Research Article

Boundary Controllability of a Pseudoparabolic Equation

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We deal with the controllability problem for the pseudoparabolic equation by means of boundary controls. Due to the unusual spectrum of this kind of equations, we prove that the null controllability property is false. Furthermore, by the explicit solution, we show that the approximate controllability holds.

1. Introduction

This paper is devoted to the study of controllability properties of the 1D pseudoparabolic equation:

$$
\begin{aligned}
y_t - y_{xx} - k y_{xxt} &= 0, \quad x \in (0,1),\ t \in (0,T), \\
y(0,t) &= 0, \quad y(1,t) = u(t), \quad t \in (0,T), \\
y(x,0) &= y_0(x), \quad x \in (0,1),
\end{aligned}
\tag{1}
$$

where $y = y(x,t)$ is the state, $u = u(t)$ is a control function acting on the boundary $\{1\} \times (0,T)$, and $k > 0$ is a constant.

The pseudoparabolic equations are a kind of Sobolev-Galpern type equations. They have occurred in numerous physical applications among which include problems involving seepage of fluids through fissured rocks [1], unsteady flows of second-order fluids [2, 3], and the theory of thermodynamics involving two temperatures [4]. They can also be used as a regularization of ill-posed transport problems, especially as a quasicontinuous approximation to discrete models for population dynamics [5]. Furthermore, pseudoparabolic equations are closely related to the well-know BBM equations [6] which are advocated as a refinement of KdV equations.

In the last two decades, important progress has been made in the controllability analysis of parabolic equations. We refer to the works [7–13] and the references therein. It is well known that the null and approximate controllability hold for the classical parabolic equations. However, for some

special models, there arise some new results. For example, in [14], the authors considered the heat equation with memory:

$$
y_t - \Delta y + \int_0^t \Delta y(x,s)\, ds = 0.
\tag{2}
$$

By establishing that the observability inequality for the heat equation with memory is not true, they proved that there exists a set of initial conditions such that the null controllability property fails by means of boundary controls. Recently, Doubova and Fernández-Cara [15] studied the approximate distributed and boundary controllability of viscoelastic fluids of the Jeffreys kind, which can be equivalently rewritten as a parabolic equation with memory

$$
y_t - \nu \Delta y - b \int_0^t e^{-a(t-s)} \Delta y(s)\, ds + \nabla \pi = e^{-at} \nabla \cdot \tau_0.
\tag{3}
$$

The main tool for proving the approximate controllability result is a unique continuation property for its adjoint system. The authors pointed out that this parabolic equation with memory transforms from a damped wave equation (see [15]):

$$
z_{tt} - a z_t - \nu \Delta z_t - b \Delta z + \nabla \left(e^{at} \pi \right) = \nabla \cdot \tau_0.
\tag{4}
$$

In recent years, there are more and more works addressing the controllability problems of damped wave equations (see e.g., [16–19] and the references cited therein).

In this paper, we focus on another kind of parabolic equation with damped term y_{txx}, that is, so-called the

pseudoparabolic equation. Inspired by the above works, whether the pseudoparabolic equation is controllable or not seems very interesting. Indeed, as the third-order term arises, some properties of (1) are quite different from those of the parabolic equations. One of the most essential differences is that, comparing with the fact that the eigenvalues of heat equations accumulate at $-\infty$, the eigenvalues of (1) have an accumulation real point $-1/k$ (it will be shown in Section 2). A similar property was pointed in [20]. This difference causes that the controllability properties of (1) become deeply different from the ones for the parabolic equations. We will show that system (1) is not null controllable under the influence of such unusual spectrum. To this end, we turn the control problem into a moment problem. Thanks to the Paley-Wiener theorem, the result is got, and our approach avoids the proof of the observability inequalities which was used in [14]. On the other hand, we establish the approximate controllability of system (1) in some Fourier definition of Sobolev spaces H_α. The proof is based on a duality method and the explicit solution of the adjoint system of (1), which we obtain by variable separations instead of the Laplace transform in [15]. The techniques we use to prove the approximate controllability contain some ideas of the unique continuation properties.

