Research Article

Existence and Multiplicity of Solutions to a Boundary Value Problem for Impulsive Differential Equations

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We investigate the existence and multiplicity of solutions to a boundary value problem for impulsive differential equations. By using critical point theory, some criteria are obtained to guarantee that the impulsive problem has at least one solution, at least two solutions, and infinitely many solutions. Some examples are given to illustrate the effectiveness of our results.

1. Introduction

In this paper, we will investigate the existence and multiplicity of solutions to the boundary value problem for impulsive differential equations:

$$
-\left(\rho(t)\left|u'(t)\right|^{p-2}u'(t)\right)' + s(t)\left|u(t)\right|^{p-2}u(t)
$$
$$
= f(t, u(t)), \quad t \neq t_i, \text{ a.e. } t \in [0, T],
$$
$$
\Delta\left(\rho(t_i)\left|u'(t_i)\right|^{p-2}u'(t_i)\right) = I_i(u(t_i)), \tag{1}
$$
$$
i = 1, 2, \ldots, k,
$$
$$
u(0) = u(T) = 0,
$$

where $p \geq 2$, $\rho(t), s(t) \in L^\infty[0, T]$, with ess $\inf_{t \in [0,T]} \rho(t) > 0$, ess $\inf_{t \in [0,T]} s(t) > 0$, and $1 \leq \rho(t) < +\infty$; $0 < s(t) < +\infty$; $0 = t_0 < t_1 \cdots < t_k < t_{k+1} = T$, $\Delta(\rho(t_i)|u'(t_i)|^{p-2}u'(t_i)) = \rho(t_i^+)|u'(t_i^+)|^{p-2}u'(t_i^+) - \rho(t_i^-)|u'(t_i^-)|^{p-2}u'(t_i^-)$, $f : [0, T] \times R \to R$ is continuous, $I_i : R \to R$ are continuous.

Recently, there have been many papers concerned with boundary value problems for impulsive differential equations. Impulsive effects exist widely in many evolution processes in which their states are changed abruptly at certain moments of time. The theory of impulsive differential systems has been developed by numerous mathematicians (see [1–6]).

Impulsive and periodic boundary value problems have been studied extensively in the literature. There have been many approaches to study periodic solutions of differential equations, such as the method of lower and upper solutions, fixed point theory, and coincidence degree theory (see [7–10]). However, the study of solutions for impulsive differential equations using variational method has received considerably less attention (see, [11–18]). Variational method is, to the best of our knowledge, novel and it may open a new approach to deal with nonlinear problems with some type of discontinuities such as impulses.

Teng and Zhang in [15] studied the existence of solutions to the boundary value problem for impulsive differential equations

$$
-\left(\left|u'(t)\right|^{p-2}u'(t)\right)' = f\left(t, u(t), u'(t)\right),
$$
$$
t \neq t_i, \text{ a.e. } t \in [0, T],
$$
$$
\Delta u'(t_i) = I_i(u(t_i)), \quad i = 1, 2, \ldots, l, \tag{2}
$$
$$
u(0) = u(T) = 0.
$$

By using variational methods and iterative methods they showed that there exists a solution for problem (2).

In this paper, we will need the following conditions.

(A) $F(t, u)$ is measurable in t for every $u \in \mathbb{R}$ and continuously differentiable in u for a.e. $t \in [0, T]$ and there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $b \in L^1(0, T; \mathbb{R}^+)$ such that

$$|F(t, u)| \le a(|u|) b(t), \qquad |f(t, u)| \le a(|u|) b(t), \quad (3)$$

for all $u \in \mathbb{R}$ and a.e. $t \in [0, T]$, where $F(t, u) = \int_0^u f(t, s) \mathrm{d}s$.

(H_0) There exist constants $a, b > 0$ and $r \in [0, p - 1)$ such that

$$|f(t, u)| \le a + b|u|^r, \quad \text{for } (t, u) \in [0, T] \times R. \quad (4)$$

(H_1) There exist constants $c > 0$ and $\tau \in [p, +\infty)$ such that

$$|f(t, u)| \le c\left(|u|^{\tau-1} + 1\right), \quad \text{for } (t, u) \in [0, T] \times R. \quad (5)$$

(H_2) There exist constants $a_i, b_i > 0$ and $r_i \in [0, p - 1)$ $(i = 1, 2, \ldots, k)$ such that

$$|I_i(u)| \le a_i + b_i|u|^{r_i}, \quad \text{for } u \in R. \quad (6)$$

(H_3) There exist constants $\mu > p$ and $M > 0$ such that

$$0 < \mu F(t, u) \le u f(t, u), \quad \text{for } t \in [0, T], \ |u| \ge M. \quad (7)$$

(H_4) $\lim_{u \to 0} f(t, u)/|u|^{p-1} = 0$ uniformly, for $t \in [0, T]$ and $u \in R$.

(H_5) there exist $\lambda > p$ and $\beta > \lambda - p$ such that

$$\limsup_{|x| \to \infty} \frac{F(t, x)}{|x|^\lambda} < \infty \quad \text{uniformly, for a.e. } t \in [0, T],$$

$$\liminf_{|x| \to \infty} \frac{x f(t, x) - 2F(t, x)}{|x|^\beta} > 0, \quad (8)$$

$$\text{uniformly for a.e. } t \in [0, T].$$

(H_6) there exists $\zeta_{ij} > 0$ such that

$$2\int_0^t I_i(s)\, \mathrm{d}s - I_i(t)\, t \ge 0 \quad \forall i \in \{1, 2, \ldots, k\}, \ |t| \ge \zeta_{ij}. \quad (9)$$

We recall some facts which will be used in the proof of our main results. It has been shown, for instance, in [19] that the set of all eigenvalues of the following problem:

$$-\left(|u'(t)|^{p-2} u'(t)\right)' + \lambda |u(t)|^{p-2} u(t) = 0,$$
$$t \ne t_i, \ t \in [0, T], \quad (10)$$
$$u(0) = u(T) = 0$$

is given by the sequence of positive numbers

$$\lambda_k = (p - 1)\left(\frac{k\pi_p}{T}\right)^p, \quad \text{for } k = 1, 2, \ldots, \quad (11)$$

where

$$\pi_p = 2 \int_0^1 \frac{1}{(1 - s^p)^{1/p}} \mathrm{d}s. \quad (12)$$

Each eigenvalue λ_k is simple with the associated eigenfunction

$$\varphi_k(x) = \sin_p\left(\frac{k\pi_p x}{T}\right), \quad \text{for } 0 \le x \le T. \quad (13)$$

X_k denotes the eigenspace associated to λ_k, then $W_0^{1,p}(0, T) = \overline{\bigoplus_{i \in N} X_i}$.

An outline of this paper is given as follows. In the next section, we present some preliminaries including some basic knowledge and critical point theory. In Section 3, by using the critical point theory, we will establish some sufficient conditions for the existence of solutions of system (1). In Section 4, some examples are given to verify and support the theoretical findings.

