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Research Article

Coefficient Estimates and Other Properties for a Class of Spirallike Functions Associated with a Differential Operator

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For $0 \le \eta < 1$, $0 \le \lambda < 1$, $-\pi/2 < \gamma < \pi/2$, $0 \le \beta \le \alpha$, and $m \in \mathbb{N} \cup \{0\}$, a new class $S^m_{\alpha,\beta}(\eta, \gamma, \lambda)$ of analytic functions defined by means of the differential operator $D^m_{\alpha,\beta}$ is introduced. Our main object is to provide sharp upper bounds for Fekete-Szegö problem in $S^m_{\alpha,\beta}(\eta, \gamma, \lambda)$. We also find sufficient conditions for a function to be in this class. Some interesting consequences of our results are pointed out.

1. Introduction

Let $\mathscr{A}$ denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk $\mathscr{U} = \{z \in \mathbb{C} : |z| < 1\}$.

Let $\mathscr{S}$ denote the subclass of $\mathscr{A}$ consisting of functions that are univalent in $\mathscr{U}$.

A function $f \in \mathscr{A}$ is said to be in the class of γ-spirallike functions of order λ in $\mathscr{U}$, denoted by $\mathscr{S}^*(\gamma, \lambda)$, if

$$\Re\left(e^{i\gamma} \frac{zf'(z)}{f(z)}\right) > \lambda \cos\gamma, \quad z \in \mathscr{U}, \tag{2}$$

for $0 \le \lambda < 1$ and some real γ with $|\gamma| < \pi/2$.

The class $\mathscr{S}^*(\gamma, \lambda)$ was studied by Libera [1] and Keogh and Merkes [2].

Note that $\mathscr{S}^*(\gamma, 0)$ is the class of spirallike functions introduced by Špaček [3], $\mathscr{S}^*(0, \lambda) = \mathscr{S}^*(\lambda)$ is the class of starlike functions of order λ, and $\mathscr{S}^*(0, 0) = \mathscr{S}^*$ is the familiar class of starlike functions.

For the constants λ, γ with $0 \le \lambda < 1$ and $|\gamma| < \pi/2$, denote

$$p_{\lambda,\gamma}(z) = \frac{1 + e^{-i\gamma}\left(e^{-i\gamma} - 2\lambda\cos\gamma\right)z}{1 - z}, \quad z \in \mathscr{U}. \tag{3}$$

The function $p_{\lambda,\gamma}(z)$ maps the open unit disk onto the half-plane $H_{\lambda,\gamma} = \{z \in \mathbb{C} : \Re(e^{i\gamma} z) > \lambda\cos\gamma\}$. If

$$p_{\lambda,\gamma}(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \tag{4}$$

then it is easy to check that

$$p_n = 2e^{-i\gamma}(1 - \lambda)\cos\gamma, \quad \forall n \ge 1. \tag{5}$$

For $f \in \mathscr{A}$ given by (1) and $g \in \mathscr{A}$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \tag{6}$$

the Hadamard product (or convolution), denoted by $f * g$, is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in \mathscr{U}. \tag{7}$$

Denote by $\mathcal{B}$ the family of all analytic functions $w(z)$ that satisfy the conditions $w(0) = 0$ and $|w(z)| < 1$, $z \in \mathcal{U}$.

A function $f \in \mathcal{A}$ is said to be subordinate to a function $g \in \mathcal{A}$, written $f \prec g$, if there exists a function $w \in \mathcal{B}$ such that $f(z) = g(w(z))$, $z \in \mathcal{U}$.

A classical theorem of Fekete and Szegö (see [4]) states that if $f \in \mathcal{S}$ is given by (1), then

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} 3 - 4\mu, & \text{if } \mu \le 0, \\ 1 + 2\exp\left(\dfrac{-2\mu}{1-\mu}\right), & \text{if } 0 \le \mu \le 1, \\ 4\mu - 3, & \text{if } \mu \ge 1. \end{cases} \tag{8}$$

This inequality is sharp in the sense that for each μ there exists a function in $\mathcal{S}$ such that the equality holds. Later Pfluger (see [5]) has considered the same problem but for complex values of μ. The problem of finding sharp upper bounds for the functional $|a_3 - \mu a_2^2|$ for different subclasses of $\mathcal{A}$ is known as the Fekete-Szegö problem. Over the years, this problem has been investigated by many authors including [6–12].

