ,μ)-fuzzy subgroup of G and a fuzzy subgroup of G in the sense of Rosenfeld as well.*

Proof. Firstly, we prove that $A_\alpha = (e \circ A)_\alpha$ for all $\alpha \in (\lambda, \mu]$.

If $x \in A_\alpha$, then $A(x) \geqslant \alpha > \lambda$. Hence,

$$(e \circ A)(x) = (A(x) \vee \lambda) \wedge \mu = A(x) \wedge \mu \geqslant \alpha, \tag{14}$$

which implies $x \in (e \circ A)_\alpha$. So $A_\alpha \subseteq (e \circ A)_\alpha$.

Conversely, if $x \in (e \circ A)_\alpha$, then $(e \circ A)(x) = (A(x) \vee \lambda) \wedge \mu \geqslant \alpha$, whence $A(x) \geqslant \alpha$; that is, $x \in A_\alpha$. Thus, $A_\alpha \supseteq (e \circ A)_\alpha$. In summary, $A_\alpha = (e \circ A)_\alpha$.

Since A is a (λ,μ)-fuzzy subgroup of G, $A_\alpha \neq \emptyset$ is the subgroup of G by Proposition 9. It follows from $A_\alpha = (e \circ A)_\alpha$ that $(e \circ A)_\alpha$ is a subgroup of G. Hence $e \circ A$ is a (λ,μ)-fuzzy subgroup of G by Proposition 9.

Since $\lambda \leqslant (e \circ A)(x) \leqslant \mu$, $\forall x \in G$, $(e \circ A)_\alpha = G$ for $\alpha \in (0, \lambda]$, and clearly it is a subgroup of G. Considering that $(e \circ A)_\alpha = \emptyset$ for $\alpha \in (\mu, 1]$, $(e \circ A)_\alpha \neq \emptyset$ is subgroup of G for all $\alpha \in (0, 1]$. As a result, $e \circ A$ is a fuzzy subgroup of G in the sense of Rosenfeld by Proposition 4. □

Proposition 31. *Let A be a (λ,μ)-fuzzy normal subgroup of G. Then $e \circ A$ is a (λ,μ)-fuzzy normal subgroup of G and a fuzzy normal subgroup of G as well.*

Proof. By Proposition 30, $e \circ A$ is a (λ, μ)-fuzzy subgroup of G and a fuzzy subgroup of G. For any $x, y \in G$,

$$\begin{aligned}(e \circ A)(xyx^{-1}) \vee \lambda &= (e \circ A)(xyx^{-1}) \\ &= (A(xyx^{-1}) \vee \lambda) \wedge \mu \\ &= ((A(xyx^{-1}) \vee \lambda) \vee \lambda) \wedge \mu \\ &\geqslant ((A(y) \wedge \mu) \vee \lambda) \wedge \mu \\ &= (A(y) \vee \lambda) \wedge \mu \\ &= (e \circ A)(y) = (e \circ A)(y) \wedge \mu.\end{aligned} \tag{15}$$

Therefore, $e \circ A$ is a (λ, μ)-fuzzy normal subgroup of G by Definition 19 and a fuzzy normal subgroup of G by Definition 16. □

Furthermore, we have the following results.

Proposition 32. *If A is a (λ, μ)-fuzzy subgroup of G and $A(e) \geqslant \mu$, then $a \in A_\mu$ if and only if $a \circ A = e \circ A$.*

Proof. Firstly, we assume $a \in A_\mu$. Since A_μ is a subgroup of G by Proposition 9, we have $a^{-1} \in A_\mu$; that is, $A(a^{-1}) \geqslant \mu$ by Proposition 10. Let $x \in G$. Consider the following three cases.

Case 1. If $A(x) \geqslant \mu$, then $x \in A_\mu$, which implies $a^{-1}x \in A_\mu$; that is, $A(a^{-1}x) \geqslant \mu$ by Proposition 11(i). Therefore,

$$\begin{aligned}(a \circ A)(x) &= (A(a^{-1}x) \vee \lambda) \wedge \mu \\ &= \mu = (A(x) \vee \lambda) \wedge \mu = (e \circ A)(x).\end{aligned} \tag{16}$$

Case 2. If $\lambda < A(x) < \mu$, then $A(a^{-1}x) = A(x)$ by Proposition 11(ii). Hence

$$\begin{aligned}(a \circ A)(x) &= (A(a^{-1}x) \vee \lambda) \wedge \mu \\ &= (A(x) \vee \lambda) \wedge \mu = (e \circ A)(x).\end{aligned} \tag{17}$$

Case 3. If $A(x) \leqslant \lambda$, then $A(a^{-1}x) \leqslant \lambda$ by Proposition 11(iii). Hence

$$\begin{aligned}(a \circ A)(x) &= (A(a^{-1}x) \vee \lambda) \wedge \mu \\ &= \lambda = (A(x) \vee \lambda) \wedge \mu = (e \circ A)(x).\end{aligned} \tag{18}$$

In summary, $a \circ A = e \circ A$.