2. Preliminaries

In this section, we recall some basic facts which will be used in the proofs of our main results. In order to apply the critical point theory, we make a variational structure. From this variational structure, we can reduce the problem of finding solutions of (1) to the one of seeking the critical points of a corresponding functional.

In [20], the Sobolev space $W_0^{1,p}(0, T)$ be the endowed with the norm

$$\|u\|_{W_0^{1,p}} = \left(\int_0^T |u'(t)|^p \mathrm{d}t\right)^{1/p}, \quad u \in W_0^{1,p}(0, T). \quad (14)$$

Throughout the paper, we also consider the norm

$$\|u\| = \left(\int_0^T \left(\rho(t) |u'(t)|^p + s(t) |u(t)|^p\right) \mathrm{d}t\right)^{1/p}, \quad u \in W_0^{1,p}(0, T). \quad (15)$$

By Poincaré inequality:

$$\int_0^T |u'(t)|^p \mathrm{d}t \ge C \int_0^T |u(t)|^p \mathrm{d}t, \quad (16)$$

where $C = \lambda_1$ is precisely the largest $C > 0$ for which the above inequality holds true. Then $\int_0^T |u'(t)|^p \mathrm{d}t - \lambda_1 \int_0^T |u(t)|^p \mathrm{d}t \ge 0$ while it minimizes and equals to zero exactly on the ray generated by the first eigenfunction $\sin_p(\pi_p t/T)$.

Let us recall that

$$|u|_p = \left(\int_0^T |u(t)|^p \mathrm{d}t\right)^{1/p}, \qquad \|u\|_\infty = \max_{t \in [0, T]} |u(t)|. \quad (17)$$

We denote by $|\cdot|_p$ the usual L^p-norm. The n-dimensional Lebesgue measure of a set $E \subseteq R^n$ is denoted by $|E|$. By the Sobolev embedding theorem, $W_0^{1,p}(0, T) \hookrightarrow L^r[0, T]$

continuously for $r \in [1, +\infty)$, and there exists $\gamma_r > 0$ such that

$$|u|_r \leq \gamma_r \|u\|, \quad \forall u \in W_0^{1,p}(0,T). \tag{18}$$

Lemma 1. *There exist $C_1, C_2 > 0$ such that*

$$C_1 \|u\|_{W_0^{1,p}} \leq \|u\| \leq C_2 \|u\|_{W_0^{1,p}}, \quad \forall u \in W_0^{1,p}(0,T). \tag{19}$$

Proof. Since ess $\inf_{t \in [0,T]} \rho(t) > 0$, ess $\inf_{t \in [0,T]} s(t) > 0$, $1 \leq \rho(t) < +\infty$, and $0 < s(t) < +\infty$, we have that ess $\inf_{t \in [0,T]} \rho(t) := m_1 \geq 1$, ess $\inf_{t \in [0,T]} s(t) := m_2 > -\lambda_1$, where λ_1 is a positive number and that there exists $n_1 \in (0,1)$ such that $-m_2 \leq \lambda_1(1 - n_1)$. Thus, by Poincaré inequality, we have

$$(1 - n_1) \int_0^T |u'(t)|^p dt \geq (1 - n_1) \lambda_1 \int_0^T |u(t)|^p dt$$
$$\geq -m_2 \int_0^T |u(t)|^p dt, \tag{20}$$

for all $u \in W_0^{1,p}(0,T)$. Thereby, for every $u \in W_0^{1,p}(0,T)$,

$$\|u\|^p = \int_0^T \rho(t) |u'(t)|^p dt + \int_0^T s(t) |u(t)|^p dt$$
$$\geq m_1 \int_0^T |u'(t)|^p dt$$
$$= m_1 \|u\|_{W_0^{1,p}}^p$$
$$= C_1^p \|u\|_{W_0^{1,p}}^p. \tag{21}$$

On the other hand, by Poincaré inequality, one has

$$\|u\|^p = \int_0^T \rho(t) |u'(t)|^p dt + \int_0^T s(t) |u(t)|^p dt$$
$$\leq \|\rho\|_\infty \int_0^T |u'(t)|^p dt + \|s\|_\infty \int_0^T |u(t)|^p dt$$
$$\leq \left(\|\rho\|_\infty + \frac{\|s\|_\infty}{\lambda_1} \right) \int_0^T |u'(t)|^p dt \tag{22}$$
$$= \left(\|\rho\|_\infty + \frac{\|s\|_\infty}{\lambda_1} \right) \|u\|_{W_0^{1,p}}^p$$
$$= C_2^p \|u\|_{W_0^{1,p}}^p.$$

Take $C_1 = (m_1 + n_1 - 1)^{1/p}$, $C_2 = (\|\rho\|_\infty + \|s\|_\infty / \lambda_1)^{1/p}$, then

$$C_1 \|u\|_{W_0^{1,p}} \leq \|u\| \leq C_2 \|u\|_{W_0^{1,p}}. \tag{23}$$

The proof is complete. □

Lemma 2. *There exists $C_3 > 0$ such that if $u \in W_0^{1,p}(0,T)$, then*

$$\|u\|_\infty \leq C_3 \|u\|. \tag{24}$$

Proof. If $u \in W_0^{1,p}(0,T)$, it follows from the mean value theorem that

$$\frac{1}{T} \int_0^T u(s) ds = u(\xi), \tag{25}$$

for some $\xi \in (0,T)$. Hence, for $t \in [0,T]$, by Hölder inequality and Poincaré inequality

$$|u(t)| = \left| u(\xi) + \int_\xi^t u'(s) ds \right|$$
$$\leq |u(\xi)| + \int_0^T |u'(s)| ds$$
$$\leq \frac{1}{T} \left| \int_0^T u(s) ds \right| + T^{1/q} \left(\int_0^T |u'(t)|^p ds \right)^{1/p} \tag{26}$$
$$\leq \left(\left(\frac{C}{T} \right)^{1/p} + T^{1/q} \right) \left(\int_0^T |u'(t)|^p ds \right)^{1/p}$$
$$\leq \left(\left(\frac{C}{T} \right)^{1/p} + T^{1/q} \right) \frac{1}{C_1} \|u\|.$$

Hence, $\|u\|_\infty \leq ((C/T)^{1/p} + T^{1/q})(1/C_1)\|u\| = C_3\|u\|$. The proof is complete. □

Take $v \in W_0^{1,p}(0,T)$ and multiply the two sides of the equality

$$-\left(\rho(t) |u'(t)|^{p-2} u'(t) \right)' + s(t) |u(t)|^{p-2} u(t) = f(t, u(t)) \tag{27}$$

by v and integrate it from 0 to T, we have

$$-\int_0^T \left(\rho(t) |u'(t)|^{p-2} u'(t) \right)' v(t) dt$$
$$+ \int_0^T s(t) |u(t)|^{p-2} u(t) v(t) dt \tag{28}$$
$$= \int_0^T f(t, u(t)) v(t) dt.$$

