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Research Article

Sliding Mode Control of the Fractional-Order Unified Chaotic System

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This paper deals with robust synchronization of the fractional-order unified chaotic systems. Firstly, control design for synchronization of nominal systems is proposed via fractional sliding mode technique. Then, systematic uncertainties and external disturbances are considered in the fractional-order unified chaotic systems, and adaptive sliding mode control is designed for the synchronization issue. Finally, numerical simulations are carried out to verify the effectiveness of the two proposed control techniques.

1. Introduction

Even though the theory of fractional calculus dates back to the end of the 17th century, the subject only really came to life over the last few decades [1]. The most significant advantage of fractional calculus is that it provides a powerful instrument of describing memory and hereditary properties of different substances [2]. In particular, fractional differential equations, as the basic theory for fractional-order control [3], have become a powerful tool in describing the dynamics of complex systems and gained great development very recently [4–6].

One of the most important areas of application is the fractional-order chaotic systems, which have wide potential applications in engineering. Since Hartley et al. firstly discovered chaotic phenomenon in fractional dynamics systems [7], there has emerged great interest in this novel and promising topic. On one hand, more and more fractional nonlinear systems which exhibit chaos have been discovered, and their chaotic behaviors have been studied with numerical simulations, such as the fractional-order Chua circuit [8], the fractional-order Van der Pol oscillator [9–11], the fractional-order Lorenz system [12, 13], the fractional-order Chen system [14–16], the fractional-order Lü system [17], the fractional-order Liu system [18], the fractional-order Rössler system [19, 20], the fractional-order Arneodo system [21], the fractional-order Lotka-Volterra system [22, 23], the fractional-order financial system [24, 25], and the discrete fractional logistic map [26]. On the other hand, control and synchronization of fractional-order dynamical systems have been attracting growing investigations. Linear-state feedback control approach has been designed in [14, 27–35], nonlinear feedback control in [36–40], fractional PID control in [41, 42], and open-plus-closed-loop control in [43]. To tackle with modeling inaccuracies and external noises which are unavoidable in the real-world application, fractional-order sliding mode control methodology has been established in [44–51].

In this paper, we investigate robust synchronization of the fractional-order unified chaotic systems. We firstly propose controllers to synchronize the nominal systems via fractional sliding mode technique. Secondly, we consider systematic uncertainties and external disturbances in the fractional-order unified chaotic systems and establish adaptive sliding mode control for synchronization of the uncertain systems.

The rest of this paper is organized as follows. Section 2 presents some basic definitions and theorems about fractional calculus and fractional-order dynamical system. Section 3 describes the general form of fractional-order unified chaotic system and presents our main objective in this paper. Section 4 proposes the sliding mode control design for synchronization of nominal systems and adaptive sliding mode control design for the uncertain system. Numerical simulations are presented to show the effectiveness of the proposed schemes in Section 5. Finally, this paper is concluded in Section 6.

2. Preliminaries

Definition 1. The most important function used in fractional calculus is Euler's Gamma function, which is defined as

$$\Gamma(n) = \int_0^{\infty} t^{n-1} e^{-t} dt. \quad (1)$$

Definition 2. Another important function is a two-parameter function of the Mittag-Leffler type defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0, \ \beta > 0. \quad (2)$$

Fractional calculus is a generalization of integration and differentiation to noninteger-order fundamental operator ${}_aD_t^{\alpha}$, where a and t are the bounds of the operation and $a \in \mathbb{R}$. The continuous integrodifferential operator is defined as

$${}_aD_t^{\alpha} = \begin{cases} \dfrac{d^{\alpha}}{dt^{\alpha}}, & \alpha > 0, \\ 1, & \alpha = 0, \\ \displaystyle\int_a^t (d\tau)^{\alpha}, & \alpha < 0. \end{cases} \quad (3)$$

The three most frequently used definitions for the general fractional calculus are the Grünwald-Letnikov definition, the Riemann-Liouville definition, and the Caputo definition [2, 23, 52, 53].

Definition 3. The Grünwald-Letnikov derivative definition of order α is described as

$${}_aD_t^{\alpha} f(t) = \lim_{h \to 0} \frac{1}{h^{\alpha}} \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} f(t - jh). \quad (4)$$

For binomial coefficients calculation, we can use the relation between Euler's Gamma function and factorial defined as

$$\binom{\alpha}{j} = \frac{\alpha!}{j!(\alpha - j)!} = \frac{\Gamma(\alpha + 1)}{\Gamma(j + 1)\Gamma(\alpha - j + 1)} \quad (5)$$

for

$$\binom{\alpha}{0} = 1. \quad (6)$$

Definition 4. The Riemann-Liouville derivative definition of order α is described as

$${}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)\, d\tau}{(t - \tau)^{\alpha - n + 1}}, \quad n - 1 < \alpha < n. \quad (7)$$

However, applied problems require definitions of fractional derivatives allowing the utilization of physically interpretable initial conditions, which contain $f(a)$, $f'(a)$, and so forth. Unfortunately, the Riemann-Liouville approach fails to meet this practical need. It is M. Caputo who solved this conflict.

Definition 5. The Caputo definition of fractional derivative can be written as

$${}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(n - \alpha)} \int_a^t \frac{f^{(n)}(\tau)\, d\tau}{(t - \tau)^{\alpha - n + 1}}, \quad n - 1 < \alpha < n. \quad (8)$$

In the following, we use the Caputo approach to describe the fractional chaotic systems and the Grünwald-Letnikov approach to propose numerical simulations. To simplify the notation, we denote the fractional-order derivative as D^{α} instead of ${}_0D_t^{\alpha}$ in this paper.

Lemma 6 (see [22]). *Consider the following commensurate fractional-order dynamics system:*

$$D^{\alpha} x = f(x), \quad (9)$$

where $0 < \alpha \le 1$ *and* $x \in \mathbb{R}^n$. *The equilibrium points of system* (9) *are calculated by solving the following equation:*

$$f(x) = 0. \quad (10)$$

These points are locally asymptotically stable if all eigenvalues λ_i *of the Jacobian matrix* $J = \partial f / \partial x$ *evaluated at the equilibrium points satisfy*

$$|\arg(\lambda)| > \frac{\alpha\pi}{2}. \quad (11)$$

Lemma 7 (see [22]). *Consider the following n-dimensional linear fractional-order dynamics system:*

$$\begin{aligned} D^{q_1} x_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n, \\ D^{q_2} x_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n, \\ &\cdots \\ D^{q_n} x_n &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n, \end{aligned} \quad (12)$$

where all α_i*'s are rational numbers between* 0 *and* 1. *Assume* M *be the lowest common multiple of the denominators* u_i*'s of* α_i*'s, where* $\alpha_i = v_i/u_i$, $(u_i, v_i) = 1$, *and* $u_i, v_i \in Z^+$, *for* $i = 1, 2, \ldots, n$. *Define*

$$\Delta(\lambda) = \begin{pmatrix} \lambda^{M\alpha_1} - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda^{M\alpha_2} - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \lambda^{M\alpha_n} - a_{nn} \end{pmatrix}. \quad (13)$$

Then, the zero solution of system (12) *is globally asymptotically stable in the Lyapunov sense if all roots of the equation* $\det(\Delta(\lambda)) = 0$ *satisfy* $|\arg(\lambda)| > \pi/2M$.