Throughout this paper, we will use the following notations. The nth Fourier coefficient (with respect to the orthonormal basis $(\sin(n\pi x))_{n\geq 1}$ of $L^2(0,1)$) of any integrable function $y : (0,1) \to \mathbb{R}$ is defined as

$$\widehat{y}_n = \int_0^1 y(x) \sin(n\pi x)\, dx. \tag{5}$$

So, y can be written in the form

$$y(x) = \sum_{n\geq 1} (\widehat{y}_n) \sin(n\pi x). \tag{6}$$

For any $\alpha \in \mathbb{R}$, let

$$H_\alpha := \left\{ y : (0,1) \longrightarrow \mathbb{R}; \sum_{n\geq 1} n^{2\alpha} \left| \widehat{y}_n \right|^2 < \infty \right\}. \tag{7}$$

Endowed with the scalar product

$$(y, z)_\alpha = \sum_{n\geq 1} n^{2\alpha} \widehat{y}_n \widehat{z}_n. \tag{8}$$

H_α is a Hilbert space. Moreover, $H_1 = H_0^1(0,1)$, $H_2 = H^2(0,1) \bigcap H_0^1(0,1)$, and $H_{-\alpha} = H_\alpha'$ (the dual space of H_α with respect to the central space $H_0 = L^2(0,1)$) for any $\alpha \geq 0$. Finally, for any $f = \sum_{n\geq 1} c_n \sin(n\pi x) \in H_{-\alpha}$ and any $g = \sum_{n\geq 1} d_n \sin(n\pi x) \in H_\alpha$, we have that

$$\langle f, g \rangle_{H_{-\alpha}, H_\alpha} = \sum_{n\geq 1} c_n d_n, \tag{9}$$

where $\langle \cdot, \cdot \rangle_{H_{-\alpha}, H_\alpha}$ stands for the duality pairing between $H_{-\alpha}$ and H_α.

The main aim of this paper is to analyze the controllability properties of (1). It will be said that (1) is approximate

controllable by boundary control at time T; if for any $y_0 \in H_0^1(0,1)$, the set of reachable states

$$R(T) = \{ y(T) = y(T; u) :$$

$$y \text{ is the solution of (1) with } u \in H_0^1(0,T) \} \tag{10}$$

is dense in H_α. And it will be said that (1) is null controllable at time T; if for any given $y_0 \in H_0^1(0,1)$, there exist controls $u \in H_0^1(0,T)$ such that the associated solutions to (1) satisfy

$$y(x,T) = 0 \quad \text{in } \Omega. \tag{11}$$

For the sake of simplicity, we will take $k = 1$ throughout this paper. All the results can be extended without difficulty to $k > 0$ arbitrary.

The rest of this paper is organized as follows. In Section 2, we will show some elementary properties for (1) and their adjoint equation. Section 3 is devoted to studying the null and approximate controllability of (1), respectively. In Section 4, some open questions related to this work are provided.

2. Preliminaries

In this section, we first consider the existence and uniqueness of the solution to problem (1).

Let $y(x,t) = z(x,t) + xu(t)$. We readily obtain that z is a solution of the system

$$z_t - z_{xx} - z_{xxt} = -xu', \quad x \in (0,1),\ t \in (0,T),$$

$$z(0,t) = z(1,t) = 0, \quad t \in (0,T), \tag{12}$$

$$z(x,0) = y_0(x) - xu(0), \quad x \in (0,1).$$

The existence and uniqueness of the solution to the problem (12) are well known (see [21, 22]); that is, for every $y_0 \in H_0^1(0,1)$ and $u \in H_0^1(0,T)$, system (12) admits a unique solution $z \in C([0,T]; L^2(0,1))$. In turn, we see that if $y_0 \in H_0^1(0,1)$ and $u \in H_0^1(0,T)$, system (1) admits a unique solution $y \in C([0,T]; L^2(0,1))$.

In order to prove the controllability of system (1), let us consider the following homogeneous initial boundary value problem:

$$y_t - y_{xx} - y_{xxt} = 0, \quad x \in (0,1),\ t \in (0,T),$$

$$y(0,t) = y(1,t) = 0, \quad t \in (0,T), \tag{13}$$

$$y(x,0) = y_0(x), \quad x \in (0,1).$$

We will give an explicit solution of the problem (13) by the method of separation of variables.

Proposition 1. *If the initial condition y_0 is given by $y_0(x) = \sum_{n\geq 1} a_n \sin(n\pi x)$, then the solution to (13) is*

$$y(x,t) = \sum_{n\geq 1} a_n e^{\lambda_n t} \sin(n\pi x), \tag{14}$$

where $\lambda_n = \mu_n/(1 - \mu_n)$ and $\mu_n = -(n\pi)^2$. Moreover, $\lambda_n < 0$, $\{\lambda_n\}_{n\geq 1}$ is a monotone decreasing sequence, and $\lambda_n \to -1$ as $n \to \infty$.

