Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2013, Article ID 395628, 14 pages
http://dx.doi.org/10.1155/2013/395628

Research Article

Exact Traveling Wave Solutions for a Nonlinear Evolution Equation of Generalized Tzitzéica-Dodd-Bullough-Mikhailov Type

Weiguo Rui

College of Mathematics of Honghe University, Mengzi Yunnan 661100, China

Correspondence should be addressed to Weiguo Rui; weiguorhhu@yahoo.com.cn

Received 18 March 2013; Accepted 9 May 2013

Academic Editor: Shiping Lu

By using the integral bifurcation method, a generalized Tzitzéica-Dodd-Bullough-Mikhailov (TDBM) equation is studied. Under different parameters, we investigated different kinds of exact traveling wave solutions of this generalized TDBM equation. Many singular traveling wave solutions with blow-up form and broken form, such as periodic blow-up wave solutions, solitary wave solutions of blow-up form, broken solitary wave solutions, broken kink wave solutions, and some unboundary wave solutions, are obtained. In order to visually show dynamical behaviors of these exact solutions, we plot graphs of profiles for some exact solutions and discuss their dynamical properties.

1. Introduction

In this paper, we consider the following nonlinear evolution equation:

$$u_{xt} = \alpha e^{mu} + \beta e^{nu}, \tag{1}$$

where α, β are two non-zero real numbers and m, n are two integers. We call (1) generalized Tzitzéica-Dodd-Bullough-Mikhailov equation because it contains Tzitzéica equation, Dodd-Bullough-Mikhailov equation, and Tzitzéica-Dodd-Bullough equation. When $\alpha = 1$, $\beta = -1$, $m = 1$, $n = -2$ or $\alpha = -1$, $\beta = 1$, $m = -2$, $n = 1$, especially (1) becomes classical Tzitzéica equation [1–3] as follows:

$$u_{xt} = e^{u} - e^{-2u}, \tag{2}$$

which was originally found in the field of geometry in 1907 by G. Tzitzéica and appeared in the fields of mathematics and physics alike. Equation (2) usually called the "Dodd-Bullough equation," which was initiated by Bullough and Dodd [4] and Žiber and Šabat [5]. Indeed, (2) has another form

$$u_{xx} - u_{tt} = e^{u} - e^{-2u}; \tag{3}$$

see [6, 7] and the references cited therein.

When $\alpha = -1$, $\beta = -1$ and $m = 1$, $n = -2$, (1) becomes Dodd-Bullough-Mikhailov equation

$$u_{xt} + e^{u} + e^{-2u} = 0. \tag{4}$$

When $\alpha = 1$, $\beta = 1$ and $m = 1$, $n = -2$, (1) becomes Tzitzéica-Dodd-Bullough equation

$$u_{xt} - e^{u} - e^{-2u} = 0. \tag{5}$$

The Dodd-Bullough-Mikhailov equation and Tzitzéica-Dodd-Bullough equation appeared in many problems varying from fluid flow to quantum field theory.

Moreover, when $m = 1$, $n = -1$, $\alpha = 1/2$, $\beta = -1/2$ or $m = -1$, $n = 1$, $\alpha = -1/2$, $\beta = 1/2$, (1) becomes the sinh-Gordon equation

$$u_{xt} = \sinh u, \tag{6}$$

which was shown in [7–10] and the references cited therein. When $m = 1$, $n = -1$, $\alpha = \beta = 1/2$ or $m = -1$, $n = 1$, $\alpha = \beta = 1/2$, (1) becomes the cosh-Gordon equation

$$u_{xt} = \cosh u, \tag{7}$$

which was shown in [11, 12] and the references cited therein. When $m = 1, n = 0$, especially (1) becomes the Liouville equation

$$u_{xt} = \alpha e^{u}. \tag{8}$$

All the equations mentioned above play very significant roles in many scientific applications. And some of them were studied by many authors in recent years; see the following brief statements.

In [13], by using the tanh method, Wazwaz considered some solitary wave and periodic wave solutions for the Dodd-Bullough-Mikhailov and Dodd-Bullough equations. In [14], Andreev studied the Bäcklund transformation for Bullough-Dodd-Jiber-Shabat equation. In [15], Cherdantzev and Sharipov obtained finite-gap solutions of the Bullough-Dodd-Jiber-Shabat equation. In [16], Cherdantzev and Sharipov investigated solitons on the finite-gap background in the Bullough-Dodd-Jiber-Shabat model. In addition, the Darboux transformation, self-dual Einstein spaces and consistency and general solution of Tzitzéica equation were studied in [17–19].