Moreover,

$$-\int_0^T \left(\rho(t) |u'(t)|^{p-2} u'(t) \right)' v(t) dt$$
$$= -\sum_{i=0}^k \int_{t_i}^{t_{i+1}} \left(\rho(t) |u'(t)|^{p-2} u'(t) \right)' v(t) dt$$

$$= \sum_{i=0}^{k} \left(\rho\left(t_{i+1}^{-}\right) \left| u'\left(t_{i+1}^{-}\right)\right|^{p-2} u'\left(t_{i+1}^{-}\right) v\left(t_{i+1}^{-}\right) \right.$$

$$- \rho\left(t_{i}^{+}\right) \left| u'\left(t_{i}^{+}\right)\right|^{p-2} u'\left(t_{i}^{+}\right) v\left(t_{i}^{+}\right)$$

$$\left. - \int_{t_i}^{t_{i+1}} \rho\left(t\right) \left| u'\left(t\right)\right|^{p-2} u'\left(t\right) v'\left(t\right) \mathrm{d}t \right)$$

$$= \sum_{i=1}^{k} \left(\rho\left(t_{i}^{+}\right) \left| u'\left(t_{i}^{+}\right)\right|^{p-2} u'\left(t_{i}^{+}\right) - \rho\left(t_{i}^{-}\right) \right.$$

$$\left. \times \left| u'\left(t_{i}^{-}\right)\right|^{p-2} u'\left(t_{i}^{-}\right)\right) - \rho\left(T\right) \left| u'\left(T\right)\right|^{p-2} u'\left(T\right) v\left(T\right)$$

$$+ \rho\left(0\right) \left| u'\left(0\right)\right|^{p-2} u'\left(0\right) v\left(0\right)$$

$$+ \int_{0}^{T} \rho\left(t\right) \left| u'\left(t\right)\right|^{p-2} u'\left(t\right) v'\left(t\right) \mathrm{d}t$$

$$= \sum_{i=1}^{k} I_i\left(u\left(t_i\right)\right) v\left(t_i\right) + \int_{0}^{T} \rho\left(t\right) \left| u'\left(t\right)\right|^{p-2} u'\left(t\right) v'\left(t\right) \mathrm{d}t. \tag{29}$$

Combining (28), we have

$$\int_{0}^{T} \rho\left(t\right) \left| u'\left(t\right)\right|^{p-2} u'\left(t\right) v'\left(t\right) \mathrm{d}t$$

$$+ \int_{0}^{T} s\left(t\right) \left| u\left(t\right)\right|^{p-2} u\left(t\right) v\left(t\right) \mathrm{d}t + \sum_{i=1}^{k} I_i\left(u\left(t_i\right)\right) v\left(t_i\right)$$

$$= \int_{0}^{T} f\left(t, u\left(t\right)\right) v\left(t\right) \mathrm{d}t. \tag{30}$$

Considering the above, we introduce the following concept solution for problem (1).

Definition 3. We say that a function $u \in W_0^{1,p}(0,T)$ is a solution of problem (1) if the identity

$$\int_{0}^{T} \rho\left(t\right) \left| u'\left(t\right)\right|^{p-2} u'\left(t\right) v'\left(t\right) \mathrm{d}t$$

$$+ \int_{0}^{T} s\left(t\right) \left| u\left(t\right)\right|^{p-2} u\left(t\right) v\left(t\right) \mathrm{d}t + \sum_{i=1}^{k} I_i\left(u\left(t_i\right)\right) v\left(t_i\right)$$

$$= \int_{0}^{T} f\left(t, u\left(t\right)\right) v\left(t\right) \mathrm{d}t \tag{31}$$

holds for any $v \in W_0^{1,p}(0,T)$.

Consider the functional $\varphi : W_0^{1,p}(0,T) \to R$ defined by

$$\varphi\left(u\right) = \frac{1}{p} \|u\|^p + \sum_{i=1}^{k} \int_{0}^{u(t_i)} I_i\left(t\right) \mathrm{d}t - \int_{0}^{T} F\left(t, u\left(t\right)\right) \mathrm{d}t. \tag{32}$$

Using the continuity of f and I_i, $i = 1, 2, \ldots, k$, one has that $\varphi \in C^1(W_0^{1,p}(0,T), R)$. For any $v \in W_0^{1,p}(0,T)$, we have

$$\varphi'\left(u\right) v = \int_{0}^{T} \rho\left(t\right) \left| u'\left(t\right)\right|^{p-2} u'\left(t\right) v'\left(t\right) \mathrm{d}t$$

$$+ \int_{0}^{T} s\left(t\right) \left| u\left(t\right)\right|^{p-2} u\left(t\right) v\left(t\right) \mathrm{d}t \tag{33}$$

$$+ \sum_{i=1}^{k} I_i\left(u\left(t_i\right)\right) v\left(t_i\right) - \int_{0}^{T} f\left(t, u\left(t\right)\right) v\left(t\right) \mathrm{d}t.$$

Thus, the solutions of problem (1) are the corresponding critical points of φ.

Definition 4. Let X be a normed space. A minimizing sequence for a function $\varphi : X \to (-\infty, +\infty)$ is a sequence u_k such that $\varphi(u_k) \to \inf \varphi$ whenever $k \to +\infty$.

Definition 5. Let X be a Banach space and let $\varphi : X \to (-\infty, +\infty)$. φ is said to be sequentially weakly lower semi-continuous if $\lim \inf_{k \to +\infty} \varphi(x_k) \geq \varphi(x)$ as $x_k \rightharpoonup x$ in X.

Definition 6. Let E be a Banach space and let $c \in R$. For any sequence $\{u_k\}$ in E, if $\varphi(u_k)$ is bounded and $\varphi'(u_k) \to 0$ as $k \to +\infty$ possesses a convergent subsequence, then we say that φ satisfies the Palais-Smale condition (denoted by PS condition for short). We say that φ satisfies the Palais-Smale condition at level c (denoted by $(\text{PS})_c$ condition for short) if there exists a sequence $\{u_k\}$ in E such that $\varphi(u_k) \to c$ and $\varphi'(u_k) \to 0$ as $k \to +\infty$ implies that c is a critical value of φ.

Definition 7. Let E be a Banach space and let $\varphi : E \to (-\infty, +\infty)$. φ is said to be coercive if $\varphi(u) \to +\infty$ as $\|u\| \to +\infty$.

Lemma 8 (see [12]). *If φ is sequentially weakly lower semi-continuous on a reflexive Banach space X and has a bounded minimizing sequence, then φ has a minimum on X.*

Definition 9. Let X be a real Banach space with a direct sum decomposition $X = X^1 \oplus X^2$. The functional $\varphi \in C^1(X, R)$ is said to have a local linking at 0, with respect to (X^1, X^2), if, for some $r > 0$,

(i) $\varphi(u) \geq 0$, $u \in X^1$, $\|u\| \leq r$,

(ii) $\varphi(u) \leq 0$, $u \in X^2$, $\|u\| \leq r$.