Lemma 8 (see [53]). *Assume that there exists a scalar function V of the state x with continuous first-order derivative such that the following are given:*

(i) $V(x)$ *is positive definite,*

(ii) $\dot{V}(x)$ *is negative definite,*

(iii) $V(x) \rightarrow \infty$ *as* $\|x\| \rightarrow \infty$.

Then, the equilibrium at the origin is globally asymptotically stable.

3. Problem Formulation

In [54], Lü et al. have considered a kind of chaotic systems and pointed out that these systems can be described in a unified form as follows:

$$\begin{aligned} \dot{x}_1 &= (25\alpha + 10)(x_2 - x_1), \\ \dot{x}_2 &= (28 - 35\alpha) x_1 - x_1 x_3 + (29\alpha - 1) x_2, \\ \dot{x}_3 &= x_1 x_2 - \frac{(8+\alpha) x_3}{3}, \end{aligned} \tag{14}$$

where x_1, x_2, and x_3 are state variables and $\alpha \in [0, 1]$ is the system parameter. Lü et al. [54] call system (14) a unified chaotic system because it is chaotic for any $\alpha \in [0, 1]$. When $\alpha \in [0, 0.8)$, system (14) is called a the generalized Lorenz chaotic system. When $\alpha = 0.8$, it is called the Lü chaotic system. And it is called the generalized Chen chaotic system when $\alpha \in (0.8, 1]$.

The fractional-order unified chaotic system has been firstly introduced and studied in [55] and reads as

$$\begin{aligned} D^{q_1} x_1 &= (25\alpha + 10)(x_2 - x_1), \\ D^{q_2} x_2 &= (28 - 35\alpha) x_1 - x_1 x_3 + (29\alpha - 1) x_2, \\ D^{q_3} x_3 &= x_1 x_2 - \frac{(8+\alpha) x_3}{3}, \end{aligned} \tag{15}$$

where $q_1, q_2, q_3 \in (0, 1]$ is the fractional order.

System (15) is considered as the drive (master) system and the response (slave) system is a controlled system as follows:

$$\begin{aligned} D^{q_1} y_1 &= (25\alpha + 10)(y_2 - y_1) + u_1, \\ D^{q_2} y_2 &= (28 - 35\alpha) y_1 - y_1 y_3 + (29\alpha - 1) y_2 + u_2, \\ D^{q_3} y_3 &= y_1 y_2 - \frac{(8+\alpha) y_3}{3} + u_3. \end{aligned} \tag{16}$$

Let us define the state errors between the response system (16) and the drive system (15) as $e_1 = y_1 - x_1$, $e_2 = y_2 - x_2$, and $e_3 = y_3 - x_3$.

By subtracting (15) from (16), one can get the following error dynamical system:

$$\begin{aligned} D^{q_1} e_1 &= (25\alpha + 10)(e_2 - e_1) + u_1, \\ D^{q_2} e_2 &= (28 - 35\alpha) e_1 + e_1 e_3 - e_1 y_3 \\ &\quad - e_3 y_1 + (29\alpha - 1) e_2 + u_2, \\ D^{q_3} e_3 &= -e_1 e_2 + e_1 y_2 + e_2 y_1 - \frac{(8+\alpha) e_3}{3} + u_3. \end{aligned} \tag{17}$$

Our main objective in this paper is to investigate the synchronization issue for the fractional-order unified chaotic system (15). It is clear that the synchronization of systems (15) and (16) is equivalent to the stabilization of the error dynamical system (17).

4. Synchronization Design

In the following, the sliding mode control technique, which can maintain low sensitivity to unmodeled dynamics and external disturbances, is applied to establish an effective control law to guarantee the synchronization of the drive system (15) and the response system (16). Two major steps are involved in the sliding mode control design: firstly, constructing an appropriate sliding surface on which the desired system dynamics is stable and, secondly, developing a suitable control law such that the sliding condition is attained.

4.1. Synchronization of the Nominal System. In this subsection, let us firstly consider a simple case: the nominal fractional-order unified chaotic system; that is, the system contains no systematic uncertainties or external disturbances. The design procedure is elaborated in the rest part of this subsection.

4.1.1. Sliding Surfaces Design. In order to achieve the stability of system (17), three sliding surfaces S_1, S_2, and S_3 are introduced as

$$s_i(t) = \left(D^{q_i} + \lambda_i\right) \int_0^t e_i(\tau)\, d\tau, \quad i = 1, 2, 3, \tag{18}$$

the time derivative of which becomes

$$\dot{s}_i(t) = D^{q_i} e_i(t) + \lambda_i e_i(t), \quad i = 1, 2, 3. \tag{19}$$

As long as system (17) operates on the sliding surfaces, it satisfies $s_i = 0$ and $\dot{s}_i = 0$, $i = 1, 2, 3$, which yields the following sliding mode dynamics:

$$\begin{aligned} D^{q_1} e_1 &= -\lambda_1 e_1, \\ D^{q_2} e_2 &= -\lambda_2 e_2, \\ D^{q_3} e_3 &= -\lambda_3 e_3. \end{aligned} \tag{20}$$

By using Lemmas 6 or 7, system (20) is asymptotically stable. As a result, the sliding mode surfaces (18) we have just constructed are appropriate for the control design.

4.1.2. Control Laws Design

Step 1. Choose the first control Lyapunov function

$$V_1(t) = \frac{1}{2} s_1^2. \tag{21}$$

Taking time derivative gives

$$\dot{V}_1(t) = s_1 \dot{s}_1 = s_1 \left(D^{q_1} e_1 + \lambda_1 e_1\right). \tag{22}$$

Substituting the first state equation of (17) into (22), one has

$$\dot{V}_1(t) = s_1\left[(25\alpha + 10)(e_2 - e_1) + u_1 + \lambda_1 e_1\right]. \quad (23)$$

Therefore, by designing the first control law as

$$u_1 = -(25\alpha + 10)(e_2 - e_1) - \lambda_1 e_1 - k_1 \operatorname{sgn}(s_1), \quad (24)$$

where $k_1 > 0$ and

$$\operatorname{sgn}(s) = \begin{cases} 1, & s > 0, \\ 0, & s = 0, \\ -1, & s < 0, \end{cases} \quad (25)$$

then, (23) becomes

$$\dot{V}_1(t) = -k_1 |s_1|. \quad (26)$$

Step 2. Choose the second control Lyapunov function

$$V_2(t) = \frac{1}{2}s_2^2. \quad (27)$$

By taking its derivative with respect to time yields

$$\dot{V}_2(t) = s_2\dot{s}_2 = s_2\left(D^{q_2}e_2 + \lambda_2 e_2\right). \quad (28)$$

Substituting the second state equation of (17) into (28), one has

$$\begin{aligned}\dot{V}_2(t) = s_2 [&(28 - 35\alpha)e_1 + e_1e_3 - e_1y_3 - e_3y_1 \\ &+ (29\alpha - 1)e_2 + u_2 + \lambda_2 e_2].\end{aligned} \quad (29)$$

Therefore, by designing the second control law as

$$\begin{aligned}u_2 = &-(28 - 35\alpha)e_1 - e_1e_3 + e_1y_3 + e_3y_1 \\ &-(29\alpha - 1)e_2 - \lambda_2 e_2 - k_2 \operatorname{sgn}(s_3),\end{aligned} \quad (30)$$

where $k_2 > 0$.