In this paper, by using integral bifurcation method [20], we will study (1). Indeed, the integral bifurcation method has successfully been combined with the computer method [21] and some transformations [22, 23] for investigating exact traveling wave solutions of some nonlinear PDEs. Therefore, using this method, we will obtain some new results which are different from those in the above references.

The rest of this paper is organized as follows: in Section 2, we will derive two-dimensional planar system which is equivalent to (1) and give its first integral equation. In Sections 3 and 4, by using the integral bifurcation method, we will obtain some new traveling wave solutions of (1) and discuss their dynamic properties.

2. Two-Dimensional Planar Dynamical System of (1) and Its First Integral

Applying the transformations $u(x,t) = \ln|v(x,t)|$, we change (1) into the following form:

$$vv_{xt} - v_x v_t = \alpha v^{m+2} + \beta v^{n+2}. \tag{9}$$

Let $v(x,t) = \phi(x-ct) = \phi(\xi)$, substituting $\phi(x-ct)$ into (9), respectively, we obtain

$$c(\phi')^2 - c\phi\phi'' = \alpha\phi^{m+2} + \beta\phi^{n+2}, \tag{10}$$

where "$'$" is the derivative with respect to ξ and c is wave speed.

Clearly, (10) is equivalent to the following two-dimensional systems:

$$\frac{d\phi}{d\xi} = y, \qquad \frac{dy}{d\xi} = \frac{cy^2 - \alpha\phi^{m+2} - \beta\phi^{2+n}}{c\phi}. \tag{11}$$

However, when $\phi = 0$, (11) is not equivalent to (10) and the $dy/d\xi$ cannot be defined, so we call the $\phi = 0$ singular line and call (11) singular system. In order to obtain an equivalent system of (10), making a scalar transformation

$$d\xi = c\phi d\tau, \tag{12}$$

equation (11) can be changed into a regular two-dimensional system as follows:

$$\frac{d\phi}{d\tau} = c\phi y, \qquad \frac{dy}{d\tau} = cy^2 - \alpha\phi^{m+2} - \beta\phi^{n+2}. \tag{13}$$

Systems (11) and (13) are two integrable systems, and they have the same first integral as follows:

$$y^2 = h\phi^2 - \frac{2\alpha}{cm}\phi^{m+2} - \frac{2\beta}{cn}\phi^{n+2}, \tag{14}$$

where h is integral constant.

Systems (11) and (13) are planar dynamical systems defined by the 5-parameter (α, β, c, m, n). Usually their first integral (14) can be rewritten as

$$H(\phi, y) \equiv \frac{2\alpha}{cm}\phi^m + \frac{2\beta}{cn}\phi^n + \frac{y^2}{\phi^2} = h. \tag{15}$$

When $n - m = 2k - 1$, $(k \in N)$, system (13) has two equilibrium points $O(0,0)$ and $A((-\alpha/\beta)^{1/(n-m)}, 0)$ in the ϕ-axes. When $n - m = 2k$, $(k \in N)$ and $\alpha\beta < 0$, system (13) has three equilibrium point $O(0,0)$ and $B_{1,2}(\pm(-\alpha/\beta)^{1/(n-m)}, 0)$ in the ϕ-axes. When $n - m = 2k$, $(k \in N)$ and $\alpha\beta > 0$, system (13) has only one equilibrium points $O(0,0)$. When $n = m = k$, $(k \in N)$, system (13) has only one equilibrium point $O(0,0)$.