If φ has a local linking at 0, then 0 is critical point (the trivial one). Suppose, furthermore, that there are two sequences of finite dimensional subspaces $X_1^1 \subset X_2^1 \subset \cdots \subset X^1$ and $X_1^2 \subset X_2^2 \subset \cdots \subset X^2$ such that

$$X^1 = \overline{\bigcup_n X_n^1}, \qquad X^2 = \overline{\bigcup_n X_n^2}. \tag{34}$$

Definition 10 (see [13, Definition 2.2]). Let $I \in C^1(X, \mathbb{R})$. The functional I satisfies the $(C)^*$ condition if every sequence (u_{α_n}) such that α_n is admissible and

$$u_{\alpha_n} \in X_{\alpha_n}, \qquad \sup \left| I\left(u_{\alpha_n}\right) \right| < \infty,$$

$$\left(1 + \left\| u_{\alpha_n} \right\| \right) I'\left(u_{\alpha_n}\right) \longrightarrow 0$$

(35)

contains a subsequence which converges to a critical point of I.

Lemma 11 (see [13]). *Suppose that $\varphi \in C^1(X, R)$ satisfies the following assumptions:*

(1) *φ satisfies the $(C)^*$ condition,*

(2) *φ has a local linking at 0,*

(3) *φ maps bounded sets into bounded sets,*

(4) *for every $m \in N$, $\varphi(u) \to -\infty$ as $\|u\| \to +\infty$, $u \in X_m^1 \oplus X^2$.*

Then φ has at least two critical points.

Lemma 12 (see [14]). *Let E be a Banach space. Let $\varphi \in C^1(X, R)$ be an even functional which satisfies the PS condition and $\varphi(0) = 0$. If $E = V \oplus Y$, where V is finite dimensional, and φ satisfies the following conditions:*

(1) *there exist constants ρ, α such that $\varphi|_{\partial B_\rho \cap Y} \geq \alpha$, where $B_\rho = \{x \in E : \|x\| < \rho\}$,*

(2) *for each finite-dimensional subspace $W \subset E$ there is $R = R(W)$ such that $\varphi(u) \leq 0$, for all $u \in W$ with $\|u\| \geq R$.*

Then φ has an unbounded sequence of critical values.

3. Existence of Periodic Solutions

Theorem 13. *Assume that (A), (H_0), and (H_2) are satisfied, then problem (1) has at least one solution.*

Proof. Let $M_1 = \max\{a_1, a_2, \ldots, a_k\}$, $M_2 = \max\{b_1, b_2, \ldots, b_k\}$. By Lemma 2, we have

$$\varphi(u) = \frac{1}{p}\|u\|^p + \sum_{i=1}^{k} \int_0^{u(t_i)} I_i(t)\, dt - \int_0^T F(t, u(t))\, dt$$

$$\geq \frac{1}{p}\|u\|^p - \sum_{i=1}^{k} \int_0^{u(t_i)} \left(a_i + b_i |u|^{r_i}\right) dt$$

$$- \int_0^T \left(a|u| + b|u|^{r+1}\right) dt$$

$$\geq \frac{1}{p}\|u\|^p - kM_1\|u\|_\infty$$

$$- M_2 \sum_{i=1}^{k} \|u\|_\infty^{r_i+1} - aT\|u\|_\infty - bT\|u\|_\infty^{r+1}$$

$$\geq \frac{1}{p}\|u\|^p - kM_1 C_3 \|u\|$$

$$- M_2 \sum_{i=1}^{k} C_3^{r_i+1}\|u\|^{r_i+1} - aTC_3\|u\| - bTC_3^{r+1}\|u\|^{r+1},$$

(36)

for all $u \in W_0^{1,p}(0, T)$. This implies that $\lim_{\|u\| \to \infty} \varphi(u) = \infty$, and φ is coercive.

On the other hand, we show that φ is weakly lower semicontinuous. If $\{u_k\}_{k \in N} \subset W_0^{1,p}(0, T)$, $u_k \rightharpoonup u$, then we have that $\{u_k\}_{k \in N}$ converges uniformly to u on $[0, T]$ and $\liminf_{k \to \infty} \|u_k\| \geq \|u\|$. Thus

$$\liminf_{k \to \infty} \varphi(u_k) = \liminf_{k \to \infty} \left(\frac{1}{p}\|u_k\|^p + \sum_{i=1}^{k} \int_0^{u_k(t_i)} I_i(t)\, dt \right.$$

$$\left. - \int_0^T F(t, u_k(t))\, dt \right)$$

$$\geq \frac{1}{p}\|u\|^p + \sum_{i=1}^{k} \int_0^{u(t_i)} I_i(t)\, dt - \int_0^T F(t, u(t))\, dt$$

$$= \varphi(u).$$

(37)

By Lemma 8, φ has a minimum point on $W_0^{1,p}(0, T)$, which is a critical point of φ. Hence, problem (1) has at least one solution. The proof is complete. □

We readily have the following corollary.

Corollary 14. *Assume that (A), (H_0), and (H_2) are satisfied and f and the impulsive functions I_i $(i = 1, 2, \ldots, k)$ are bounded. Then problem (1) has at least one solution.*

Lemma 15. *Assume that (H_1), (H_2), and (H_3) are satisfied, then $\varphi(u)$ satisfies the PS condition.*

Proof. Assume that $\{u_n\} \subset W_0^{1,p}(0, T)$ satisfies that $\varphi(u_n)$ is bounded and $\varphi'(u_n) \to 0$ as $n \to +\infty$. We will prove that the sequence $\{u_n\}$ is bounded.

It follows from (H_1), (H_2), (H_3), and Lemma 2, we have

$$\mu\varphi(u_n) - \varphi'(u_n) u_n$$

$$= \left(\frac{\mu}{p} - 1\right)\|u_n\|^p$$

$$+ \sum_{i=1}^{k} \left[\mu \int_0^{u_n(t_i)} I_i(t)\, dt - I_i(u_n(t_i)) u_n(t_i) \right]$$

$$-\int_0^T \left(\mu F\left(t, u_n(t)\right) - f\left(t, u_n(t)\right) u_n(t)\right) dt$$

$$\geq \left(\frac{\mu}{p} - 1\right) \|u_n\|^p - (\mu + 1)$$

$$\times \left(k M_1 C_3 \|u_n\| + M_2 \sum_{i=1}^k C_3^{r_i+1} \|u_n\|^{r_i+1}\right)$$

$$-\int_{\{|u_n| \leq u_0\}} \left(\mu F\left(t, u_n(t)\right) - f\left(t, u_n(t)\right) u_n(t)\right) dt$$

$$\geq \left(\frac{\mu}{p} - 1\right) \|u_n\|^p - (\mu + 1)$$

$$\times \left(k M_1 C_3 \|u_n\| + M_2 \sum_{i=1}^k C_3^{r_i+1} \|u_n\|^{r_i+1}\right) - C_4.$$

$$(38)$$

Hence, $\{u_n\}$ is bounded in $W_0^{1,p}(0, T)$.