Conversely, assume $a \circ A = e \circ A$. Then we have the following implications successively:

$$\begin{aligned}&(a \circ A)(a) = (e \circ A)(a) \\ &\Longrightarrow (A(a^{-1}a) \vee \lambda) \wedge \mu = (A(a) \vee \lambda) \wedge \mu \\ &\Longrightarrow (A(e) \vee \lambda) \wedge \mu = (A(a) \vee \lambda) \wedge \mu \\ &\Longrightarrow \mu = (A(a) \vee \lambda) \wedge \mu \quad (\text{since } A(e) \geqslant \mu) \\ &\Longrightarrow A(a) \geqslant \mu. \\ &\Longrightarrow a \in A_\mu.\end{aligned} \tag{19}$$

□

Similarly, if A is a (λ, μ)-fuzzy subgroup of G and $A(e) \geqslant \mu$, then $a \in A_\mu$ if and only if $A \circ a = A \circ e$.

Corollary 33. *If A is a (λ, μ)-fuzzy subgroup of G and $a \in A_\mu$, then*

$$a \circ A = A \circ a = e \circ A. \tag{20}$$

Proof. Since $a \in A_\mu$, we have $A(e) \geqslant \mu$ by Corollary 6, and hence $a \circ A = e \circ A$ and $A \circ a = A \circ e$. It follows from $e \circ A = A \circ e$ that $a \circ A = A \circ a = e \circ A$. □

Corollary 34. *Let A be a (λ, μ)-fuzzy subgroup of G and $a, b \in G$. Then $aA_\mu = bA_\mu$ if and only if $a \circ A = b \circ A$ provided $A_\mu \neq \emptyset$.*

Proof. The desired result follows from the following equivalences:

$$\begin{aligned}&aA_\mu = bA_\mu \\ &\Longleftrightarrow b^{-1}a \in A_\mu \\ &\Longleftrightarrow (b^{-1}a) \circ A = e \circ A \quad (\text{by Proposition 32}) \\ &\Longleftrightarrow a \circ A = b \circ A \quad (\text{by Corollary 28}).\end{aligned} \tag{21}$$

□

Similarly, under the conditions stated in Corollary 34, we have the equivalence $A_\mu a = A_\mu b$ if and only if $A \circ a = A \circ b$.

Proposition 35. *If A is a (λ, μ)-fuzzy subgroup of G and $\lambda < A(e) < \mu$, then $A(a) = A(e)$ if and only if $a \circ A = e \circ A$.*

Proof. Firstly, we assume $A(a) = A(e)$; that is, $a \in A_{A(e)}$. By Proposition 9, $A_{A(e)}$ is a subgroup of G. Hence we have $a^{-1} \in A_{A(e)}$; that is, $A(a^{-1}) \geqslant A(e)$, so $A(a^{-1}) = A(e)$ by Corollary 6. Let $x \in G$. We have $A(x) \leqslant A(e)$ by Corollary 6. Consider the following three cases.

Case 1 ($A(x) \leqslant \lambda$). In this case, $A(a^{-1}x) \leqslant \lambda$ by Proposition 11(iii). So

$$\begin{aligned}(a \circ A)(x) &= (A(a^{-1}x) \vee \lambda) \wedge \mu = \lambda \\ &= (A(x) \vee \lambda) \wedge \mu = (e \circ A)(x).\end{aligned} \tag{22}$$

Case 2 ($\lambda < A(x) < A(e) = A(a^{-1})$). In this case, $A(a^{-1}x) = A(x)$ by Proposition 11(ii). So

$$\begin{aligned}(a \circ A)(x) &= (A(a^{-1}x) \vee \lambda) \wedge \mu \\ &= (A(x) \vee \lambda) \wedge \mu = (e \circ A)(x).\end{aligned} \tag{23}$$

Case 3 ($A(x) = A(e)$). Since $A_{A(e)}$ is a subgroup of G by Proposition 9, we have $x, a^{-1} \in A_{A(e)}$. Hence $a^{-1}x \in A_{A(e)}$; that is, $A(a^{-1}x) \geqslant A(e)$. Thus $A(a^{-1}x) = A(e)$ by Corollary 6. As a result, $A(a^{-1}x) = A(x) = A(e)$. Therefore,

$$\begin{aligned}(a \circ A)(x) &= (A(a^{-1}x) \vee \lambda) \wedge \mu \\ &= (A(x) \vee \lambda) \wedge \mu = (e \circ A)(x).\end{aligned} \tag{24}$$

In summary, $a \circ A = e \circ A$.