Equation (29) becomes

$$\dot{V}_2(t) = -k_2 |s_2|. \quad (31)$$

Step 3. Choose the third control Lyapunov function

$$V_3(t) = \frac{1}{2}s_3^2. \quad (32)$$

Its time derivative is given by

$$\dot{V}_3(t) = s_3\dot{s}_3 = s_3\left(D^{q_3}e_3 + \lambda_3 e_3\right). \quad (33)$$

Substituting the third state equation of (17) into (33), one has

$$\dot{V}_3(t) = s_3\left[-e_1e_2 + e_1y_2 + e_2y_1 - \frac{(8+\alpha)e_3}{3} + u_3 + \lambda_3 e_3\right]. \quad (34)$$

We are, then, in the position to design the third control law as follows:

$$u_3 = e_1e_2 - e_1y_2 - e_2y_1 + \frac{(8+\alpha)e_3}{3} - \lambda_3 e_3 - k_3 \operatorname{sgn}(s_3), \quad (35)$$

where $k_3 > 0$.

With this choice, (34) can be rewritten as

$$\dot{V}_3(t) = -k_3 |s_3|. \quad (36)$$

Step 4. Finally, we gather the above three control functions as

$$V(s_1, s_2, s_3) = \frac{1}{2}s_1^2 + \frac{1}{2}s_2^2 + \frac{1}{2}s_3^2. \quad (37)$$

It is clear from (26), (31), and (36) that

$$\dot{V}(s_1, s_2, s_3) = -\left(k_1|s_1| + k_2|s_2| + k_3|s_3|\right). \quad (38)$$

There exists some $k > 0$ such that

$$k_1|s_1| + k_2|s_2| + k_3|s_3| > k\|s\|, \quad (39)$$

where

$$\|s\| = \sqrt{s_1^2 + s_2^2 + s_3^2}. \quad (40)$$

The resulting derivative of V is

$$\dot{V}(s_1, s_2, s_3) < -k\|s\|. \quad (41)$$

In terms of Lemma 8, the Lyapunov function (37) provides the proof of globally asymptotical stability with the control laws (24), (30), and (35).

4.2. Synchronization of the Uncertain System. In this subsection, we will proceed to study the synchronization of the fractional-order unified chaotic system in the presence of systematic uncertainties and external disturbances which can be hardly ignored in the real-world application. It is assumed that systematic uncertainties Δf_1, Δf_2, and Δf_3 and external disturbances d_1, d_2, and d_3 are all bounded; that is, $|\Delta f_i| < \rho_i$ and $|d_i(t)| < \theta_i$, where ρ_i and θ_i are unknown positive constants, $i = 1, 2, 3$. Let us denote that $\widehat{\rho}_i$ is an estimate of ρ_i, while $\widehat{\theta}_i$ is an estimate of θ_i. Since ρ_i and θ_i are unknown, our task in this subsection is fulfilled with an adaptive controller consisting of control laws and update laws to obtain $\widehat{\rho}_i$ and $\widehat{\theta}_i$, $i = 1, 2, 3$.

The uncertain fractional-order unified chaotic system can be described as

$$\begin{aligned}D^{q_1}e_1 &= (25\alpha + 10)(e_2 - e_1) + \Delta f_1 + d_1 + u_1, \\ D^{q_2}e_2 &= (28 - 35\alpha)e_1 + e_1e_3 - e_1y_3 - e_3y_1 \\ &\quad + (29\alpha - 1)e_2 + \Delta f_2 + d_2 + u_2, \\ D^{q_3}e_3 &= -e_1e_2 + e_1y_2 + e_2y_1 - \frac{(8+\alpha)e_3}{3} + \Delta f_3 + d_3 + u_3.\end{aligned} \quad (42)$$

The adaptive sliding mode design of system (42) consists of four steps which are elaborated as follows.

Step 1. Consider the first control Lyapunov function

$$V_1(t) = \frac{1}{2}\left[s_1^2 + \frac{1}{\mu_1}\left(\widehat{\rho}_1 - \rho_1\right)^2 + \frac{1}{\gamma_1}\left(\widehat{\theta}_1 - \theta_1\right)^2\right], \quad (43)$$

the time derivative of which becomes

$$\begin{aligned}\dot{V}_1(t) &= s_1\dot{s}_1 + \frac{1}{\mu_1}(\widehat{\rho}_1-\rho_1)\dot{\widehat{\rho}}_1 + \frac{1}{\gamma_1}(\widehat{\theta}_1-\theta_1)\dot{\widehat{\theta}}_1 \\ &= s_1(D^{q_1}e_1+\lambda_1 e_1) + \frac{1}{\mu_1}(\widehat{\rho}_1-\rho_1)\dot{\widehat{\rho}}_1 \\ &\quad + \frac{1}{\gamma_1}(\widehat{\theta}_1-\theta_1)\dot{\widehat{\theta}}_1.\end{aligned} \tag{44}$$

Substituting the first state equation of (17) into (44), one has

$$\begin{aligned}\dot{V}_1(t) = s_1&[(25\alpha+10)(e_2-e_1)+\Delta f_1+d_1+u_1+\lambda_1 e_1] \\ &+\frac{1}{\mu_1}(\widehat{\rho}_1-\rho_1)\dot{\widehat{\rho}}_1 + \frac{1}{\gamma_1}(\widehat{\theta}_1-\theta_1)\dot{\widehat{\theta}}_1.\end{aligned} \tag{45}$$

By designing the first control law and adaptive law as

$$u_1 = -(25\alpha+10)(e_2-e_1)-\lambda_1 e_1-(\widehat{\rho}_1+\widehat{\theta}_1+k_1)\operatorname{sgn}(s_1), \tag{46}$$

$$\begin{aligned}\dot{\widehat{\rho}}_1 &= \mu_1|s_1|, \\ \dot{\widehat{\theta}}_1 &= \gamma_1|s_1|,\end{aligned} \tag{47}$$

then, (45) becomes

$$\begin{aligned}\dot{V}_1 &= s[(\Delta f_1+d_1)-(\widehat{\rho}_1+\widehat{\theta}_1+k_1)\operatorname{sgn}(s_1)] \\ &\quad +(\widehat{\rho}_1-\rho_1)|s_1|+(\widehat{\theta}_1-\theta_1)|s_1| \\ &\le (|\Delta f_1|+|d_1|)|s_1|-(\widehat{\rho}_1+\widehat{\theta}_1+k_1)|s_1| \\ &\quad +(\widehat{\rho}_1-\rho_1)|s_1|+(\widehat{\theta}_1-\theta_1)|s_1| \\ &< (\rho_1+\theta_1)|s_1|-(\widehat{\rho}_1+\widehat{\theta}_1+k_1)|s_1| \\ &\quad +(\widehat{\rho}_1-\rho_1)|s_1|+(\widehat{\theta}_1-\theta_1)|s_1| = -k_1|s_1|.\end{aligned} \tag{48}$$