For a function $f \in \mathcal{A}$, we consider the following differential operator introduced by Răducanu and Orhan [13]:

(i) $D^0_{\alpha,\beta} f(z) = f(z)$,

(ii) $D^1_{\alpha,\beta} f(z) = D_{\alpha,\beta} f(z) = \alpha\beta z^2 f''(z) + (\alpha - \beta) z f'(z) + (1 - \alpha + \beta) f(z)$,

(iii) $D^m_{\alpha,\beta} f(z) = D_{\alpha,\beta}(D^{m-1}_{\alpha,\beta} f(z))$, $z \in \mathcal{U}$,

where $0 \le \beta \le \alpha$ and $m \in \mathbb{N}_0 = \{0, 1, \ldots\}$.

If the function f is given by (1), then, from the definition of the operator $D^m_{\alpha,\beta} f$, it is easy to observe that

$$D^m_{\alpha,\beta} f(z) = z + \sum_{n=2}^{\infty} \Phi_n(\alpha, \beta, m) a_n z^n, \tag{9}$$

where

$$\Phi_n(\alpha, \beta, m) = \left[1 + (\alpha\beta n + \alpha - \beta)(n-1)\right]^m, \quad n \ge 2. \tag{10}$$

It should be remarked that the operator $D^m_{\alpha,\beta} f$ generalizes other differential operators considered earlier. For $f \in \mathcal{A}$, we have

(i) $D^m_{1,0} f(z) = D^m f(z)$, the operator introduced by Sălăgean [14];

(ii) $D^m_{\alpha,0} f(z) = D^m_\alpha f(z)$, the operator studied by Al-Oboudi [15].

In view of (9), $D^m_{\alpha,\beta} f(z)$ can be written in terms of convolution as

$$D^m_{\alpha,\beta} f(z) = \left(g_{\alpha,\beta} * f\right)(z), \quad z \in \mathcal{U}, \tag{11}$$

where

$$g_{\alpha,\beta}(z) = z + \sum_{n=2}^{\infty} \Phi_n(\alpha, \beta, m) z^n, \quad z \in \mathcal{U}. \tag{12}$$

Define the function $g^{(-1)}_{\alpha,\beta}$ such that

$$g^{(-1)}_{\alpha,\beta}(z) * g_{\alpha,\beta}(z) = \frac{z}{1-z}, \quad z \in \mathcal{U}. \tag{13}$$

It is easy to observe that

$$f(z) = g^{(-1)}_{\alpha,\beta}(z) * D^m_{\alpha,\beta} f(z). \tag{14}$$

Making use of the differential operator $D^m_{\alpha,\beta} f$, we define the following class of functions.

Definition 1. For $0 \le \eta < 1$, $0 \le \lambda < 1$, and $|\gamma| < \pi/2$, denote by $\mathcal{S}^m_{\alpha,\beta}(\eta, \gamma, \lambda)$ the class of functions $f \in \mathcal{A}$ which satisfy the condition

$$\Re\left(e^{i\gamma} \frac{z\left(D^m_{\alpha,\beta} f(z)\right)'}{(1-\eta) D^m_{\alpha,\beta} f(z) + \eta z\left(D^m_{\alpha,\beta} f(z)\right)'}\right) > \lambda \cos\gamma, \quad z \in \mathcal{U}. \tag{15}$$

The class $\mathcal{S}^m_{\alpha,\beta}(\eta, \gamma, \lambda)$ contains as particular cases the following classes of functions:

$$\mathcal{S}^0_{\alpha,\beta}(0, \gamma, \lambda) = \mathcal{S}^*(\gamma, \lambda), \qquad \mathcal{S}^0_{\alpha,\beta}(0, \gamma, 0) = \mathcal{S}^*(\gamma), \qquad \mathcal{S}^0_{\alpha,\beta}(0, 0, 0) = \mathcal{S}^*. \tag{16}$$

