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Research Article

A Sharp RIP Condition for Orthogonal Matching Pursuit

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A restricted isometry property (RIP) condition $\delta_K + \sqrt{K}\theta_{K,1} < 1$ is known to be sufficient for orthogonal matching pursuit (OMP) to exactly recover every K-sparse signal x from measurements $y = \Phi x$. This paper is devoted to demonstrate that this condition is sharp. We construct a specific matrix with $\delta_K + \sqrt{K}\theta_{K,1} = 1$ such that OMP cannot exactly recover some K-sparse signals.

1. Introduction

Compressive sampling (compressed sensing, CS) is known as a new type of sampling theory that one can reconstruct a high dimensional spare signal from a small number of linear measurements at the sub-Nyquist rate [1–3]. Nowadays, the CS technique has attracted considerable attention from across a wide array of fields like applied mathematics, statistics, and engineering, including signal processing areas such as MR imaging, speech processing, and analog to digital conversion. The basic problem in CS is to reconstruct the unknown sparse signal x from measurements:

$$y = \Phi x, \tag{1}$$

where Φ is an $M \times N$ $(M \ll N)$ sampling matrix. Suppose $\Phi = (\Phi_1, \Phi_2, \ldots, \Phi_N)$, where Φ_i denotes the ith column of Φ. Throughout the paper, we will assume that the columns of Φ are normalized; that is, $\|\Phi_i\|_2 = 1$ for $i = 1, 2, \ldots, N$.

It is well understood that under some assumptions on the sampling matrix Φ, the unknown sparse signal x can be reconstructed by solving the l_0-minimization problem:

$$\min \|x\|_0 \quad \text{subject to } y = \Phi x, \tag{2}$$

where $\|x\|_0$ denotes the number of nonzero entries of x. We say a signal x is K-sparse when $\|x\|_0 \le K$.

However, the optimization problem is NP-hard, so one seeks computationally efficient algorithms to approximate the sparse signal x, such as greedy algorithm, l_1 minimization, and l_p $(0 < p < 1)$ minimization [4–6].

Orthogonal matching pursuit (OMP), which is a canonical greedy algorithm, has receive much attention in solving the problem (2), due to its ease of implementation and low complexity. Algorithm 1 can be described below. Until recently, many popular generalizations of OMP are introduced, for example, OMMP and KOMP; for details, see [7, 8].

The mutual incoherence property (MIP) introduced in [9] is an important tool to analyze the performance of OMP. The MIP requires the mutual coherence μ of the sampling matrix Φ to be small, where μ is defined as

$$\mu = \max_{i \neq j} \left| \Phi_i^T \Phi_j \right|. \tag{3}$$

Tropp has shown that the MIP condition $(2K-1)\mu < 1$ is sufficient for OMP to exactly recover every K-sparse signal [6]. This condition is proved to be sharp in [10].

The restricted isometry property (RIP) is also widely used in studying a large number of algorithms for sparse recovery in CS, which is introduced in [11]. A matrix Φ satisfies the RIP of order K with the restricted isometry constant (RIC) δ_K if δ_K is the smallest constant such that

$$\left(1 - \delta_K\right) \|x\|_2^2 \le \|\Phi x\|_2^2 \le \left(1 + \delta_K\right) \|x\|_2^2 \tag{4}$$

holds for all K-sparse signal x. A related quantity of the restricted orthogonality constant (ROC) $\theta_{K,K'}$ is defined as the smallest quantity such that

$$\left| \left\langle \Phi x, \Phi x' \right\rangle \right| \le \theta_{K,K'} \|x\|_2 \cdot \left\| x' \right\|_2 \tag{5}$$

Input: Sampling matrix Φ, observation y
Output: Reconstructed sparse vector x^* and index set
INITIALIZATION: Let the index set $\Omega_0 = \varnothing$ and the residual $r_0 = y$. Let the iteration counter $t = 1$.
IDENTIFICATION: Choose the index i subject to $|\Phi_i^T r_{t-1}| > \max_{j \neq i} |\Phi_j^T r_{t-1}|$.
UPDATE: Add the new index i to the index set: $\Omega_t = \Omega_{t-1} \cup i$, and update the signal and the residual

$$x_t|_{\Omega_t} = \arg\min_z \|y - \Phi_{\Omega_t} z\|_2, \quad x_t|_{\overline{\Omega}_t} = 0;$$
$$r_t = y - \Phi x_t.$$

If $r_t = 0$, stop the algorithm. Otherwise, update the iteration counter $t = t + 1$ and return to Step **IDENTIFICATION**.

Algorithm 1: Orthogonal matching pursuit—OMP (Φ, y).

holds for all disjoint support K-sparse signal x and K'-sparse signal x'. It is first shown by Davenport and Wakin that the RIP condition

$$\delta_{K+1} < \frac{1}{3\sqrt{K}} \tag{6}$$

can guarantee that OMP will exactly recover every K-sparse signal [12]. The sufficient condition is then improved to $\delta_{K+1} < 1/(1 + 2\sqrt{K})$ [13], $\delta_{K+1} < 1/\sqrt{2K}$ [14], $\delta_{K+1} < 1/(1 + \sqrt{K})$ [7], and $\delta_K + \sqrt{K}\theta_{K,1} < 1$ [15, 16]. By contrast, Mo and Shen have given a counterexample, a matrix with $\delta_{K+1} = 1/\sqrt{K}$ where OMP fails for some K-sparse signals [17]. The main result of this note is to show that the sufficient RIP condition

$$\delta_K + \sqrt{K}\theta_{K,1} < 1 \tag{7}$$

is sharp for OMP.