Proof. Let $y(x,t) = T(t)X(x) \neq 0$. From the first equality of (13), we have

$$\frac{T'(t)}{T(t) + T'(t)} \equiv \frac{X''(x)}{X(x)}. \tag{15}$$

We can see that the identity is true if and only if both sides of it are equal to one constant. Let the constant be μ. Then we get

$$X''(x) = \mu X(x), \quad 0 < x < 1, \\ T'(t) - \mu T(t) - \mu T'(t) = 0, \quad t > 0. \tag{16}$$

Since the solution $y(x,t)$ satisfies the boundary condition, $X(0) = X(1) = 0$ is necessary. Thus, we obtain an eigenvalue problem

$$X''(x) = \mu X(x), \quad 0 < x < 1, \\ X(0) = X(1) = 0. \tag{17}$$

By using a simple calculation, we have that $\mu = \mu_n = -(n\pi)^2$ and

$$X_n(x) = B_0 \sin(n\pi x), \tag{18}$$

where B_0 is an arbitrary constant. Now, let us turn to the second equation of (16)

$$T'(t) - \mu_n T'(t) - \mu_n T(t) = 0, \quad t > 0. \tag{19}$$

It is easy to see that

$$T_n(t) = B_1 e^{\lambda_n t} \tag{20}$$

is the solution to (19), where $\lambda_n = \mu_n/(1 - \mu_n)$ and B_1 is an arbitrary constant. Combining the initial condition y_0, we can write the solution of (13) as

$$y(x,t) = \sum_{n\geq 1} a_n e^{\lambda_n t} \sin(n\pi x). \tag{21}$$

The expressions of λ_n imply that $\lambda_n < 0$, $\{\lambda_n\}_{n\geq 1}$ is a monotone decreasing sequence, and $\lambda_n \to -1$ as $n \to \infty$. □

Remark 2. It is important to observe that the spectrum of (1) is quite different from that of heat equation. This will be essential when dealing with the controllability problem of (1).

As an easy consequence of the above representation formula, we have the following result.

Proposition 3. *Let $\alpha \in \mathbb{R}$. If $y_0 \in H_\alpha$, then $y \in C(\mathbb{R}^+; H_\alpha)$. In addition, if $\alpha > 3/2$, then $\sum_{n\geq 1} n|a_n| < +\infty$ and $y_x(1,\cdot) \in C(\mathbb{R}^+)$.*

Proof. If $y_0 \in H_\alpha$, we have $\sum_{n\geq 1} n^{2\alpha}|a_n|^2 < +\infty$. It holds that

$$n^{2\alpha}\left|a_n e^{\lambda_n t}\right|^2 \leq n^{2\alpha}|a_n|^2 e^{2\lambda_n t} \leq n^{2\alpha}|a_n|^2, \tag{22}$$

and hence, $y \in C(\mathbb{R}^+; H_\alpha)$.

On the other hand, taking the derivative with respect to x in (14), we obtain that

$$y_x(1,t) = \sum_{n\geq 1} a_n e^{\lambda_n t} (n\pi)(-1)^n. \tag{23}$$

Since

$$\sum_{n\geq 1} n\left|a_n e^{\lambda_n t}\right| \leq \sum_{n\geq 1} n|a_n|$$
$$\leq \left(\sum_{n\geq 1} n^{2\alpha}|a_n|^2\right)^{1/2} \cdot \left(\sum_{n\geq 1} n^{-2(\alpha-1)}\right)^{1/2}, \tag{24}$$

we see that $\sum_{n\geq 1} n|a_n| < +\infty$ and $y_x(1,\cdot) \in C(\mathbb{R}^+)$, provided that $\alpha > 3/2$. □

Now, let us turn to the adjoint system to system (1) as follows:

$$\varphi_t + \varphi_{xx} - \varphi_{xxt} = 0, \quad x \in (0,1), \ t \in (0,T), \\ \varphi(0,t) = \varphi(1,t) = 0, \quad t \in (0,T), \\ \varphi(x,T) = \varphi^T(x), \quad x \in (0,1). \tag{25}$$

Based on the method in Proposition 1, we have that if φ^T is decomposed as

$$\varphi^T(x) = \sum_{n\geq 1} b_n \sin(n\pi x), \tag{26}$$

the solution of (25) is given by

$$\varphi(x,t) = \sum_{n\geq 1} b_n e^{\lambda_n(T-t)} \sin(n\pi x), \tag{27}$$

which yields

$$\varphi_x(1,t) = \sum_{n\geq 1} b_n e^{\lambda_n(T-t)} (n\pi)(-1)^n. \tag{28}$$

Corresponding to Proposition 3, we have the following result.