Also, the class $\mathcal{S}^0_{\alpha,\beta}(\eta, \gamma, \lambda)$ consists of functions $f \in \mathcal{A}$ satisfying the inequality

$$\Re\left(e^{i\gamma} \frac{z f'(z)}{(1-\eta) f(z) + \eta z f'(z)}\right) > \lambda \cos\gamma, \quad z \in \mathcal{U}. \tag{17}$$

An analogous of the class $\mathcal{S}^0_{\alpha,\beta}(\eta, \gamma, \lambda)$ has been recently studied by Murugusundaramoorthy [16].

The main object of this paper is to obtain sharp upper bounds for the Fekete-Szegö problem for the class $\mathcal{S}^m_{\alpha,\beta}(\eta, \gamma, \lambda)$. We also find sufficient conditions for a function to be in this class.

2. Membership Characterizations

In this section, we obtain several sufficient conditions for a function $f \in \mathcal{A}$ to be in the class $\mathcal{S}^m_{\alpha,\beta}(\eta, \gamma, \lambda)$.

Theorem 2. *Let $f \in \mathcal{A}$, and let δ be a real number with $0 \le \delta < 1$. If*

$$\left|\frac{z\left(D^m_{\alpha,\beta} f(z)\right)'}{(1-\eta) D^m_{\alpha,\beta} f(z) + \eta z\left(D^m_{\alpha,\beta} f(z)\right)'} - 1\right| \le 1 - \delta, \quad z \in \mathcal{U}, \tag{18}$$

then $f \in \mathcal{S}^m_{\alpha,\beta}(\eta, \gamma, \lambda)$ provided that

$$|\gamma| \le \cos^{-1}\left(\frac{1-\delta}{1-\lambda}\right). \tag{19}$$

Proof. From (18), it follows that

$$\frac{z\left(D^m_{\alpha,\beta}f(z)\right)'}{(1-\eta)\,D^m_{\alpha,\beta}f(z)+\eta z\left(D^m_{\alpha,\beta}f(z)\right)'}=1+(1-\delta)\,w(z), \tag{20}$$

where $w(z)\in\mathscr{B}$. We have

$$\begin{aligned}\Re&\left(e^{i\gamma}\frac{z\left(D^m_{\alpha,\beta}f(z)\right)'}{(1-\eta)\,D^m_{\alpha,\beta}f(z)+\eta z\left(D^m_{\alpha,\beta}f(z)\right)'}\right)\\&=\Re\left[e^{i\gamma}(1+(1-\delta)\,w(z))\right]\\&=\cos\gamma+(1-\delta)\,\Re\left(e^{i\gamma}w(z)\right)\\&\geq\cos\gamma-(1-\delta)\left|e^{i\gamma}w(z)\right|\\&>\cos\gamma-(1-\delta)\geq\lambda\cos\gamma,\end{aligned} \tag{21}$$

provided that $|\gamma|\leq\cos^{-1}((1-\delta)/(1-\lambda))$. Thus, the proof is completed. □

If in Theorem 2 we take $\delta=1-(1-\lambda)\cos\gamma$, we will obtain the following result.

Corollary 3. *Let $f\in\mathscr{A}$. If*

$$\left|\frac{z\left(D^m_{\alpha,\beta}f(z)\right)'}{(1-\eta)\,D^m_{\alpha,\beta}f(z)+\eta z\left(D^m_{\alpha,\beta}f(z)\right)'}-1\right|\leq(1-\lambda)\cos\gamma,\quad z\in\mathscr{U} \tag{22}$$

then $f\in\mathscr{S}^m_{\alpha,\beta}(\eta,\gamma,\lambda)$.

A sufficient condition for a function $f\in\mathscr{A}$ to be in the class $\mathscr{S}^m_{\alpha,\beta}(\eta,\gamma,\lambda)$, in terms of coefficients inequality, is obtained in the next theorem.