2. Main Result

Theorem 1. *For any given positive integer $K \geq 1$, there exist a K-sparse signal x and a matrix Φ with the restricted isometry constant*

$$\delta_K + \sqrt{K}\theta_{K,1} = 1 \tag{8}$$

for which OMP fails in K iterations.

Proof. For any given positive integer $K \geq 1$, let

$$\Phi = \left(\Phi_{ij}\right)_{(2K-1)\times(2K-1)}, \tag{9}$$

where

$$\Phi_{ij} = \begin{cases} 0 & (i < j), \\ \sqrt{\dfrac{2K}{2K-1}} \cdot \left(-\dfrac{i}{\sqrt{i(i+1)}}\right) & (i = j), \\ \sqrt{\dfrac{2K}{2K-1}} \cdot \dfrac{1}{\sqrt{i(i+1)}} & (i > j). \end{cases} \tag{10}$$

By simple calculation, we can get

$$\begin{aligned} \left\|\Phi_j\right\|_2^2 &= \frac{2K}{2K-1} \cdot \left(\frac{j^2}{j(j+1)} + \sum_{i=j+1}^{2K-1} \frac{1}{i(i+1)}\right) \\ &= \frac{2K}{2K-1} \\ &\quad \cdot \left(\frac{j}{j+1} + \frac{1}{j+1} - \frac{1}{j+2}\right. \\ &\qquad \left. + \cdots + \frac{1}{2K-1} - \frac{1}{2K}\right) \\ &= 1, \\ \left\langle \Phi_l, \Phi_j \right\rangle &= \frac{2K}{2K-1} \cdot \left(-\frac{j}{j(j+1)} + \sum_{i=j+1}^{2K-1} \frac{1}{i(i+1)}\right) \\ &= \frac{2K}{2K-1} \\ &\quad \cdot \left(-\frac{1}{j+1} + \frac{1}{j+1} - \frac{1}{j+2}\right. \\ &\qquad \left. + \cdots + \frac{1}{2K-1} - \frac{1}{2K}\right) \\ &= -\frac{1}{2K-1} \end{aligned} \tag{11}$$

for any integers $1 \leq l < j \leq 2K - 1$.

Thus, for any index set Λ whose cardinality is K, we have

$$\Phi_\Lambda^T \Phi_\Lambda = \begin{pmatrix} 1 & -\dfrac{1}{2K-1} & \cdots & -\dfrac{1}{2K-1} \\ & & \ddots & \\ -\dfrac{1}{2K-1} & \cdots & -\dfrac{1}{2K-1} & 1 \end{pmatrix}. \tag{12}$$

It is obvious that the eigenvalues $\{\lambda_i\}_{i=1}^K$ of $\Phi_\Lambda^T \Phi_\Lambda$ are

$$\begin{aligned} \lambda_1 &= \cdots = \lambda_{K-1} = 1 + \frac{1}{2K-1}, \\ \lambda_K &= 1 - \frac{K-1}{2K-1}. \end{aligned} \tag{13}$$

Therefore, the restricted isometry constant δ_K of Φ is $(K-1)/(2K-1)$.

Now, we turn to calculate the restricted orthogonality constant $\theta_{K,1}$. In view of (11), we may, without loss of generality, assume that $x = (x_1, \dots, x_K, 0, \dots, 0)^T$ and $x' = (0, \dots, 0, x'_{K+1} 0, \dots, 0)^T$. We have

$$\begin{aligned} \theta_{K,1} &= \max \frac{\left|\left\langle \Phi x, \Phi x' \right\rangle\right|}{\|x\|_2 \cdot \|x'\|_2} \\ &= \max \frac{\left|\left\langle \Phi x, \Phi_{K+1} \right\rangle\right|}{\|x\|_2} \\ &= \max \frac{1}{2K-1} \cdot \frac{\left|\sum_{i=1}^{K} x_i\right|}{\|x\|_2} \\ &= \frac{\sqrt{K}}{2K-1}. \end{aligned} \tag{14}$$

The last equality holds when $x_1 = \cdots = x_K$. It is easy to check that

$$\delta_K + \sqrt{K}\theta_{K,1} = \frac{K-1}{2K-1} + \sqrt{K} \cdot \frac{\sqrt{K}}{2K-1} = 1. \tag{15}$$

Let $x = (\underbrace{1, \dots, 1}_{K}, 0, \dots, 0)^T \in \mathbf{R}^{2K-1}$; we have

$$\left|S_j\right| = \left|\left\langle \Phi x, \Phi_j \right\rangle\right| = \frac{K}{2K-1}, \quad \forall j \in \{1, 2, \dots, 2K-1\}. \tag{16}$$

This implies that OMP fails in the first iteration. The proof is complete. □

3. Discussion

In this paper, we showed that the RIP condition $\delta_K + \sqrt{K}\theta_{K,1} < 1$ is sharp for orthogonal matching pursuit to exactly recover every K-sparse signal x from measurements $y = \Phi x$. It is worth discussing the relations between our sharp RIP condition and that in two relative papers [10, 17]. First of all, it follows from the facts that $\delta_K < (K-1)\mu$ and $\theta_{K,1} < \sqrt{K}\mu$ that the sharp RIP condition $\delta_K + \sqrt{K}\theta_{K,1} < 1$ in this paper is weaker than the sharp MIP condition $(2K-1)\mu < 1$ in [10]. Moreover, our result is also stronger than the previous RIP condition. The condition $\delta_K + \sqrt{K}\theta_{K,1} < 1$ in this paper is necessary and sufficient for OMP, while the previous necessary RIP condition $\delta_K < 1/\sqrt{K}$ in [17] is not sufficient. Therefore, the result in the paper may guide the practitioners to apply OMP properly in sparse recovery.

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