Proposition 4. *Let $\alpha \in \mathbb{R}$. If $\varphi^T \in H_{-\alpha}$, then $\varphi \in C(\mathbb{R}^+; H_{-\alpha})$. In addition, if $\alpha < -3/2$, then $\sum_{n\geq 1} n|b_n| < +\infty$ and $\varphi_x(1,\cdot) \in C(\mathbb{R}^+)$.*

Proof. If $\varphi^T \in H_{-\alpha}$, we have $\sum_{n\geq 1} n^{-2\alpha}|b_n|^2 < \infty$. It follows that

$$n^{-2\alpha}\left|b_n e^{\lambda_n(T-t)}\right|^2 \leq n^{-2\alpha}|b_n|^2. \tag{29}$$

Hence $\varphi \in C(\mathbb{R}^+; H_{-\alpha})$. On the other hand, we have by the Cauchy-Schwarz inequality that

$$\sum_{n\geq 1} n|b_n| \leq \left(\sum_{n\geq 1} n^{2\alpha+2}\right)^{1/2} \cdot \left(\sum_{n\geq 1} n^{-2\alpha}\left(|b_n|^2\right)\right)^{1/2}. \tag{30}$$

It is clear that $\sum_{n\geq 1} n|b_n| < +\infty$, provided that $\alpha < -3/2$. □

3. The Main Results

This section is devoted to the study of the controllability of system (1). We first do some transformation by the duality principle.

Multiplying (formally) the first equation in (1) by φ which is the solution of (25) and integrating on $(0, 1) \times (0, T)$, we have

$$\int_0^T \int_0^1 y_t \varphi \, dx \, dt - \int_0^T \int_0^1 y_{xx} \varphi \, dx \, dt - \int_0^T \int_0^1 y_{xxt} \varphi \, dx \, dt = 0. \tag{31}$$

Integrating by parts, we get

$$\begin{aligned}
\int_0^1 & y(x, T) \, \varphi(x, T) \, dx - \int_0^T \int_0^1 y \varphi_t \, dx \, dt \\
& + \int_0^T y(1, t) \, \varphi_x(1, t) \, dt - \int_0^T \int_0^1 y \varphi_{xx} \, dx \, dt \\
& + \int_0^T y_t(1, t) \, \varphi_x(1, t) \, dt \\
& - \int_0^1 y(x, T) \, \varphi_{xx}(x, T) \, dx \\
& + \int_0^T \int_0^1 y \varphi_{xxt} \, dx \, dt \\
& + \int_0^1 y_0(x) \left(\varphi_{xx}(x, 0) - \varphi(x, 0) \right) dx = 0.
\end{aligned} \tag{32}$$

Noticing the first equation in (25), we obtain

$$\begin{aligned}
\int_0^1 & y(x, T) \left(\varphi_{xx}(x, T) - \varphi(x, T) \right) dx \\
& - \int_0^1 y_0(x) \left(\varphi_{xx}(x, 0) - \varphi(x, 0) \right) dx \\
& = \int_0^T y_t(1, t) \, \varphi_x(1, t) \, dt + \int_0^T y(1, t) \, \varphi_x(1, t) \, dt.
\end{aligned} \tag{33}$$

It is clear that (33) can be rewritten as

$$-\left\langle \varphi - \varphi_{xx}, y \right\rangle_{H_{-\alpha}, H_\alpha} \Big|_0^T = \int_0^T \left(u(t) + u'(t) \right) \varphi_x(1, t) \, dt. \tag{34}$$

Now, we are ready to show the controllability of system (1).

3.1. Null Controllability. Following some of the key ideas developed by Micu [23], we are able to show that the null controllability property fails.