Since $W_0^{1,p}(0, T)$ is a reflexive Banach space, passing to a subsequence if necessary, we may assume that there is a $u \in W_0^{1,p}(0, T)$ such that

$$u_n \rightharpoonup u \quad \text{in } W_0^{1,p}(0, T),$$

$$u_n \longrightarrow u \quad \text{in } L^p[0, T], \tag{39}$$

$$\{u_n\} \text{ converges uniformly to } u \text{ on } [0, T].$$

Notice that

$$\left(\varphi'(u_n) - \varphi'(u), u_n - u\right)$$

$$= \int_0^T \rho(t) \left(\left|u_n'(t)\right|^{p-2} u_n'(t) - \left|u'(t)\right|^{p-2} u'(t)\right)$$

$$\times \left(u_n'(t) - u'(t)\right) dt$$

$$+ \int_0^T s(t) \left(\left|u_n(t)\right|^{p-2} u_n(t) - \left|u(t)\right|^{p-2} u(t)\right)$$

$$\times \left(u_n(t) - u(t)\right) dt$$

$$+ \sum_{i=1}^k \left(I_i\left(u_n(t_i)\right) - I_i\left(u(t_i)\right)\right) \left(u_n(t_i) - u(t_i)\right)$$

$$- \int_0^T \left(f\left(t, u_n(t)\right) - f\left(t, u(t)\right)\right) \left(u_n(t) - u(t)\right) dt.$$

$$(40)$$

Recalling the following well-known inequality: for any $x, y \in R^N$,

$$\left(|x|^{p-2} x - |y|^{p-2} y\right)(x - y) \geq c_p |x - y|^p, \quad p \geq 2, \tag{41}$$

for some constant c_p (Lemma 4.2 in [21]) and using Schwarz inequality, we have

$$c_p \int_0^T \rho(t) \left|u_n'(t) - u'(t)\right|^p dt + c_p \int_0^T s(t) \left|u_n(t) - u(t)\right|^p dt$$

$$\leq \left\|\varphi'(u_n) - \varphi'(u)\right\| \|u_n - u\|$$

$$- \sum_{i=1}^k \left(I_i\left(u_n(t_i)\right) - I_i\left(u(t_i)\right)\right) \left(u_n(t_i) - u(t_i)\right)$$

$$+ \int_0^T \left(f\left(t, u_n(t)\right) - f\left(t, u(t)\right)\right) \left(u_n(t) - u(t)\right) dt.$$

$$(42)$$

Since

$$\left(I_i\left(u_n(t_i)\right) - I_i\left(u(t_i)\right)\right) \left(u_n(t_i) - u(t_i)\right)$$

$$\leq \left(\left|I_i\left(u_n(t_i)\right)\right| + \left|I_i\left(u(t_i)\right)\right|\right) \|u_n - u\|_\infty. \tag{43}$$

By the assumption (H_1), we have

$$\int_0^T \left(f\left(t, u_n(t)\right) - f\left(t, u(t)\right)\right) \left(u_n(t) - u(t)\right) dt$$

$$\leq \int_0^T \left(\left|f\left(t, u_n(t)\right)\right| + \left|f\left(t, u(t)\right)\right|\right) \left|u_n(t) - u(t)\right| dt$$

$$\leq c \int_0^T \left(2 + |u_n|^{\tau-1} + |u|^{\tau-1}\right) \left|u_n(t) - u(t)\right| dt$$

$$\leq C_5 \|u_n - u\|_\infty \left(1 + |u_n|_{\tau-1}^{\tau-1} + |u|_{\tau-1}^{\tau-1}\right).$$

$$(44)$$

From (39), it follows that $u_n \to u$ in $W_0^{1,p}(0, T)$. Thus, $\varphi(u)$ satisfies the PS condition. The proof is complete. □

Lemma 16. *Assume that* (A), (H_2), (H_5), *and* (H_6) *are satisfied, then* φ *satisfies the* $(C)^*$ *condition.*

Proof. Let $\{u_{\alpha_n}\}$ be a sequence in $W_0^{1,p}(0, T)$ such that α_n is admissible and

$$u_{\alpha_n} \in X_{\alpha_n}, \quad \sup \left|\varphi\left(u_{\alpha_n}\right)\right| < +\infty,$$

$$\left(1 + \|u_{\alpha_n}\|\right) \varphi'\left(u_{\alpha_n}\right) \longrightarrow 0, \tag{45}$$

then there exist a constant $C_4 > 0$ such that

$$\left|\varphi\left(u_{\alpha_n}\right)\right| \leq C_4, \quad \left(1 + \|u_{\alpha_n}\|\right) \varphi'\left(u_{\alpha_n}\right) \leq C_4, \tag{46}$$

for all large n. On the other hand, by (H_5), there are constants $C_5 > 0$ and $\rho_1 > 0$ such that

$$F(t, x) \leq C_5 |x|^\lambda, \tag{47}$$

for all $|x| \geq \rho_1$ and a.e. $t \in [0, T]$. By (A) one has

$$|F(t, x)| \leq \max_{s \in [0, \rho_1]} a(s) b(t), \tag{48}$$

for all $|x| \leq \rho_1$ and a.e. $t \in [0, T]$. It follows from (47) and (48) that

$$|F(t, x)| \leq \max_{s \in [0, \rho_1]} a(s) b(t) + C_5 |x|^\lambda. \qquad (49)$$

From (H_2) and Lemma 2, we have that

$$\left| \sum_{i=1}^{k} \int_0^{|u(t_i)|} I_i(t) \, dt \right| \leq \sum_{i=1}^{k} \int_0^{|u(t_i)|} \left(a_i + b_i |t|^{r_i} \right) dt$$

$$\leq M_1 k \|u\|_\infty + M_2 \sum_{i=1}^{k} \|u\|_\infty^{r_i + 1} \qquad (50)$$

$$\leq M_1 k C_3 \|u\| + M_2 C_3 \sum_{i=1}^{k} \|u\|^{r_i + 1}$$

for all $u \in W_0^{1,p}(0, T)$, where $M_1 = \{a_1, a_2, \ldots, a_k\}$, $M_2 = \{b_1, b_2, \ldots, b_k\}$. Combining (49), (50), and Hölder's inequality, we have

$$\frac{1}{p} \left\| u_{\alpha_n} \right\|^p = \varphi\left(u_{\alpha_n} \right) - \sum_{i=1}^{k} \int_0^{u_{\alpha_n}(t_i)} I_i(t) \, dt$$