Respectively, substituting every equilibrium point (the point $O(0,0)$ is exception) into (15), we have

$$h_A = H\left(\left(-\frac{\alpha}{\beta}\right)^{1/(n-m)}, 0\right) = \frac{2\alpha(n-m)}{cmn}\left(-\frac{\alpha}{\beta}\right)^{m/(n-m)},$$

$$h_{B1} = H(B_1) = H\left(\left(-\frac{\alpha}{\beta}\right)^{1/(n-m)}, 0\right) = \frac{2\alpha(n-m)}{cmn}\left(-\frac{\alpha}{\beta}\right)^{m/(n-m)},$$

$$h_{B2} = H(B_2) = H\left(-\left(-\frac{\alpha}{\beta}\right)^{1/(n-m)}, 0\right) = (-1)^m h_{B1}. \tag{16}$$

From the transformation $u(x,t) = \ln|\phi(x-ct)|$, we know that $\ln|\phi(x-ct)|$ approaches to $-\infty$ if a solution $\phi(\xi)$ approaches to 0 in (14). In other words, this solution determines an unboundary wave solution or blow-up wave solution of (1). It is easy to see that the second equation of system (11) is not continuous when $\phi = 0$. In other words, on such straight line $\phi = 0$ in the phase plane (ϕ, y), the function ϕ'' is not defined, and this implies that the smooth wave solutions of (1) sometimes become nonsmooth wave solutions.

3. Exact Solutions of (1) and Their Properties under the Conditions $n = \pm m, \pm 2m, m \in \mathbf{Z}^+$, and $m = 2, n = -1$

In this section, we investigate exact solutions of (1) under different kinds of parametric conditions and their properties.

3.1. Different Kinds of Solitary Wave Solutions and Unboundary Wave Solutions in the Special Cases $n = 2m \in \mathbf{Z}^+$ or $n = m \in \mathbf{Z}^+$. (i) When $h > 0$, $\beta c < 0$, $n = 2m$ and m is even, then (14) can be reduced to

$$y = \pm\phi\sqrt{h - \frac{2\alpha}{cm}\phi^m - \frac{\beta}{cm}(\phi^m)^2}. \tag{17}$$

Taking $\xi_0 = 0$ and $\phi(0) = \phi_1$ as initial values, substituting (17) into the $d\phi/d\xi = y$ of (11), and then integrating it yield

$$v(x,t) = \pm\left(\frac{\gamma_1}{\Omega_1 \cosh(\omega_1(x-ct)) + \delta_1}\right)^{1/m}, \tag{18}$$

where $\phi_1 = (-\alpha/\beta + \sqrt{(\alpha^2 + \beta cmh)/\beta^2})^{1/m}$, $\Omega_1 = \phi_1^m(\alpha^2 + \beta cmh)$, $\omega_1 = -m\sqrt{h}$, $\gamma_1 = cmh(cmh - \alpha\phi_1^m)$, $\delta_1 = \alpha(cmh - \alpha\phi_1^m)$. By using program of *Maple*, it is easy to validate that (18) is the solution equation (9) when m is given. Substituting (18) into the transformation $u(x,t) = \ln|v(x,t)|$, we obtain a solitary wave solutions of (1) as follows:

$$u(x,t) = \frac{1}{m}\ln\left|\frac{\gamma_1}{\Omega_1 \cosh(\omega_1(x-ct)) + \delta_1}\right|. \tag{19}$$

(ii) When $n = m$, $h > 0$, $c/(\alpha+\beta) > 0$ and m is even, (14) can be reduced to

$$y = \pm\phi\sqrt{h - \frac{2(\alpha+\beta)}{cm}\phi^m}. \tag{20}$$

Taking $\xi_0 = 0$ and $\phi(0) = (mhc/2(\alpha+\beta))^{1/m}$ as initial values, substituting (20) into the $d\phi/d\xi = y$ of (11), and then integrating it yield

$$v(x,t) = \pm\left(\frac{mhc\,\mathrm{sech}^2\omega_2(x-ct)}{2(\alpha+\beta)}\right)^{1/m}, \tag{21}$$

where $\omega_2 = (1/2)m\sqrt{h}$. Substituting (21) into the transformation $u(x,t) = \ln|v(x,t)|$, we obtain a solution without boundary of (1) as follows:

$$u(x,t) = \frac{1}{m}\ln\left|\frac{mhc\,\mathrm{sech}^2\omega_2(x-ct)}{2(\alpha+\beta)}\right|. \tag{22}$$

Equation (21) is smooth solitary wave solution of (9) but (22) is an exact solution without boundary of (1) because $u \to -\infty$ as $v \to 0$. In order to visually show dynamical behaviors of solutions v and u of (21) and (22), we plot graphs of their profiles which are shown in Figures 1(a) and 1(b), respectively.