Theorem 4. *If a function $f\in\mathscr{A}$ given by* (1) *satisfies the inequality*

$$\begin{aligned}\sum_{n=2}^{\infty}&\left[(1-\eta)(n-1)\sec\gamma+(1-\lambda)(1+\eta(n-1))\right]\\&\times\Phi_n(\alpha,\beta,m)\left|a_n\right|\leq1-\lambda,\end{aligned} \tag{23}$$

where $0\leq\eta<1$, $0\leq\lambda<1$, $|\gamma|<\pi/2$, and $\Phi_n(\alpha,\beta,m)$ is defined by (10), *then it belongs to the class $\mathscr{S}^m_{\alpha,\beta}(\eta,\gamma,\lambda)$.*

Proof. In virtue of Corollary 3, it suffices to show that the condition (22) is satisfied. We have

$$\begin{aligned}&\left|\frac{z\left(D^m_{\alpha,\beta}f(z)\right)'}{(1-\eta)\,D^m_{\alpha,\beta}f(z)+\eta z\left(D^m_{\alpha,\beta}f(z)\right)'}-1\right|\\&\quad=(1-\eta)\left|\frac{\sum_{n=2}^{\infty}(n-1)\,\Phi_n(\alpha,\beta,m)\,a_nz^{n-1}}{1+\sum_{n=2}^{\infty}(1-\eta+\eta n)\,\Phi_n(\alpha,\beta,m)\,a_nz^{n-1}}\right|\\&\quad<(1-\eta)\frac{\sum_{n=2}^{\infty}(n-1)\,\Phi_n(\alpha,\beta,m)\left|a_n\right|}{1-\sum_{n=2}^{\infty}(1-\eta+\eta n)\,\Phi_n(\alpha,\beta,m)\left|a_n\right|}.\end{aligned} \tag{24}$$

The last expression is bounded previously by $(1-\lambda)\cos\gamma$, if

$$\begin{aligned}\sum_{n=2}^{\infty}&(1-\eta)(n-1)\,\Phi_n(\alpha,\beta,m)\left|a_n\right|\\&\leq(1-\lambda)\cos\gamma\left(1-\sum_{n=2}^{\infty}(1-\eta+\eta n)\,\Phi_n(\alpha,\beta,m)\left|a_n\right|\right),\end{aligned} \tag{25}$$

which is equivalent to

$$\begin{aligned}\sum_{n=2}^{\infty}&\left[(1-\eta)(n-1)\sec\gamma+(1-\lambda)(1+\eta(n-1))\right]\\&\times\Phi_n(\alpha,\beta,m)\left|a_n\right|\leq1-\lambda.\end{aligned} \tag{26}$$

□

For special values of m, η, γ, and λ, from Theorem 4, we can derive the following sufficient conditions for a function $f\in\mathscr{A}$ to be in the classes $\mathscr{S}^0_{\alpha,\beta}(\eta,\gamma,\lambda)$, $\mathscr{S}^0_{\alpha,\beta}(0,\gamma,\lambda)=\mathscr{S}^*(\gamma,\lambda)$, and $\mathscr{S}^0_{\alpha,\beta}(0,\gamma,0)=\mathscr{S}^*(\gamma)$, respectively.

Corollary 5. *Let $f\in\mathscr{A}$. If*

$$\begin{aligned}\sum_{n=2}^{\infty}&\left[(1-\eta)(n-1)\sec\gamma+(1-\lambda)(1+\eta(n-1))\right]\left|a_n\right|\\&\leq1-\lambda,\end{aligned} \tag{27}$$

where $0\leq\eta<1$, $0\leq\lambda<1$, and $|\gamma|<\pi/2$, then $f\in\mathscr{S}^0_{\alpha,\beta}(\eta,\gamma,\lambda)$.

Corollary 6 (see [17]). *Let $f\in\mathscr{A}$. If*

$$\sum_{n=2}^{\infty}\left[(n-1)\sec\gamma+1-\lambda\right]\left|a_n\right|\leq1-\lambda, \tag{28}$$

where $0\leq\lambda<1$, $|\gamma|<\pi/2$, then $f\in\mathscr{S}^*(\gamma,\lambda)$.