$$+ \int_0^T F\left(t, u_{\alpha_n}(t) \right) dt$$

$$\leq C_4 + M_1 k C_3 \|u\| + M_2 C_3 \sum_{i=1}^{k} \|u\|^{r_i + 1}$$

$$+ C_5 \int_0^T \left| u_{\alpha_n}(t) \right|^\lambda dt + \max_{s \in [0, \rho_1]} a(s) \int_0^T b(t) \, dt$$

$$\leq C_4 + M_1 k C_3 \|u\| + M_2 C_3 \sum_{i=1}^{k} \|u\|^{r_i + 1}$$

$$+ C_5 \int_0^T \left| u_{\alpha_n}(t) \right|^\lambda dt + C_6, \qquad (51)$$

for all large n, where $C_6 = \max_{s \in [0, \rho_1]} a(s) \int_0^T b(t) dt$. On the other hand, by (H_5), there exist $C_7 > 0$ and $\rho_2 > 0$ such that

$$xf(t, x) - 2F(t, x) \geq C_7 |x|^\beta, \qquad (52)$$

for all $|x| \geq \rho_2$ and a.e. $t \in [0, T]$. By (A),

$$|xf(t, x) - 2F(t, x)| \leq C_8 b(t), \qquad (53)$$

for all $|x| \leq \rho_2$ and a.e. $t \in [0, T]$, where $C_8 = (2 + \rho_2) \max_{s \in [0, \rho_2]} a(s)$. Combining (52) and (53), one has

$$(f(t, x), x) - 2F(t, x) \geq C_7 |x|^\beta - C_7 \rho_2^\beta - C_8 b(t), \qquad (54)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$. According to (H_6), there exists $C_9 > 0$ such that

$$2 \int_0^t I_{ij}(s) \, ds - I_{ij}(t) t \geq -C_9 \quad \forall i = 1, 2, \ldots, k, \ t \in \mathbb{R}. \qquad (55)$$

Thus by (46), (54), and (55), we obtain

$$(p + 1) C_4 \geq p\varphi\left(u_{\alpha_n} \right) - \left\langle \varphi'\left(u_{\alpha_n} \right), u_{\alpha_n} \right\rangle$$

$$= p \sum_{i=1}^{k} \int_0^{u_{\alpha_n}(t_i)} I_i(t) \, dt - u_{\alpha_n} I_i\left(u_{\alpha_n} \right)$$

$$+ \int_0^T \left[u_{\alpha_n}(t) f\left(t, u_{\alpha_n}(t) \right) - pF\left(t, u_{\alpha_n}(t) \right) \right] dt$$

$$= \sum_{i=1}^{k} \left(p \int_0^{u_{\alpha_n}(t_i)} I_i(t) \, dt - I_i\left(u_{\alpha_n}(t_i) \right) u_{\alpha_n}(t_i) \right)$$

$$+ \int_0^T \left[\left(\nabla F\left(t, u_{\alpha_n}(t) \right), u_{\alpha_n}(t) \right) \right.$$

$$\left. - pF\left(t, u_{\alpha_n}(t) \right) \right] dt$$

$$\geq -kC_9 + C_7 \int_0^T \left| u_{\alpha_n} \right|^\beta dt - C_7 \rho_2^\beta T - C_8$$

$$\times \int_0^T b(t) \, dt, \qquad (56)$$

for all large n. From (56), $\int_0^T |u_{\alpha_n}|^\beta dt$ is bounded. If $\beta > \lambda$, by Hölder's inequality, we have

$$\int_0^T \left| u_{\alpha_n} \right|^\lambda dt \leq T^{(\beta - \lambda)/\beta} \left(\int_0^T \left| u_{\alpha_n} \right|^\beta dt \right)^{\lambda/\beta}. \qquad (57)$$

Since $\xi_{ij} \in [0, 1)$ for all $i = 1, 2, \ldots, k$, by (51) and (57), $\{u_{\alpha_n}\}$ is bounded in $W_0^{1,p}(0, T)$. If $\beta \leq \lambda$, by Lemma 2, we obtain

$$\int_0^T \left| u_{\alpha_n}(t) \right|^\lambda dt = \int_0^T \left| u_{\alpha_n}(t) \right|^\beta \left| u_{\alpha_n}(t) \right|^{\lambda - \beta} dt$$

$$\leq \left\| u_{\alpha_n} \right\|_\infty^{\lambda - \beta} \int_0^T \left| u_{\alpha_n}(t) \right|^\beta dt \qquad (58)$$

$$\leq C_1^{\lambda - \beta} \left\| u_{\alpha_n} \right\|^{\lambda - \beta} \int_0^T \left| u_{\alpha_n}(t) \right|^\beta dt.$$

Since $\xi_{ij} \in [0, 1)$, $\lambda - \beta < 2$, by (51) and (58), $\{u_{\alpha_n}\}$ is also bounded in $W_0^{1,p}(0, T)$. Hence, $\{u_{\alpha_n}\}$ is also bounded in $W_0^{1,p}(0, T)$. Going if necessary to a subsequence, we can assume that $u_{\alpha_n} \rightharpoonup u$ in $W_0^{1,p}(0, T)$. As the same the proof of Lemma 15, Therefore, $u_{\alpha_n} \to u$ in $W_0^{1,p}(0, T)$. Hence φ satisfies the $(C)^*$ condition. $\quad\square$

Theorem 17. *Assume that* (A), (H_1), (H_2), (H_3), (H_5), *and* (H_6) *are satisfied and the following conditions hold.*

(H_7) *There exist constants $\delta > 0$, $K > 0$, $\gamma > 0$ such that*

$$\frac{\gamma}{p}|u|^p + \frac{2kM_1}{T}|u| \le F(t,u) \le \frac{1}{2p\gamma_p^p}|u|^p + K|u|^{p+1},$$

(59)

$$\text{for } t \in [0,T], \ |u| \le \delta.$$

(H_8) $I_i(u)$ $(i = 1, 2, \ldots, k)$ *is nondecreasing.*

Then the problem (1) *has at least two critical points.*

Proof. Let $X_n^1 = \text{span}\{\lambda_{k+1}, \lambda_{k+2}, \ldots, \lambda_{k+n}\}$, $X_n^2 = X^2 = (X^1)^\perp$ for $n \in N$. Then $X^j = \bigcup_{n \in N} X_n^j$, $j = 1, 2$. If $u \in X^1$ one has $|u|_p \le \gamma_p \|u\|$, and if $u \in X^2$, we have $\|u\|^p \le \gamma|u|_p^p$.

Step 1. φ has a local linking at 0 with respect to (X^1, X^2).