Step 2. Choose the second control Lyapunov function

$$V_2(t) = \frac{1}{2}\left[s_2^2+\frac{1}{\mu_2}(\widehat{\rho}_2-\rho_2)^2+\frac{1}{\gamma_2}(\widehat{\theta}_2-\theta_2)^2\right] \tag{49}$$

whose derivative is

$$\begin{aligned}\dot{V}_2(t) &= s_2\dot{s}_2 + \frac{1}{\mu_2}(\widehat{\rho}_2-\rho_2)\dot{\widehat{\rho}}_2 + \frac{1}{\gamma_2}(\widehat{\theta}_2-\theta_2)\dot{\widehat{\theta}}_2 \\ &= s_2(D^{q_2}e_2+\lambda_2 e_2) + \frac{1}{\mu_2}(\widehat{\rho}_2-\rho_2)\dot{\widehat{\rho}}_2 \\ &\quad + \frac{1}{\gamma_1}(\widehat{\theta}_2-\theta_2)\dot{\widehat{\theta}}_2.\end{aligned} \tag{50}$$

Substituting the second state equation of (17) into (50), one has

$$\begin{aligned}\dot{V}_2(t) = s_2&[(28-35\alpha)e_1+e_1e_3-e_1y_3 \\ &-e_3y_1+(29\alpha-1)e_2] \\ &+s_2[\Delta f_2+d_2+u_2+\lambda_2 e_2] \\ &+\frac{1}{\mu_2}(\widehat{\rho}_2-\rho_2)\dot{\widehat{\rho}}_2+\frac{1}{\gamma_2}(\widehat{\theta}_2-\theta_2)\dot{\widehat{\theta}}_2.\end{aligned} \tag{51}$$

We choose the control law and the adaptive law

$$\begin{aligned}u_2 = &-(28-35\alpha)e_1-e_1e_3+e_1y_3+e_3y_1 \\ &-(29\alpha-1)e_2-\lambda_2 e_2-(\widehat{\rho}_2+\widehat{\theta}_2+k_2)\operatorname{sgn}(s_2),\end{aligned} \tag{52}$$

$$\dot{\widehat{\rho}}_2 = \mu_2|s_2|, \tag{53}$$

$$\dot{\widehat{\theta}}_2 = \gamma_2|s_2|. \tag{54}$$

With this choice, (51) becomes

$$\dot{V}_2(t) < -k_2|s_2|. \tag{55}$$

Step 3. Choose the third Lyapunov function

$$V_3(t) = \frac{1}{2}\left[s_3^2+\frac{1}{\mu_3}(\widehat{\rho}_3-\rho_3)^2+\frac{1}{\gamma_3}(\widehat{\theta}_3-\theta_3)^2\right]. \tag{56}$$

Taking time derivative gives

$$\begin{aligned}\dot{V}_3(t) &= s_3\dot{s}_3 + \frac{1}{\mu_3}(\widehat{\rho}_3-\rho_3)\dot{\widehat{\rho}}_3 + \frac{1}{\gamma_3}(\widehat{\theta}_3-\theta_3)\dot{\widehat{\theta}}_3 \\ &= s_3(D^{q_3}e_3+\lambda_3 e_3) + \frac{1}{\mu_3}(\widehat{\rho}_3-\rho_3)\dot{\widehat{\rho}}_3 \\ &\quad + \frac{1}{\gamma_3}(\widehat{\theta}_3-\theta_3)\dot{\widehat{\theta}}_3.\end{aligned} \tag{57}$$

Substituting the third state equation of (17) into (57), one has

$$\begin{aligned}\dot{V}_3(t) = s_3&\left[-e_1e_2+e_1y_2+e_2y_1-\frac{(8+\alpha)e_3}{3}\right] \\ &+s_3[\Delta f_3+d_3+u_3+\lambda_3 e_3] \\ &+\frac{1}{\mu_3}(\widehat{\rho}_3-\rho_3)\dot{\widehat{\rho}}_3+\frac{1}{\gamma_3}(\widehat{\theta}_3-\theta_3)\dot{\widehat{\theta}}_3.\end{aligned} \tag{58}$$

We choose the third control law and adaptive law

$$\begin{aligned}u_3 = &\ e_1e_2-e_1y_2-e_2y_1+\frac{(8+\alpha)e_3}{3}-\lambda_3 e_3 \\ &-(\widehat{\rho}_3+\widehat{\theta}_3+k_3)\operatorname{sgn}(s_3),\end{aligned} \tag{59}$$

$$\begin{aligned}\dot{\widehat{\rho}}_3 &= \mu_3|s_3|, \\ \dot{\widehat{\theta}}_3 &= \gamma_3|s_3|.\end{aligned} \tag{60}$$