(iii) When $n = 2m$, $h = 0$, $\beta c > 0$, $\alpha c < 0$ and m is even, (14) can be reduced to

$$y = \pm\phi\sqrt{-\frac{2\alpha}{cm}\phi^m - \frac{\beta}{cm}(\phi^m)^2}. \tag{23}$$

Taking $\xi_0 = 0$ and $\phi(0) = 0$ as initial values, substituting (23) into the $d\phi/d\xi = y$ of (11), and then integrating it yield

$$v(x,t) = \pm\left(\frac{2(-\alpha c)}{\beta c + m\alpha^2(x-ct)^2}\right)^{1/m}. \tag{24}$$

Substituting (24) into the transformation $u(x,t) = \ln|v(x,t)|$, we obtain an exact solutions of rational function type of (1) as follows:

$$u(x,t) = \frac{1}{m}\ln\left|\frac{2\alpha c}{\beta c + m\alpha^2(x-ct)^2}\right|. \tag{25}$$

(iv) When $n = 2m$, $h > 0$, $\alpha\beta > 0$, $\beta c > 0$ (or $\alpha\beta < 0$, $\beta c > 0$), and m is odd, (14) can be reduced to

$$y = \pm\sqrt{\frac{\beta}{cm}}\phi\sqrt{\frac{cmh}{\beta} - \frac{2\alpha}{\beta}\phi^m - (\phi^m)^2}. \tag{26}$$

Respectively, taking $\xi_0 = 0$, $\phi(0) = \phi_1$, and $\xi_0 = 0$, $\phi(0) = \phi_2$ as initial values, substituting (26) into the $d\phi/d\xi = y$ of (11), and then integrating them yield

$$v(x,t) = \left(\frac{\gamma_1}{\Omega_1 \cosh(\omega_1(x-ct)) + \delta_1}\right)^{1/m}, \tag{27}$$

$$v(x,t) = \left(\frac{\gamma_2}{\Omega_2 \cosh(\omega_2(x-ct)) + \delta_2}\right)^{1/m}, \tag{28}$$

where ϕ_1 is given above and $\phi_2 = (-\alpha/\beta - \sqrt{(\alpha^2 + \beta cmh)/\beta^2})^{1/m}$, $\Omega_2 = \phi_2^m(\alpha^2 + \beta cmh)$, $\omega_2 = m\sqrt{h}$, $\gamma_2 = cmh(cmh - \alpha\phi_2^m)$, $\delta_2 = \alpha(cmh - \alpha\phi_2^m)$. Respectively, substituting (27) and (28) into the transformation $u(x,t) = \ln|v(x,t)|$, we obtain two smooth unboundary wave solutions of (1) as follows:

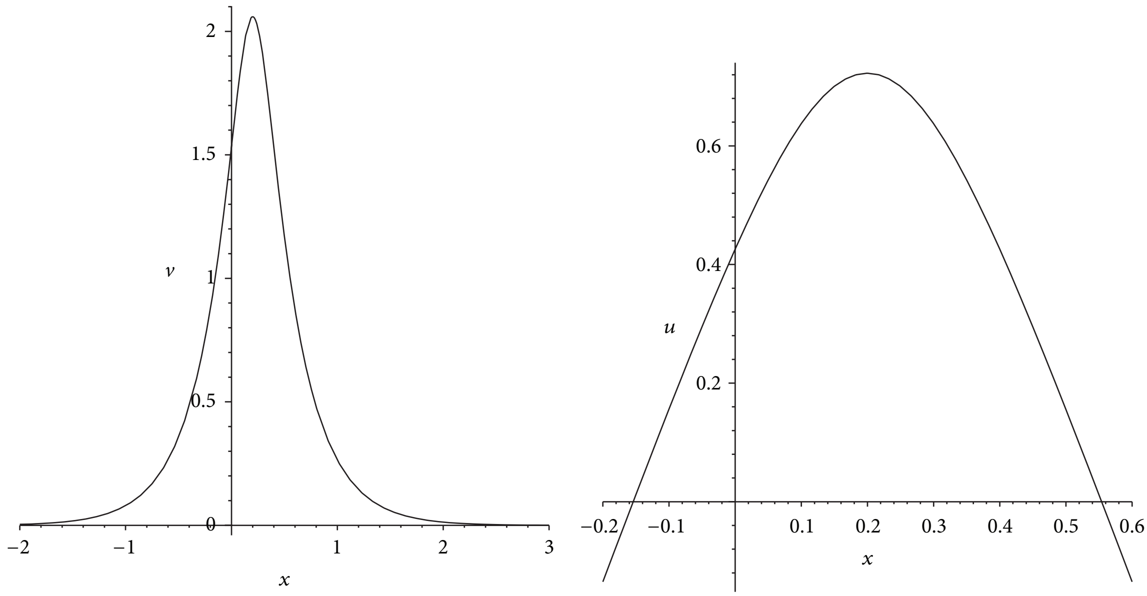
$$u(x,t) = \frac{1}{m}\ln\left|\frac{\gamma_1}{\Omega_1 \cosh(\omega_1(x-ct)) + \delta_1}\right|, \tag{29}$$