For $u \in X^1$, using (H_7) we have

$$\sum_{i=1}^{k} \int_0^{u(t_i)} I_i(t)\,dt \ge 0.$$

(60)

It follows from (H_8) that

$$\varphi(u) = \frac{1}{p}\|u\|^p + \sum_{i=1}^{k} \int_0^{u(t_i)} I_i(t)\,dt - \int_0^T F(t, u(t))\,dt$$

$$\ge \frac{1}{p}\|u\|^p - \frac{1}{2p\gamma_p^p}|u|_p^p - K|u|_{p+1}^{p+1}$$

$$\ge \frac{1}{p}\|u\|^p - \frac{1}{2p\gamma_p^p}\gamma_p^p\|u\|^p - K\gamma_{p+1}^{p+1}\|u\|^{p+1}$$

$$= \frac{1}{2p}\|u\|^p - K\gamma_{p+1}^{p+1}\|u\|^{p+1}.$$

(61)

Thus, $\varphi(u) \ge 0$ for $u \in X^1$ with $\|u\| \le r_1$, where $r_1 > 0$ is small enough.

For $u \in X^2$, with $\|u\| \le r_2 := \delta/C_3$, we have $|u| \le |u|_\infty \le C_3\|u\| \le \delta$ since $\dim X^2 = k < +\infty$. Thus, From (H_2) and (H_8) it follows that

$$\varphi(u) = \frac{1}{p}\|u\|^p + \sum_{i=1}^{k} \int_0^{u(t_i)} I_i(t)\,dt$$

$$- \int_0^T F(t, u(t))\,dt \le \frac{1}{p}\|u\|^p$$

$$+ \sum_{i=1}^{k} \int_0^{u(t_i)} \left(a_i + b_i|u|^{r_i}\right)$$

$$- \int_0^T \left(\frac{\gamma}{p}|u|^p + \frac{2kM_1}{T}|u|\right) dt$$

$$\le \frac{\gamma}{p}|u|_p^p + kM_1\|u\|_\infty + M_2$$

$$\times \sum_{i=1}^{k} \|u\|_\infty^{r_i+1} - \frac{\gamma}{p}|u|_p^p - 2kM_1\|u\|_\infty$$

$$\le -kM_1\|u\|_\infty + M_2\sum_{i=1}^{k} \|u\|_\infty^{r_i+1} \le 0.$$

(62)

Take $r = \min\{r_1, r_2\}$. We know that φ has a local linking at 0 with respect to (X^1, X^2).

Step 2. φ maps bounded sets into bounded sets.

Assume $\|u\| \le R_0$, where R_0 is a constant. By (H_8), one has

$$\varphi(u) \le \frac{1}{p}\|u\|^p + \sum_{i=1}^{k} \int_0^{u(t_i)} I_i(t)\,dt$$

$$+ \int_0^T |F(t, u(t))|\,dt \le \frac{1}{p}\|u\|^p + kM_1\|u\|_\infty$$

$$+ M_2\sum_{i=1}^{k} \|u\|_\infty^{r_i+1} + \frac{1}{2p}\|u\|^p + K\gamma_{p+1}^{p+1}\|u\|^{p+1}$$

$$\le \frac{3}{2p}\|u\|^p + kM_1C_3\|u\| + M_2$$

(63)

$$\times \sum_{i=1}^{k} C_3^{r_i+1}\|u\|^{r_i+1} + K\gamma_{p+1}^{p+1}\|u\|^{p+1}$$

$$\le \frac{3}{2p}\|R_0\|^p + kM_1C_3\|R_0\| + M_2$$

$$\times \sum_{i=1}^{k} C_3^{r_i+1}\|R_0\|^{r_i+1} + K\gamma_{p+1}^{p+1}\|R_0\|^{p+1} < \infty,$$

which implies that φ maps bounded sets into bounded sets.

Step 3. For every $m \in N$, $\varphi(u) \to -\infty$ as $\|u\| \to +\infty$, $u \in X_m^1 \oplus X^2$.

By (H_5), for any $M_0 > 0$, there exists a constant $h(M_0)$ such that $F(t, u(t)) \ge M|u|^p - h(M_0)$ for all $(t, u) \in [0,T] \times R$. Since $\dim(X_m^1 \oplus X^2)$ is of finite dimension, there exists $\gamma > 0$ such that $\|u\|^p \le \gamma|u|_p^p$ for all $u \in X_m^1 \oplus X^2$, which implies

$$\varphi(u) \le \frac{1}{p}\|u\|^p + \sum_{i=1}^{k} \int_0^{u(t_i)} I_i(t)\,dt$$

$$- M|u|_p^p + h(M)T$$

$$\le \frac{1}{p}\|u\|^p + kM_1\|u\|_\infty + M_2$$

$$\times \sum_{i=1}^{k} \|u\|_{\infty}^{r_i+1} - \frac{M}{\gamma} \|u\|^p + h(M) T$$

$$\leq \left(\frac{1}{p} - \frac{M}{\gamma} \right) \|u\|^p + k M_1 C_3 \|u\|$$

$$+ M_2 \sum_{i=1}^{k} C_3^{r_i+1} \|u\|^{r_i+1} + h(M) T. \tag{64}$$

Choosing $M > \gamma/p$, we have $\varphi(u) \to -\infty$ as $\|u\| \to +\infty$, $u \in X_m^1 \oplus X^2$.

Summing up the above, and by Lemma 16, φ satisfies all the assumptions of Lemma 11. Hence by Lemma 11, problem (1) has at least one nontrivial solution. The proof of Theorem 17 is completed. □

Theorem 18. *Assume that (A), (H_1), (H_2), (H_3), and (H_4) are satisfied and the following conditions hold.*

(H_9) $f(t, u) = -f(t, -u)$ for $(t, u) \in [0, T] \times R$.

(H_{10}) $I_i(u) = -I_i(-u)$ $(i = 1, 2, \ldots, k)$ and nondecreasing.

Then the problem (1) has an infinite number of nontrivial solutions.

Proof. $\varphi \in C^1(W_0^{1,p}(0, T), R)$, by (H_9) and (H_{10}), φ is an even functional and $\varphi(0) = 0$.