Then, the resulting derivative of V_3 is

$$\dot{V}_3(t) < -k_3|s_3|. \tag{61}$$

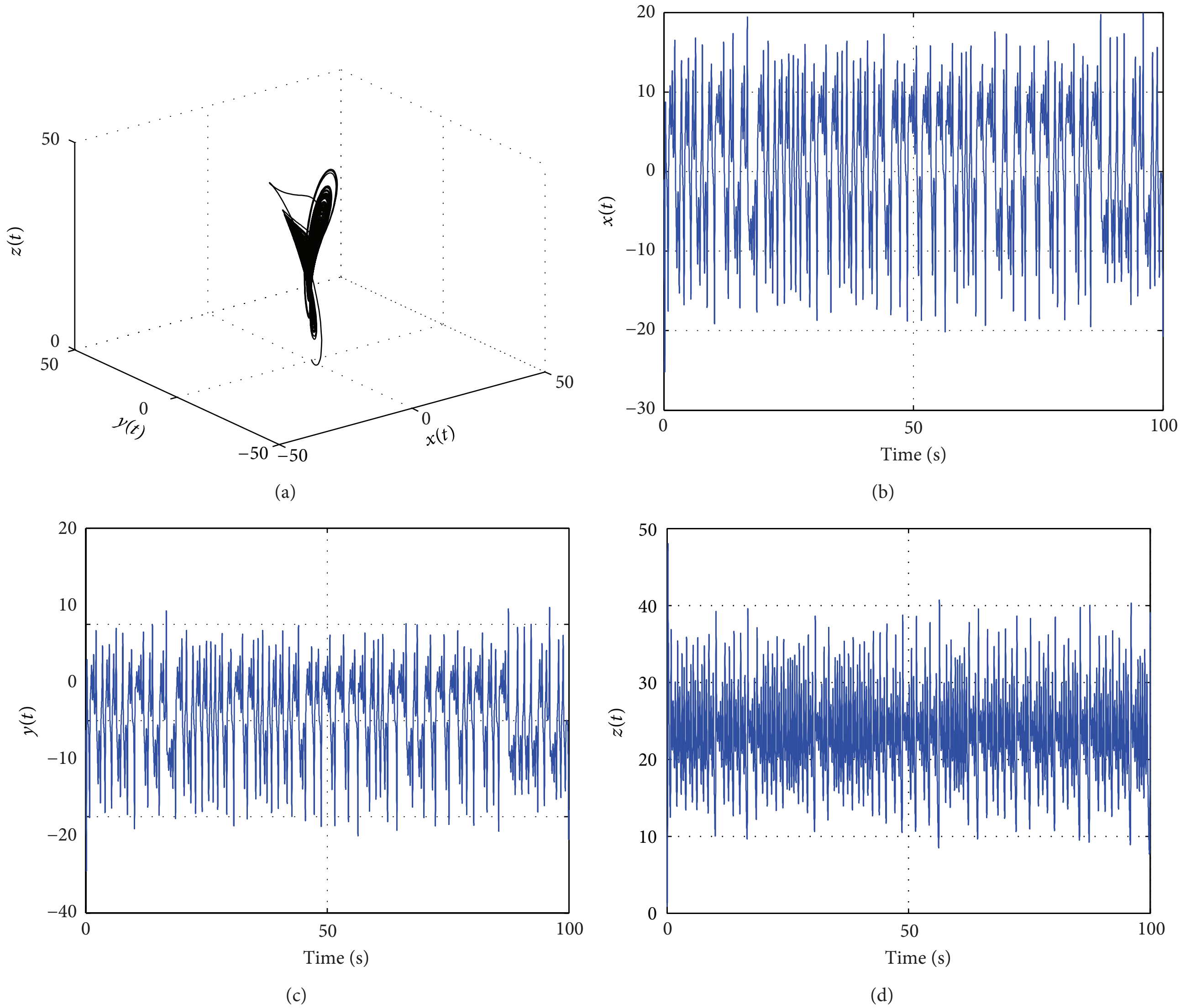

Figure 1: Chaotic trajectories with $\alpha = 0.5$.

Step 4. Finally, we gather the above three control functions as

$$V(t) = \frac{1}{2}\sum_{i=1}^{3} s_i^2 + \sum_{i=1}^{3} \frac{1}{2\mu_i}(\widehat{\rho}_i - \rho_i)^2 + \sum_{i=1}^{3} \frac{1}{2\gamma_i}\left(\widehat{\theta}_i - \theta_i\right)^2. \quad (62)$$

It is clear from (48), (55), and (61) that

$$\dot{V}(t) = -\left(k_1 |s_1| + k_2 |s_2| + k_3 |s_3|\right). \quad (63)$$

There exists some $k > 0$ such that

$$k_1 |s_1| + k_2 |s_2| + k_3 |s_3| > k \|s\|. \quad (64)$$

The resulting derivative of $\dot{V}$ is

$$\dot{V}(s_1, s_2, s_3) < -k \|s\|, \quad (65)$$

where

$$\|s\| = \sqrt{s_1^2 + s_2^2 + s_3^2}. \quad (66)$$

By using Lemma 8, this Lyapunov function provides the proof of globally asymptotical stability with the control laws (46), (52), and (59) and adaptive laws (47), (53), and (60).

5. Numerical Simulations

5.1. Chaotic Behaviors of Fractional-Order Chaotic System. In [55], the authors have provided us with numerical methods of fractional calculus. In [56], the chaotic behaviors of the fractional-order unified system were numerically investigated, where it is found that the lowest order to exhibit chaos is 2.76.

The chaotic behaviors are presented in Figures 1 and 2 with fractional orders of $q_1 = 0.93$, $q_2 = 0.94$, and

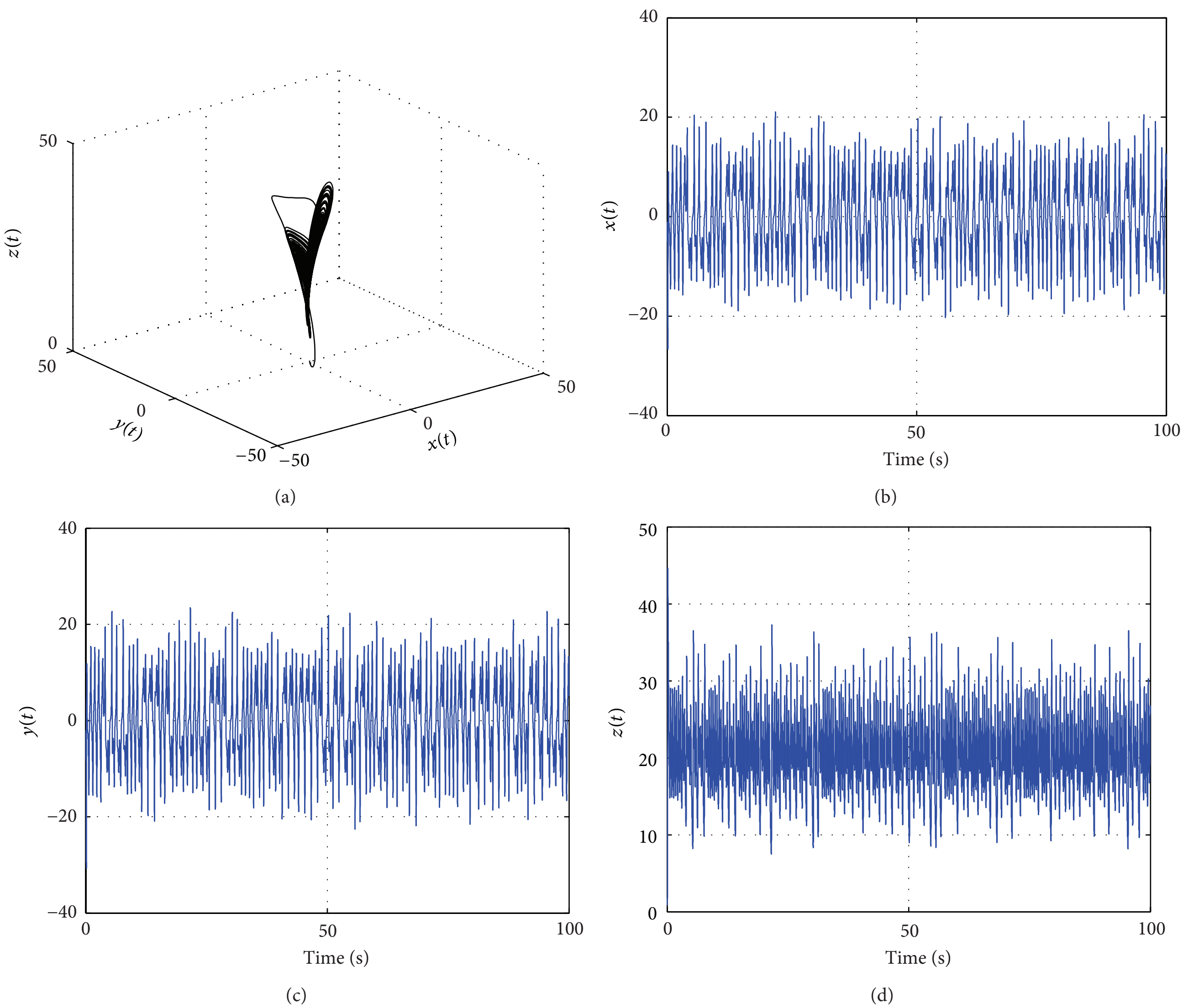

FIGURE 2: Chaotic trajectories with $\alpha = 1$.