Theorem 5. *For $T > 0$, there exist initial conditions $y_0 \in H_0^1(0, 1)$ such that for any control function $u \in H_0^1(0, T)$, the associated solution y to system (1) is not identically equal zero at time T.*

Proof. It follows from (34) that the null controllability problem is equivalent to the existence of a control function $u = u(t)$ such that

$$\begin{aligned}
& \left\langle \varphi(x, 0) - \varphi_{xx}(x, 0), y_0(x) \right\rangle_{H_{-\alpha}, H_\alpha} \\
& = \int_0^T \left(u(t) + u'(t) \right) \varphi_x(1, t) \, dt.
\end{aligned} \tag{35}$$

Using (9), (27), and (28), we can transform the control problem into a moments problem. In other words, we need to find the control function $u = u(t)$ such that

$$\int_0^T \left(u(t) + u'(t) \right) e^{-\lambda_n t} (n\pi) (-1)^n dt = a_n \left(1 + n^2 \pi^2 \right). \tag{36}$$

To this end, we consider that any initial data y_0 with the sequence $\{a_n\}_{n \geq 1}$ satisfies $a_n = 0$ for $n > N$. Suppose that for this kind of y_0, there exists a u such that (36) holds. Let

$$F(z) := \int_0^T \left(u(t) + u'(t) \right) e^{izt} dt. \tag{37}$$

By Paley-Wiener theorem, F is an entire function and

$$F(i\lambda_n) = \int_0^T \left(u(t) + u'(t) \right) e^{-\lambda_n t} dt = 0, \quad \text{for } n > N. \tag{38}$$

The fact that $i\lambda_n \to -i$ as $n \to +\infty$ implies that F is zero on a set with a finite accumulation point. Therefore, $F \equiv 0$. It follows that $a_n = 0$ for each $n \geq 1$. Thus the proof of Theorem 5 is completed. $\qquad\square$

3.2. Approximate Controllability. Because of the lack of null controllability, the approximate controllability of system (1) becomes much more interesting. We will study the approximate controllability in this part. Without loss of generality, we assume that $y_0 = 0$ and by (34) we have

$$\begin{aligned}
& - \left\langle \varphi^T(x) - \varphi_{xx}^T(x), y(x, T) \right\rangle_{H_{-\alpha}, H_\alpha} \\
& = \int_0^T \left(u(t) + u'(t) \right) \varphi_x(1, t) \, dt.
\end{aligned} \tag{39}$$

In order to show the approximate controllability, two lemmas are needed later. The first one is an equivalent condition for the approximate controllability.

Lemma 6. *For system (25), if we assume that $\varphi^T = 0$ if and only if $\int_0^T (u(t) + u'(t)) \varphi_x(1, t) dt = 0$ for any function $u = u(t)$, then system (1) is approximatively controllable in H_α.*

Proof. Recall that the approximate controllability of system (1) in H_α is equivalent to that $R(T)$ that is dense in H_α. Therefore, in order to conclude the proof, we only need to show that $R(T)$ is dense in H_α. It follows from Hahn-Banach theorem that every continuous linear functional on H_α which vanishes on $R(T)$, must vanish everywhere on H_α.

Now, suppose that $R(T)$ is not dense in H_α. Then by Hahn-Banach theorem there exists $\varphi^T \in H_{-\alpha}$ with $\varphi^T - \varphi^T_{xx} \neq 0$, such that for any $y(x, T) \in R(T)$, we have

$$\left\langle \varphi^T(x) - \varphi^T_{xx}(x), y(x, T) \right\rangle_{H_{-\alpha}, H_\alpha} = 0. \tag{40}$$

By (39), we get that

$$\int_0^T \left(u(t) + u'(t) \right) \varphi_x(1, t)\, dt = 0. \tag{41}$$

But from the condition $\varphi^T = 0$ if and only if $\int_0^T (u(t) + u'(t))\varphi_x(1, t)dt = 0$ for any function $u = u(t)$, we have that 0 is the only initial value $\varphi^T \in H_{-\alpha}$ for which the solution $\varphi(x, t)$ of (25) fulfills $\int_0^T (u(t) + u'(t))\varphi_x(1, t)dt = 0$ for any function $u = u(t)$. Therefore $\varphi^T = 0$, and it is contrary to the choice of φ^T. This completes the proof of Lemma 6. □

Lemma 6 implies that, in order to prove the approximate controllability of system (1) with boundary control, we only need to check the condition about the solution of the dual problem (25) in Lemma 6.