Corollary 7 (see [18]). *Let $f\in\mathscr{A}$. If*

$$\sum_{n=2}^{\infty}\left[1+(n-1)\sec\gamma\right]\left|a_n\right|\leq1, \tag{29}$$

where $|\gamma|<\pi/2$, then $f\in\mathscr{S}^*(\gamma)$.

A necessary and sufficient condition for a function to be in the class $\mathcal{S}^m_{\alpha,\beta}(\eta,\gamma,\lambda)$ can be given in terms of integral representation.

Theorem 8. *A function $f \in \mathcal{A}$ is in the class $\mathcal{S}^m_{\alpha,\beta}(\eta,\gamma,\lambda)$ if and only if there exists $w \in \mathcal{B}$ such that*

$$f(z) = g^{(-1)}_{\alpha,\beta}(z) * z\exp\left(\int_0^z \left[\frac{p_{\lambda,\gamma}(w(\zeta)) - 1}{1 - \eta p_{\lambda,\gamma}(w(\zeta))}\right]\frac{d\zeta}{\zeta}\right), \quad z \in \mathcal{U}, \tag{30}$$

where $p_{\lambda,\gamma}(z)$ and $g^{(-1)}_{\alpha,\beta}(z)$ are defined by (3) *and* (13), *respectively.*

Proof. In virtue of (15), $f \in \mathcal{S}^m_{\alpha,\beta}(\eta,\gamma,\lambda)$ if and only if there exists $w \in \mathcal{B}$ such that

$$\frac{z\left(D^m_{\alpha,\beta}f(z)\right)'}{(1-\eta)D^m_{\alpha,\beta}f(z) + \eta z\left(D^m_{\alpha,\beta}f(z)\right)'} = p_{\lambda,\gamma}(w(z)). \tag{31}$$

From the last equality, we obtain

$$D^m_{\alpha,\beta}f(z) = z\exp\left(\int_0^z \left[\frac{p_{\lambda,\gamma}(w(\zeta)) - 1}{1 - \eta p_{\lambda,\gamma}(w(\zeta))}\right]\frac{d\zeta}{\zeta}\right). \tag{32}$$

Making use of (14) and (32), we have

$$f(z) = g^{(-1)}_{\alpha,\beta}(z) * z\exp\left(\int_0^z \left[\frac{p_{\lambda,\gamma}(w(\zeta)) - 1}{1 - \eta p_{\lambda,\gamma}(w(\zeta))}\right]\frac{d\zeta}{\zeta}\right), \quad z \in \mathcal{U}, \tag{33}$$

and thus, the proof is completed. □

For $0 \le \theta \le 2\pi$, $0 \le \tau \le 1$, define the function

$$\Psi(z,\theta,\tau) = g^{(-1)}_{\alpha,\beta}(z) * z\exp\left(\int_0^z \left[\frac{p_{\lambda,\gamma}\left(e^{i\theta}\zeta(\zeta+\tau)/(1+\tau\zeta)\right) - 1}{1 - \eta p_{\lambda,\gamma}\left(e^{i\theta}\zeta(\zeta+\tau)/(1+\tau\zeta)\right)}\right]\frac{d\zeta}{\zeta}\right), \tag{34}$$

where $p_{\lambda,\gamma}(z)$ and $g^{(-1)}_{\alpha,\beta}(z)$ are defined by (3) and (13), respectively.

In virtue of Theorem 8, the function $\Psi(z,\theta,\tau)$ belongs to the class $\mathcal{S}^m_{\alpha,\beta}(\eta,\gamma,\lambda)$. Note that $\Psi(z,0,0)$ is an odd function.

3. The Fekete-Szegö Problem

In order to obtain sharp upper bounds for the Fekete-Szegö functional for the class $\mathcal{S}^m_{\alpha,\beta}(\eta,\gamma,\lambda)$, the following lemma is required (see, e.g., [19, page 108]).