Conversely, assume $a \circ A = e \circ A$. Then we have the following implications successively:

$$\begin{aligned}(a \circ A)(a) &= (e \circ A)(a) \\ &\Longrightarrow \left(A\left(a^{-1}a\right) \vee \lambda\right) \wedge \mu = (A(a) \vee \lambda) \wedge \mu \\ &\Longrightarrow A(e) = A(a) \vee \lambda \quad (\text{since } \lambda < A(e) < \mu) \\ &\Longrightarrow A(e) = A(a).\end{aligned} \tag{25}$$

□

Similarly, if A is a (λ, μ)-fuzzy subgroup of G and $\lambda < A(e) < \mu$, then $A(a) = A(e)$ if and only if $A \circ a = A \circ e$.

Corollary 36. *Let A be a (λ, μ)-fuzzy subgroup of G and $a, b \in G$. Then $aA_{A(e)} = bA_{A(e)}$ if and only if $a \circ A = b \circ A$ provided $\lambda < A(e) < \mu$.*

Proof. The desired result follows from the following equivalencies:

$$\begin{aligned}aA_{A(e)} &= bA_{A(e)} \\ &\Longleftrightarrow b^{-1}a \in A_{A(e)} \\ &\Longleftrightarrow \left(b^{-1}a\right) \circ A = e \circ A \quad (\text{by Proposition 35}) \\ &\Longleftrightarrow a \circ A = b \circ A \quad (\text{by Corollary 28}).\end{aligned} \tag{26}$$

□

Similarly, under the conditions stated in Corollary 36, we have the equivalence $A_{A(e)}a = A_{A(e)}b$ if and only if $A \circ a = A \circ b$.

Proposition 37. *Let A be a (λ, μ)-fuzzy subgroup of G. Then $a \circ A$ is a (λ, μ)-fuzzy subgroup of G if and only if $a \circ A = e \circ A$.*

Proof. Suppose that $a \circ A$ is a (λ, μ)-fuzzy subgroup of G.

If $A(e) \geqslant \mu$, then $(a \circ A)(a) = (A(e) \vee \lambda) \wedge \mu = \mu$, which implies $a \in (a \circ A)_\mu$, and thus $(a \circ A)_\mu \neq \emptyset$. Since $(a \circ A)_\mu$ is a subgroup of G by Proposition 9, we have $e \in (a \circ A)_\mu$. Hence $(a \circ A)(e) = (A(a^{-1}) \vee \lambda) \wedge \mu = \mu$, which implies $A(a^{-1}) \geqslant \mu$; that is, $a^{-1} \in A_\mu$. Hence we have $a \in A_\mu$ by Proposition 10(i). Therefore, $a \circ A = e \circ A$ by Proposition 32.

If $\lambda < A(e) < \mu$, then $A(e) \geqslant A(a^{-1})$ by Corollary 6 and $(a \circ A)(a) = (A(e) \vee \lambda) \wedge \mu = A(e)$, which implies $a \in (a \circ A)_{A(e)}$. Since $a \circ A$ is a (λ, μ)-fuzzy subgroup of G, $(a \circ A)_{A(e)}$ is a subgroup of G by Proposition 9, which implies $e \in (a \circ A)_{A(e)}$. Hence

$$A(e) \leq (a \circ A)(e) = \left(A\left(a^{-1}\right) \vee \lambda\right) \wedge \mu = A\left(a^{-1}\right) \vee \lambda, \tag{27}$$

and thus $A(a^{-1}) \geqslant A(e)$; that is, $a^{-1} \in A_{A(e)}$, which implies $a \in A_{A(e)}$ since $A_{A(e)}$ is a subgroup of G. Therefore, $A(a) \geqslant A(e)$, and hence $A(a) = A(e)$ by Corollary 6. It follows $a \circ A = e \circ A$ from Proposition 35.

If $A(e) \leqslant \lambda$, then $A(x) \leqslant \lambda$ for all $x \in G$ by Corollary 6. Consider

$$\begin{aligned}(a \circ A)(x) &= \left(A\left(a^{-1}x\right) \vee \lambda\right) \wedge \mu = \lambda \\ &= (A(x) \vee \lambda) \wedge \mu = (e \circ A)(x).\end{aligned} \tag{28}$$

Hence $a \circ A = e \circ A$.