The following elementary lemma can be found in [24, 25].

Lemma 7. *Let* $(\beta_n)_{n \geq 1}$ *and* $(\lambda_n)_{n \geq 1}$ *be two sequences of complex numbers such that* $\sum_{n \geq 1} |\beta_n| < \infty$ *and* $\operatorname{Re} \lambda_n < \Lambda$ *for each* $n \geq 1$ *and some number* $\Lambda \in \mathbb{R}$. *Assume that the* $\lambda_n s$ *are pairwise distinct and that* $\sum_{n \geq 1} \beta_n e^{\lambda_n t} = 0$ *for a.e.* $t \in (0, T)$. *Then* $\beta_n = 0$ *for all* $n \geq 1$.

We are now in a position to present the proof of the approximate controllability of (1) with boundary control.

Theorem 8. *Let* $T > 0$. *System* (1) *is approximately controllable in* H_α *with* $\alpha < -3/2$ *at time* T.

Proof. First, we prove that there is a control function u such that $R(T)$ is dense in H_α with $\alpha < -3/2$. Now, take $\varphi^T \in H_{-\alpha}$ with $\alpha < -3/2$ decomposed as in (26).

Set $\theta = u + u'$ and denote

$$G := \left\{ g \in L^2(0, T) ; \int_0^T e^t g(t)\, dt = 0 \right\}. \tag{42}$$

Then it follows that when u ranges over $H_0^1(0, T)$ and θ ranges over G.

Assume that, for any $u \in H_0^1(0, T)$,

$$\int_0^T \theta(t) \varphi_x(1, t)\, dt = \int_0^T \left(u(t) + u'(t) \right) \varphi_x(1, t)\, dt = 0. \tag{43}$$

By Lemma 6, we only need to prove that $\varphi^T = 0$ or equivalently that $b_n = 0$ for each n (see (26)). Noticing that for any $\theta \in \operatorname{Span}(e^t)^\perp$,

$$\left(\theta, \varphi_x(1, \cdot) \right)_{L^2(0, T)} = 0. \tag{44}$$

We have $\varphi_x(1, \cdot) \in \operatorname{Span}(e^t)^{\perp\perp} = \operatorname{Span}(e^t)$. Therefore, from (28), there exists a constant $C \in \mathbb{R}$ such that

$$\sum_{n \geq 1} b_n e^{\lambda_n(T-t)} (n\pi) (-1)^n = Ce^t, \tag{45}$$

for a.e. $t \in (0, T)$. If taking $\tau = T - t$, $c_n = b_n(n\pi)(-1)^n$, and $c_0 := -Ce^T$, we obtain

$$\sum_{n \geq 1} c_n e^{\lambda_n \tau} + c_0 e^{-\tau} = 0, \tag{46}$$

for a.e. $\tau \in (0, T)$.

On the other hand, since $\varphi^T \in H_{-\alpha}$ with $\alpha < -3/2$ and by Proposition 4, we have that

$$\sum_{n \geq 1} |c_n| = \pi \sum_{n \geq 1} n |b_n| < +\infty. \tag{47}$$

Then by Lemma 7, we obtain that $c_n = 0$ for each $n \geq 1$. It implies that $\varphi^T = 0$. The proof of Theorem 8 is completed. □

4. Concluding Remark

In this paper, the controllability of the pseudoparabolic equation is studied. With a boundary control, it is proved that the system is not null controllable, but that an approximate controllability result is obtained in some appropriate functional spaces. Below is a list of unsolved and interesting questions related to our work.

(1) It has been got that the approximate controllability holds in H_α with $\alpha < -3/2$. The question whether the approximate controllability holds for $\alpha \geq -3/2$ remains open. The method in this paper does not work for that case.

(2) It seems natural to expect that the controllability for multidimensional pseudoparabolic equations through a boundary controller or a locally distributed one. We will consider these problems in the forthcoming papers.

(3) It would be quite interesting to study the controllability properties of (1) for the case variable coefficient (i.e., $k = k(x, t)$). However, it seems very difficult, and many classical methods such as moment problem and strongly continuous semigroups may be false. Indeed, controllability of equation with variable coefficients always bring us much more difficult than that with constant coefficient. One needs a highly innovative way to obtain the observability inequalities or unique continuation properties. An example was presented in [26] to establish some controllability results for wave equations with variable coefficients by a Riemannian geometry method.

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