Lemma 9. *Let the function $w \in \mathcal{B}$ be given by*

$$w(z) = \sum_{n=1}^{\infty} w_n z^n, \quad z \in \mathcal{U}. \tag{35}$$

Then

$$|w_1| \le 1, \qquad |w_2| \le 1 - |w_1|^2, \tag{36}$$

$$\left|w_2 - s w_1^2\right| \le \max\{1, |s|\}, \quad \text{for any complex number } s. \tag{37}$$

The functions $w(z) = z$ and $w(z) = z^2$, or one of their rotations, show that both inequalities (36) *and* (37) *are sharp.*

First we obtain sharp upper bounds for the Fekete-Szegö functional $|a_3 - \mu a_2^2|$ with μ real parameter.

Theorem 10. *Let $f \in \mathcal{S}^m_{\alpha,\beta}(\eta,\gamma,\lambda)$ be given by* (1), *and let μ be a real number. Then*

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} \dfrac{(1-\lambda)\cos\gamma}{(1-\eta)^2\Phi_3(\alpha,\beta,m)} \times \left[\eta + 3 - 2\lambda(1+\eta) - \mu\dfrac{4(1-\lambda)\Phi_3(\alpha,\beta,m)}{\Phi_2^2(\alpha,\beta,m)}\right], & \text{if } \mu \le \sigma_1, \\ \dfrac{(1-\lambda)\cos\gamma}{(1-\eta)\Phi_3(\alpha,\beta,m)}, & \text{if } \sigma_1 \le \mu \le \sigma_2, \\ \dfrac{(1-\lambda)\cos\gamma}{(1-\eta)^2\Phi_3(\alpha,\beta,m)} \times \left[\mu\dfrac{4(1-\lambda)\Phi_3(\alpha,\beta,m)}{\Phi_2^2(\alpha,\beta,m)} + 2\lambda(1+\eta) - \eta - 3\right], & \text{if } \mu \ge \sigma_2, \end{cases} \tag{38}$$

where

$$\sigma_1 = (1+\eta)\frac{\Phi_2^2(\alpha,\beta,m)}{2\Phi_3(\alpha,\beta,m)}, \tag{39}$$

$$\sigma_2 = \frac{2 - \lambda(1+\eta)}{1-\lambda}\,\frac{\Phi_2^2(\alpha,\beta,m)}{2\Phi_3(\alpha,\beta,m)}, \tag{40}$$

and $\Phi_2(\alpha,\beta,m)$, $\Phi_3(\alpha,\beta,m)$ are defined by (10) *with $n = 2$ and $n = 3$, respectively.*

All estimates are sharp.

Proof. Suppose that $f \in \mathcal{S}^m_{\alpha,\beta}(\eta,\gamma,\lambda)$ is given by (1). Then, from the definition of the class $\mathcal{S}^m_{\alpha,\beta}(\eta,\gamma,\lambda)$, there exist $w \in \mathcal{B}$, $w(z) = w_1 z + w_2 z^2 + w_3 z^3 + \cdots$ such that

$$\frac{z\left(D^m_{\alpha,\beta} f(z)\right)'}{(1-\eta) D^m_{\alpha,\beta} f(z) + \eta z\left(D^m_{\alpha,\beta} f(z)\right)'} = p_{\lambda,\gamma}(w(z)), \quad z \in \mathcal{U}. \tag{41}$$

Set $p_{\lambda,\gamma}(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$. Equating the coefficients of z and z^2 on both sides of (41), we obtain

$$a_2 = \frac{p_1 w_1}{(1-\eta)\Phi_2(\alpha,\beta,m)}, \qquad a_3 = \frac{1}{2(1-\eta)\Phi_3(\alpha,\beta,m)}\left[\left(\frac{1+\eta}{1-\eta}p_1^2 + p_2\right)w_1^2 + p_1 w_2\right]. \tag{42}$$