First, we verify the condition (2) of Lemma 12.

$$\frac{\mu}{u} \leq \frac{f(t, u)}{F(t, u)}, \quad u \geq M_0,$$

$$\frac{\mu}{u} \geq \frac{f(t, u)}{F(t, u)}, \quad u \leq -M_0. \tag{65}$$

Integrating (65) for u from $[M_0, u]$ and $[u, -M_0]$, respectively, we have

$$\mu \ln \frac{u}{M_0} \leq \ln \frac{F(t, u)}{F(t, u_0)}, \quad u \geq M_0,$$

$$\mu \ln \frac{M_0}{-u} \geq \ln \frac{F(t, u)}{F(t, -u_0)}, \quad u \leq -M_0. \tag{66}$$

That is,

$$F(t, u) \geq F(t, M_0) \left(\frac{u}{M_0} \right)^{\mu}, \quad u \geq M_0, \tag{67}$$

$$F(t, u) \geq F(t, -M_0) \left(\frac{-u}{M_0} \right)^{\mu}, \quad u \leq -M_0. \tag{68}$$

Combining (67) and (68), we have

$$F(t, u) \geq \alpha_1 |u|^{\mu}, \quad |u| \geq M_0, \tag{69}$$

where

$$\alpha_1 = u_0^{-\mu} \min \left\{ \min_{t \in [0,T]} F(t, M_0), \min_{t \in [0,T]} F(t, -M_0) \right\} > 0. \tag{70}$$

On the other hand, by the continuity of $F(t, u)$, $F(t, u)$ is bounded on $[0, T] \times [-M_0, M_0]$, there exists $K_1 > 0$ such that

$$F(t, u) \geq -K_1 \geq \alpha_1 |u|^{\mu} - \alpha_1 |M_0|^{\mu} - K_1, \quad |u| \leq M_0. \tag{71}$$

Combining (69) and (71), we have

$$F(t, u) \geq \alpha_1 |u|^{\mu} - \alpha_2, \quad \forall (t, u) \in [0, T] \times R, \tag{72}$$

where $\alpha_2 = \alpha_1 + K_1$.

For arbitrary finite-dimensional subspace $W \subset W_0^{1,p}(0, T)$, and any $u \in W$, there exists $C_{10} = C_{10}(W) > 0$ such that

$$|u|_{\mu} \geq C_{10} \|u\|. \tag{73}$$

By (H_2), (72), (73), and Lemma 2, we have

$$\varphi(u) = \frac{1}{p} \|u\|^p + \sum_{i=1}^{k} \int_0^{u(t_i)} I_i(t) \, dt$$

$$- \int_0^T F(t, u(t)) \, dt \leq \frac{1}{p} \|u\|^p$$

$$+ \sum_{i=1}^{k} \int_0^{u(t_i)} \left(a_i + b_i |u|^{r_i} \right) dt$$

$$- \int_0^T \left(\alpha_1 u^{\mu} - \alpha_2 \right) dt \leq \frac{1}{p} \|u\|^p \tag{74}$$

$$+ k M_1 \|u\|_{\infty} + M_2 \sum_{i=1}^{k} \|u\|_{\infty}^{r_i+1}$$

$$- \alpha_1 |u|_{\mu}^{\mu} + \alpha_2 T \leq \frac{1}{p} \|u\|^p + k M_1 C_3 \|u\|$$

$$+ M_2 \sum_{i=1}^{k} C_3^{r_i+1} \|u\|^{r_i+1} - \alpha_1 C_{10}^{\mu} \|u\|^{\mu} + \alpha_2 T,$$

for every $u \in W$. This implies that $\varphi(u) \to -\infty$ as $u \in W$ and $\|u\| \to \infty$. So there exists $R(W) > 0$ such that $\varphi \leq 0$ on for all $u \in W$ with $\|u\| \geq R$.

In the following, we verify the condition (1) of Lemma 12.

Let $V = X_1 \bigoplus X_2$, $Y = \overline{\bigoplus_{i=3}^{\infty} X_i}$, then $W_0^{1,p}(0, T) = V \bigoplus Y$ and V is finite dimensional. Using (H_{10}) we have

$$\sum_{i=1}^{k} \int_0^{u(t_i)} I_i(t) \, dt \geq 0. \tag{75}$$

By (H_3) and (H_4), we have

$$\lim_{u \to 0} \frac{F(t, u)}{u^p} = 0. \tag{76}$$

Hence, for $\epsilon = 1/2 p \gamma_p^p$, there exists $\delta > 0$ such that for every u with $|u| \leq \delta$,

$$|F(t, u)| \leq \frac{1}{2 p \gamma_p^p} |u|^p. \tag{77}$$

Hence, for any $u \in Y$ with $\|u\| \le \delta/C_3$, $\|u\|_\infty \le \delta$, by (18), (54) and (55), we have

$$
\begin{aligned}
\varphi(u) &= \frac{1}{p}\|u\|^p + \sum_{i=1}^{k} \int_0^{u(t_i)} I_i(t)\, dt \\
&\quad - \int_0^T F(t, u(t))\, dt \ge \frac{1}{p}\|u\|^p \\
&\quad - \int_0^T F(t, u(t))\, dt \ge \frac{1}{p}\|u\|^p - \frac{1}{2p\gamma_p^p}|u|_p^p \\
&\ge \frac{1}{p}\|u\|^p - \frac{1}{2p\gamma_p^p}\gamma_p^p\|u\|^p = \frac{1}{2p}\|u\|^p.
\end{aligned}
\tag{78}
$$

Take $\alpha = (1/2p)(\delta^p/C_3^p)$, $\rho = \delta/C_3$, then

$$
\varphi(u) \ge \alpha, \quad \forall u \in \partial B_\rho \cap Y.
\tag{79}
$$

Hence, by Lemma 12 and Lemma 15, φ possesses infinite critical points, that is, problem (1) has infinite nontrivial solutions. The proof is complete. $\square$

4. Example

Example 19. Let $p = 2$, $\rho(t) = 1$, $s(t) = t$, $t_1 = 1/2$. Consider the boundary value problem

$$
-u''(t) + tu(t) = t + \sqrt{u(t)}, \quad \text{a.e. } t \in [0, 1],
$$
$$
u(0) = u(1) = 0,
\tag{80}
$$
$$
\Delta u'(t_1) = u'(t_1^+) - u'(t_1^-) = 1 + \sqrt{u(t)}.
$$

It is easy to see that conditions (H_0) and (H_2) of Theorem 13 hold. According to Theorem 13, problem (80) has at least one solution.

Example 20. Let $p = 2$, $\rho(t) = 1$, $s(t) = 2 - t$, $t_1 = 1/3$. Consider the boundary value problem

$$
-u''(t) + (2 - t)u(t) = (u(t))^2, \quad \text{a.e. } t \in [0, 1],
$$
$$
u(0) = u(1) = 0,
\tag{81}
$$
$$
\Delta u'(t_1) = u'(t_1^+) - u'(t_1^-) = 1 + \sqrt{u(t)}.
$$

It is easy to check that all the conditions of Theorem 17 are satisfied. Thus, according to Theorem 17, problem (81) has at least two critical points.

Example 21. Let $p = 3$, $\rho(t) = 1 + 2t$, $s(t) = 2 + t$, $t_1 = 1/2$. Consider the boundary value problem

$$
-\left((1 + 2t)\left|u'(t)\right|u'(t)\right)' + (2 + t)u(t) = (u(t))^3,
$$
$$
\text{a.e. } t \in [0, 1], \ u(0) = u(1) = 0,
\tag{82}
$$
$$
\Delta u'(t_1) = u'(t_1^+) - u'(t_1^-) = \sqrt[3]{u(t)}.
$$

It is easy to check that all the conditions of Theorem 18 are satisfied. Therefore, according to Theorem 18, problem (82) has infinite nontrivial solutions.

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