Conversely, if $a \circ A = e \circ A$, then $a \circ A$ is a (λ, μ)-fuzzy subgroup of G by Proposition 30. □

Proposition 38. *Let A be a (λ, μ)-fuzzy subgroup of G, $a, b \in G$, $\alpha \in (\lambda, \mu]$ and $A_\alpha \neq \emptyset$. If $a \circ A = b \circ A$, then $aA_\alpha = bA_\alpha$.*

Proof. Since $A_\alpha \neq \emptyset$, A_α is a subgroup of G by Proposition 9, and thus $A(e) \geqslant \alpha > \lambda$. Hence, from $a \circ A = b \circ A$, we know that $(a \circ A)(b) = (b \circ A)(b)$; that is,

$$\left(A\left(a^{-1}b\right) \vee \lambda\right) \wedge \mu = (A(e) \vee \lambda) \wedge \mu = A(e) \wedge \mu. \tag{29}$$

If $A(e) \geqslant \mu$, then $A(a^{-1}b) \geqslant \mu$; that is, $a^{-1}b \in A_\mu \subseteq A_\alpha$. Hence $aA_\alpha = bA_\alpha$.

If $A(e) < \mu$, then $A(a^{-1}b) = A(e)$; that is, $a^{-1}b \in A_{A(e)}$. By $A_{A(e)} \subseteq A_\alpha$, thus $a^{-1}b \in A_\alpha$. Hence $aA_\alpha = bA_\alpha$. □

Similarly, we have the following conclusion.

Let A be a (λ, μ)-fuzzy subgroup of G, $a, b \in G$, $\alpha \in (\lambda, \mu]$ and $A_\alpha \neq \emptyset$. If $A \circ a = A \circ b$, then $A_\alpha a = A_\alpha b$.

Proposition 39. *Let A be a (λ, μ)-fuzzy subgroup of G and $a, b \in G$. If $a \circ A = b \circ A$ and $\lambda < A(a) < \mu$, then $A(a) = A(b)$.*

Proof. Since $a \circ A = b \circ A$, $(a^{-1}b) \circ A = e \circ A$ by Corollary 28.

Case 1. If $A(e) \geqslant \mu$, then by Proposition 32, $a^{-1}b \in A_\mu$; that is, $A(a^{-1}b) \geqslant \mu$. By Proposition 11 and $\lambda < A(a) < \mu$, $A(a) = A(a(a^{-1}b)) = A(b)$.

Case 2. If $A(e) < \mu$ and $\lambda < A(a) < \mu$, then $A(a) \leqslant A(e) < \mu$ and then $A(a^{-1}b) = A(e)$ by Proposition 35.

If $A(a) < A(e) = A(a^{-1}b)$, then by Proposition 11, $A(a) = A(a(a^{-1}b)) = A(b)$.

If $A(a) = A(e) = A(a^{-1}b)$, then $a, a^{-1}b \in A_{A(e)}$. Since A is a (λ, μ)-fuzzy subgroup of G, $A_{A(e)}$ is a subgroup of G by Proposition 9. Hence $b = a(a^{-1}b) \in A_{A(e)}$; that is, $A(b) \geqslant A(e)$; hence $A(b) = A(e) = A(a)$ by Corollary 7. □

It is noteworthy that the converse of Proposition 39 is not true as shown by the following example.

Example 40. Let G be a cyclic group of order 3 with generator a. Define A by: $A(a) = A(a^2) = 0.5$, $A(e) = 0.7$. Then A is a $(0.3, 0.6)$-fuzzy subgroup of G. In this example, however, $(a \circ A)(a) = 0.6$, $(a^2 \circ A)(a) = 0.5$. Hence $a \circ A \neq a^2 \circ A$.

5. Concluding Remarks

In this paper, we present a further investigation into properties of (λ, μ)-fuzzy subgroups, (λ, μ)-fuzzy normal subgroups, and (λ, μ)-fuzzy left and right cosets. It should be noted that we concentrate on the case $A(e) > \lambda$ in studying fuzzy left and right cosets. The reason is that $(a \circ A)(x) = \lambda$ is valid for all $x \in G$ when $A(e) \leqslant \lambda$, which is of little mathematical significance. Certainly, some other topics of (λ, μ)-fuzzy subgroups such as the characterization of (λ, μ)-fuzzy subgroups and operations of (λ, μ)-fuzzy subgroups are still open for the research in the future.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this article.

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