From (5), we have $p_1 = p_2 = 2e^{-i\gamma}(1-\lambda)\cos\gamma$, and thus we obtain

$$a_2 = \frac{2e^{-i\gamma}(1-\lambda)\cos\gamma}{(1-\eta)\Phi_2(\alpha,\beta,m)} w_1, \qquad a_3 = \frac{e^{-i\gamma}(1-\lambda)\cos\gamma}{(1-\eta)\Phi_3(\alpha,\beta,m)} \times \left[\left(2e^{-i\gamma}(1-\lambda)\cos\gamma\frac{1+\eta}{1-\eta} + 1\right)w_1^2 + w_2\right]. \tag{43}$$

It follows that

$$\left|a_3 - \mu a_2^2\right| \le \frac{(1-\lambda)\cos\gamma}{(1-\eta)\Phi_3(\alpha,\beta,m)} \times \left\{\left|\frac{2e^{-i\gamma}(1-\lambda)\cos\gamma}{1-\eta}\left(1+\eta-\mu\frac{2\Phi_3(\alpha,\beta,m)}{\Phi_2^2(\alpha,\beta,m)}\right)+1\right| \times \left|w_1\right|^2 + \left|w_2\right|\right\}. \tag{44}$$

Making use of Lemma 9 (36), we have

$$\left|a_3 - \mu a_2^2\right| \le \frac{(1-\lambda)\cos\gamma}{(1-\eta)\Phi_3(\alpha,\beta,m)} \times \left\{1 + \left[\left|\frac{2e^{-i\gamma}(1-\lambda)\cos\gamma}{1-\eta} \times \left(1+\eta-\mu\frac{2\Phi_3(\alpha,\beta,m)}{\Phi_2^2(\alpha,\beta,m)}\right)+1\right| - 1\right] \times \left|w_1\right|^2\right\} \tag{45}$$

or

$$\left|a_3 - \mu a_2^2\right| \le \frac{(1-\lambda)\cos\gamma}{(1-\eta)\Phi_3(\alpha,\beta,m)} \times \left[1 + \left(\sqrt{1 + M(2+M)\cos^2\gamma} - 1\right)\left|w_1\right|^2\right], \tag{46}$$

where

$$M = \frac{2(1-\lambda)}{1-\eta}\left(1+\eta-\mu\frac{2\Phi_3(\alpha,\beta,m)}{\Phi_2^2(\alpha,\beta,m)}\right). \tag{47}$$

Denote

$$F(x,y) = 1 + \left(\sqrt{1 + M(2+M)x^2} - 1\right)y^2, \tag{48}$$

where $x = \cos\gamma$, $y = |w_1|$, and $(x,y) \in [0,1]\times[0,1]$.

Simple calculation shows that the function $F(x,y)$ does not have a local maximum at any interior point of the open rectangle $(0,1)\times(0,1)$. Thus, the maximum must be attained at a boundary point. Since $F(x,0) = 1$, $F(0,y) = 1$, and $F(1,1) = |1+M|$, it follows that the maximal value of $F(x,y)$ may be $F(0,0) = 1$ or $F(1,1) = |1+M|$.

Therefore, from (46), we obtain

$$\left|a_3 - \mu a_2^2\right| \le \frac{(1-\lambda)\cos\gamma}{(1-\eta)\Phi_3(\alpha,\beta,m)} \max\{1, |1+M|\}, \tag{49}$$

where M is given by (47).

Consider first the case $|1+M| \ge 1$. If $\mu \le \sigma_1$, where σ_1 is given by (39), then $M \ge 0$, and from (49), we obtain

$$\left|a_3 - \mu a_2^2\right| \le \frac{(1-\lambda)\cos\gamma}{(1-\eta)^2\Phi_3(\alpha,\beta,m)} \times \left[\eta + 3 - 2\lambda(1+\eta) - \mu\frac{4(1-\lambda)\Phi_3(\alpha,\beta,m)}{\Phi_2^2(\alpha,\beta,m)}\right], \tag{50}$$

which is the first part of the inequality (38). If $\mu \ge \sigma_2$, where σ_2 is given by (40), then $M \le -2$, and it follows from (49) that

$$\left|a_3 - \mu a_2^2\right| \le \frac{(1-\lambda)\cos\gamma}{(1-\eta)^2\Phi_3(\alpha,\beta,m)} \times \left[\mu\frac{4(1-\lambda)\Phi_3(\alpha,\beta,m)}{\Phi_2^2(\alpha,\beta,m)} + 2\lambda(1+\eta) - \eta - 3\right], \tag{51}$$

and this is the third part of (38).