$q_3 = 0.95$ and the initial conditions of $(x_1(0), x_2(0), x_3(0)) = (-1, -2, 1)$. The numerical algorithm is based on the following Grünwald-Letnikov's definition:

$$
\begin{aligned}
x\left(t_k\right) &= \left(25\alpha + 10\right)\left(x_2\left(t_{k-1}\right) - x_1\left(t_{k-1}\right)\right) h^{q_1} \\
&\quad - \sum_{j=2}^{k} c_j^{(q_1)} x_1\left(t_{k-j}\right), \\
x_2\left(t_k\right) &= \left[\left(28 - 35\alpha\right) x_1\left(t_k\right) - x_1\left(t_k\right) x_3\left(t_{k-1}\right)\right. \\
&\qquad \left. + \left(29\alpha - 1\right) x_2\left(t_{k-1}\right)\right] h^{q_2} - \sum_{j=2}^{k} c_j^{(q_2)} x_2\left(t_{k-j}\right), \\
x_3\left(t_k\right) &= \left[x_1\left(t_k\right) x_2\left(t_k\right) - \frac{\alpha + 8}{3} x_3\left(t_{k-1}\right)\right] h^{q_3} \\
&\quad - \sum_{j=2}^{k} c_j^{(q_3)} x_3\left(t_{k-j}\right),
\end{aligned}
\tag{67}
$$

where T_{sim} is the simulation time, $k = 1, 2, \ldots, N$, for $N = [T_{\text{sim}}/h]$.

5.2. Simulations of Synchronization of the Nominal System. In this subsection, numerical simulations are presented to demonstrate the effectiveness of the proposed sliding model control in Section 4.1. In the numerical simulations, the fractional orders are chosen as $q_1 = 0.93$, $q_2 = 0.94$, and $q_3 = 0.95$. The initial conditions of the drive system (15) and the response system (16) are chosen as $(x_1(0), x_2(0), x_3(0)) = (-1, -2, 1)$ and $(y_1(0), y_2(0), y_3(0)) = (4, -4, 4)$, respectively. Parameters in (18) are chosen as $\lambda_1 = \lambda_2 = \lambda_3 = 4$. Gains of the control inputs in (24), (30), and (35) are chosen as $k_1 = k_2 = k_2 = 1$.

When $\alpha = 0.5$ and $\alpha = 1$, numerical simulations of synchronization of system (15) are presented in Figures 3, 4, 5, 6, 7, and 8 with control inputs (24), (30), and (35). For interpretations of the references to colors in these figure legends, the reader is referred to the web version of this paper.

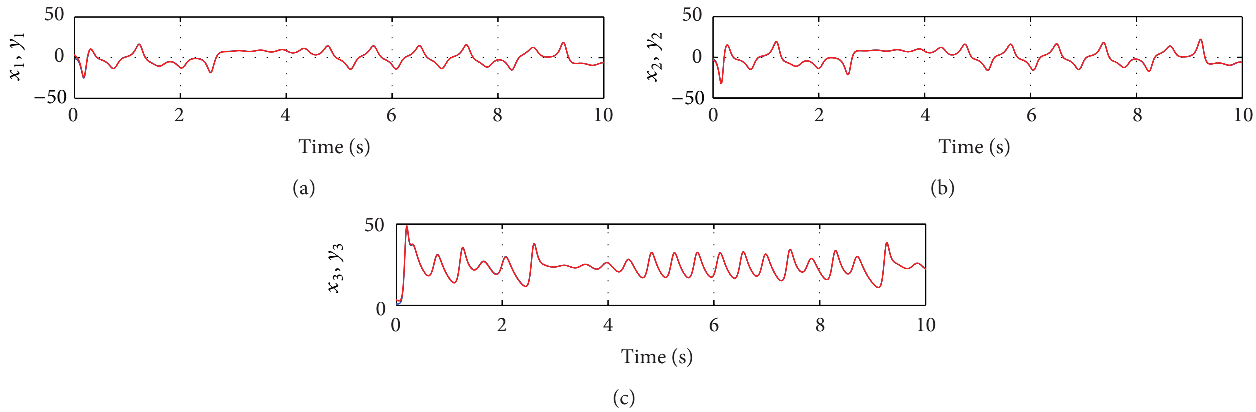

Figure 3: Synchronization of determined fractional-order unified chaotic system with $\alpha = 0.5$ (blue line represents the trajectories of the drive system, while red line represents the trajectories of the response system).

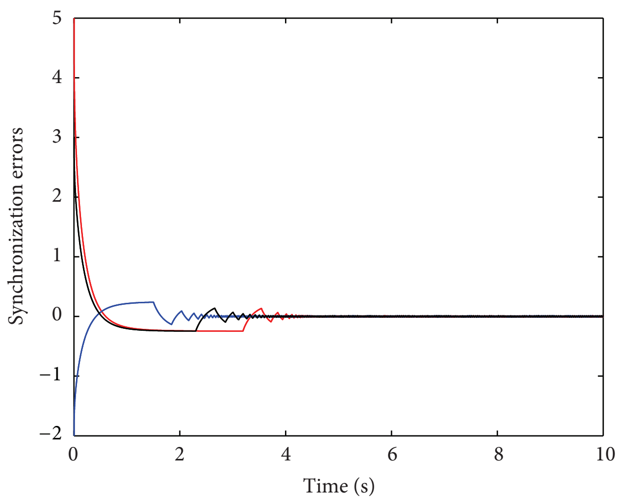

Figure 4: Synchronization errors states with $\alpha = 0.5$ (red line represents the first error state e_1, blue line represents the second error state e_2, and black line represents the third error state e_3).

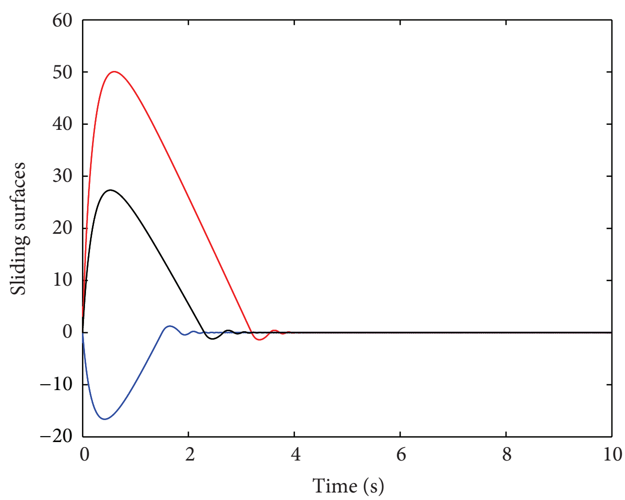

Figure 5: Sliding surfaces states with $\alpha = 0.5$ (red line represents the first sliding surface state s_1, blue line represents the second sliding surface state s_2, and black line represents the third sliding surface state s_3).