Next, suppose that $\sigma_1 \le \mu \le \sigma_2$. Then, $|1+M| \le 1$, and thus, from (49), we obtain

$$\left|a_3 - \mu a_2^2\right| \le \frac{(1-\lambda)\cos\gamma}{(1-\eta)\Phi_3(\alpha,\beta,m)}, \tag{52}$$

which is the second part of the inequality (38).

In view of Lemma 9, the results are sharp for $w(z) = z$ and $w(z) = z^2$ or one of their rotations. From (41), we obtain that the extremal functions are $\Psi(z, \theta, 1)$ and $\Psi(z, \theta, 0)$ defined by (34) with $\tau = 1$ and $\tau = 0$. □

Next, we consider the Fekete-Szegö problem for the class $\mathcal{S}^m_{\alpha,\beta}(\eta, \gamma, \lambda)$ with μ complex parameter.

Theorem 11. *Let $f \in \mathcal{S}^m_{\alpha,\beta}(\eta, \gamma, \lambda)$ be given by (1), and let μ be a complex number. Then,*

$$\begin{aligned} &\left|a_3 - \mu a_2^2\right| \\ &\quad \le \frac{(1-\lambda)\cos\gamma}{(1-\eta)\,\Phi_3(\alpha,\beta,m)} \\ &\qquad \times \max\left\{1, \left|\frac{2(1-\lambda)\cos\gamma}{1-\eta} \right.\right. \\ &\qquad\qquad \left.\left. \times \left(\mu \frac{2\Phi_3(\alpha,\beta,m)}{\Phi_2^2(\alpha,\beta,m)} - 1 - \eta\right) - e^{i\gamma}\right|\right\}. \end{aligned} \tag{53}$$

The result is sharp.

Proof. Assume that $f \in \mathcal{S}^m_{\alpha,\beta}(\eta, \gamma, \lambda)$. Making use of (43), we obtain

$$\begin{aligned} &\left|a_3 - \mu a_2^2\right| \\ &\quad \le \frac{(1-\lambda)\cos\gamma}{(1-\eta)\,\Phi_3(\alpha,\beta,m)} \\ &\qquad \times \left| w_2 - \left[\frac{2e^{-i\gamma}(1-\lambda)\cos\gamma}{1-\eta} \right.\right. \\ &\qquad\qquad \left.\left. \times \left(\mu \frac{2\Phi_3(\alpha,\beta,m)}{\Phi_2^2(\alpha,\beta,m)} - 1 - \eta\right) - 1\right] w_1^2 \right|. \end{aligned} \tag{54}$$

The inequality (53) follows as an application of Lemma 9 (37) with

$$s = \frac{2e^{-i\gamma}(1-\lambda)\cos\gamma}{1-\eta}\left(\mu \frac{2\Phi_3(\alpha,\beta,m)}{\Phi_2^2(\alpha,\beta,m)} - 1 - \eta\right) - 1. \tag{55}$$

The functions $\Psi(z, \theta, 1)$ and $\Psi(z, \theta, 0)$ defined by (34) with $\tau = 1$ and $\tau = 0$ show that the inequality (53) is sharp. □

Our Theorems 10 and 11 include several various results for special values of m, η, γ, and λ. For example, taking $m = \eta = \gamma = \lambda = 0$, in Theorem 10, we obtain the Fekete-Szegö inequalities for the class $\mathcal{S}^*$ (see [2, 11]). The special case $m = \eta = \lambda = 0$ leads to the Fekete-Szegö inequalities for the class $\mathcal{S}^*(\gamma)$ (see [2]). The Fekete-Szegö inequalities for the class $\mathcal{S}^*(\gamma, \lambda)$ (see [2]) are also included in Theorems 10 and 11.

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