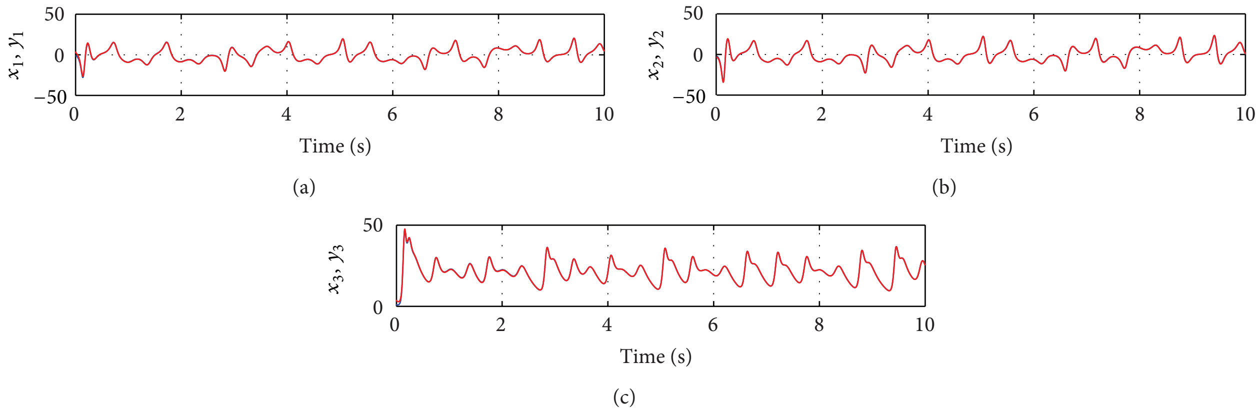

Figure 6: Synchronization of determined fractional-order unified chaotic system with $\alpha = 1$ (blue line represents the trajectories of the drive system, while red line represents the trajectories of the response system).

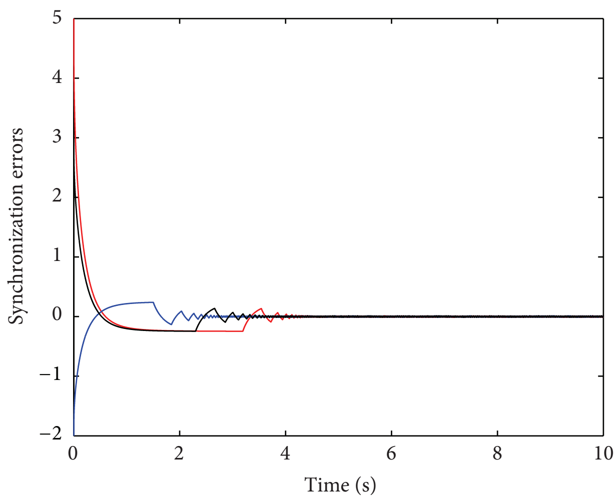

Figure 7: Synchronization errors states with $\alpha = 1$ (red line represents the first error state e_1, blue line represents the second error state e_2, and black line represents the third error state e_3).

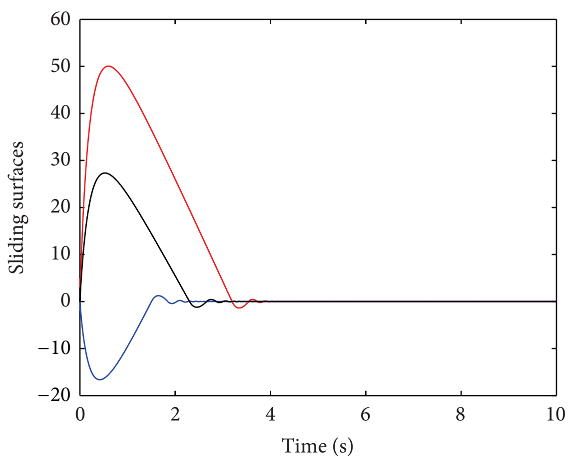

Figure 8: Sliding surfaces states with $\alpha = 1$ (red line represents the first sliding surface state s_1, blue line represents the second sliding surface state s_2, and black line represents the third sliding surface state s_3).

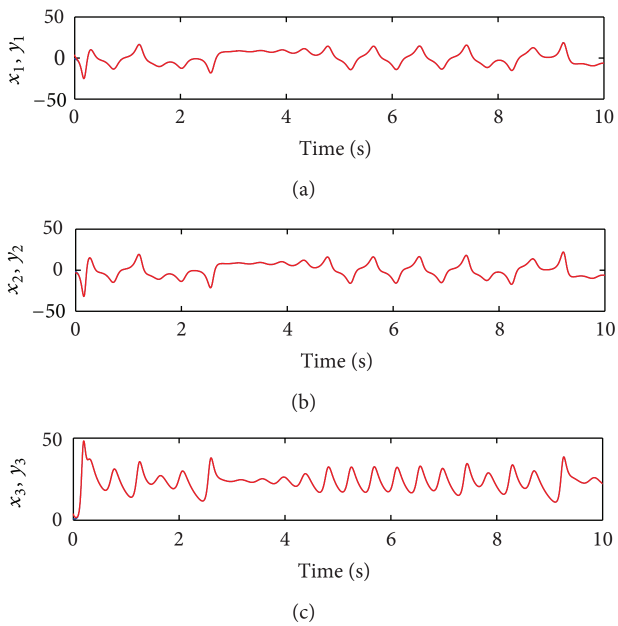

FIGURE 9: Synchronization of fractional-order unified chaotic system in the presence of systematic uncertainties and external disturbances with $\alpha = 0.5$ (blue line represents the trajectories of the drive system, while red line represents the trajectories of the response system).

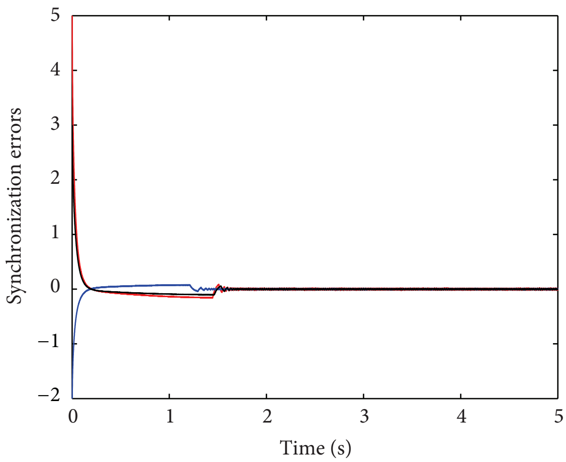

FIGURE 10: Synchronization errors states in the presence of systematic uncertainties and external disturbances with $\alpha = 0.5$ (red line represents the first error state e_1, blue line represents the second error state e_2, and black line represents the third error state e_3).

5.3. Simulations of Synchronization of the Uncertain System. In this subsection, numerical simulations are presented to demonstrate the effectiveness of the proposed adaptive sliding model control in Section 4.2. In the numerical simulations, the fractional orders are always chosen as $q_1 = 0.93$, $q_2 = 0.94$, and $q_3 = 0.95$. The initial conditions of the drive system (15) and the response system (16) are also chosen as $(x_1(0), x_2(0), x_3(0)) = (-1, -2, 1)$ and $(y_1(0), y_2(0), y_3(0)) = (4, -4, 4)$, respectively. Parameters in (18) are chosen as

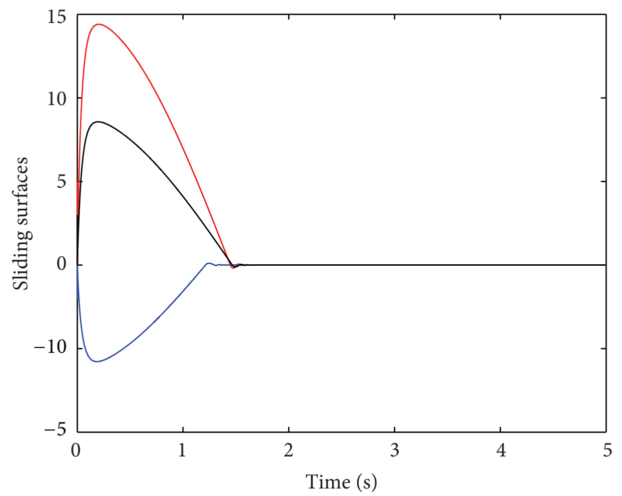

FIGURE 11: Sliding surfaces states in the presence of systematic uncertainties and external disturbances with $\alpha = 0.5$ (red line represents the first sliding surface state s_1, blue line represents the second sliding surface state s_2, and black line represents the third sliding surface state s_3).

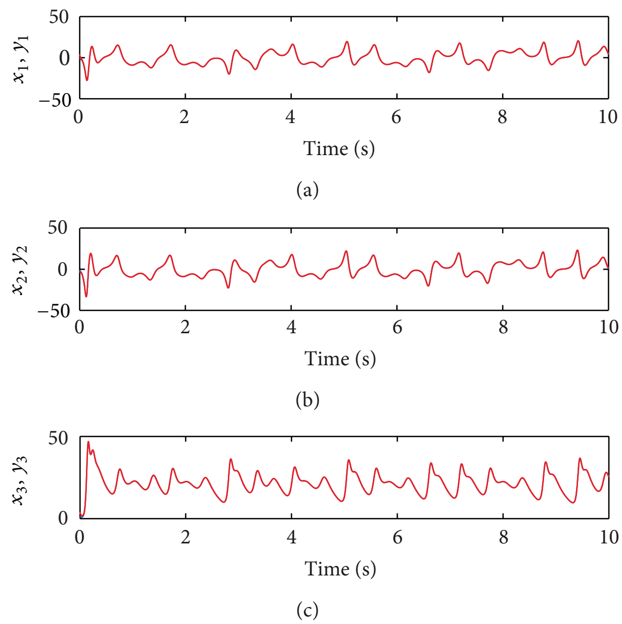

FIGURE 12: Synchronization of fractional-order unified chaotic system in the presence of systematic uncertainties and external disturbances with $\alpha = 1$ (blue line represents the trajectories of the drive system, while red line represents the trajectories of the response system).

$\lambda_1 = \lambda_2 = \lambda_3 = 20$. Gains of the control laws (46), (52), and (59) are chosen as $k_1 = k_2 = k_2 = 0.5$. Gains of the adaptive laws (47), (53), and (60) are chosen as $\mu_1 = \gamma_1 = \mu_2 = \gamma_2 = \mu_3 = \gamma_3 = 0.1$. Systematic uncertainties and external disturbances are assumed to be $\Delta f_1 = 0$, $d_1 = -3\cos(\pi(t-0.1))$, $\Delta f_2 = -0.6\sin(2(y_1 - x_1))$, $d_2 = 5\sin(t)$, $\Delta f_3 = 0$, and $d_3 = 2\cos(t-0.1)$.

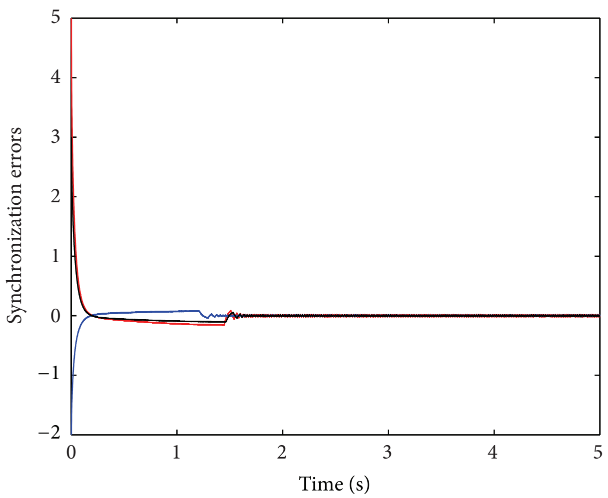

Figure 13: Synchronization errors states in the presence of systematic uncertainties and external disturbances with $\alpha = 1$ (red line represents the first error state e_1, blue line represents the second error state e_2, and black line represents the third error state e_3).

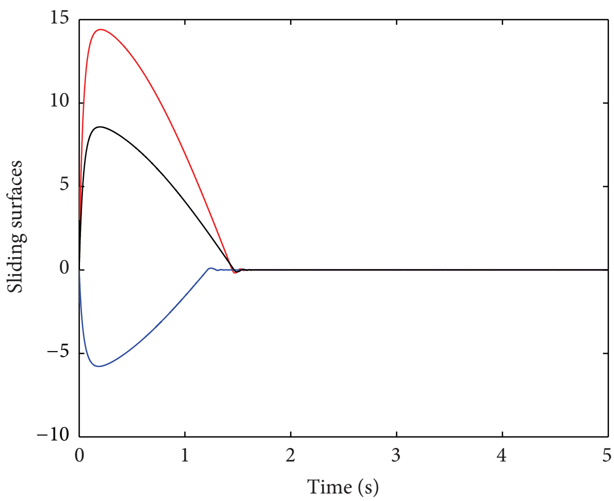

Figure 14: Sliding surfaces states in the presence of systematic uncertainties and external disturbances with $\alpha = 1$ (red line represents the first sliding surface state s_1, blue line represents the second sliding surface state s_2, and black line represents the third sliding surface state s_3).

When $\alpha = 0.5$ and $\alpha = 1$, numerical simulations are presented in Figures 9, 10, 11, 12, 13, and 14 with control inputs (46), (52), and (59) and adaptive laws (47), (53), and (60). For interpretations of the references to colors in these figure legends, the reader is referred to the web version of this paper.

6. Conclusions

This work is concerned with robust synchronization of the fractional-order unified chaotic system. The sliding mode control technique was applied to propose the control design of nominal system and adaptive sliding mode control scheme was designed to develop the control laws and adaptive laws for uncertain system with systematic uncertainties and external disturbances whose bounds are unknown. Numerical simulations were presented to demonstrate the effectiveness of the two kinds of techniques.

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