$$u(x,t) = \frac{1}{m}\ln\left|\frac{\gamma_2}{\Omega_2 \cosh(\omega_2(x-ct)) + \delta_2}\right|. \tag{30}$$

(v) When $n = 2m$, $0 < h < h_A$, $\alpha\beta < 0$, $\beta c < 0$ (or $\alpha\beta > 0$, $\beta c < 0$) and m is odd, (14) can be reduced to

$$y = \pm\sqrt{-\frac{\beta}{cm}}\phi\sqrt{-\frac{cmh}{\beta} + \frac{2\alpha}{\beta}\phi^m + (\phi^m)^2}. \tag{31}$$

Similarly, by using (31), we obtain two smooth solitary wave solutions of (1) which are the same as the solutions (29) and (30).



(a) Smooth solitary wave
(b) Unboundary wave

Figure 1: The phase portraits of solutions (21) and (22) for $m = 4$, $\alpha = 3$, $\beta = -1$, $c = 2$, $h = 9$, $t = 0.1$.

(vi) When $n = m$, $h > 0$ and m is odd, (14) can be reduced to (20), so the obtained solution is as the same as the solution (22).

(vii) When $n = 2m$, $h = 0$, $\beta c > 0$ and m is odd, (14) can be reduced to

$$y = \pm\sqrt{\frac{\beta}{cm}}\phi\sqrt{-\frac{2\alpha}{\beta}\phi^m - (\phi^m)^2}. \quad (32)$$

Taking $\xi_0 = 0$ and $\phi(0) = (-2\alpha/\beta)^{1/m}$ as initial values, substituting (32) into the $d\phi/d\xi = y$ of (11), and then integrating it yield

$$v(x,t) = \left(\frac{-2\alpha}{\beta\left[1 + (m\alpha^2/\beta c)(x - ct)^2\right]}\right)^{1/m}. \quad (33)$$

Substituting (33) into the transformation $u(x,t) = \ln|v(x,t)|$, we obtain a smooth unboundary wave solution of rational function type of (1) as follows:

$$u(x,t) = \frac{1}{m}\ln\left|\frac{2\alpha}{\beta\left[1 + (m\alpha^2/\beta c)(x - ct)^2\right]}\right|. \quad (34)$$

3.2. Periodic Blow-Up Wave Solutions, Broken Kink Wave, and Antikink Wave Solutions in the Special Case $n = 2m \in \mathbf{Z}^+$. (1) When $n = 2m$, $\alpha\beta < 0$, $\beta c > 0$, $h_{B1} < h < 0$ (or $h_{B2} < h < 0$) and m is even, (14) can be reduced to

$$y = \pm\sqrt{\frac{\beta}{cm}}\phi\sqrt{\frac{cmh}{\beta} - \frac{2\alpha}{\beta}\phi^m - (\phi^m)^2}. \quad (35)$$

Taking $\xi_0 = 0$, $\phi(0) = \left(-\alpha/\beta + \sqrt{\alpha^2/\beta^2 + mhc/\beta}\right)^{1/m}$ and $\xi_0 = 0$, $\phi(0) = \left(-\alpha/\beta - \sqrt{\alpha^2/\beta^2 + mhc/\beta}\right)^{1/m}$ as the initial values, substituting (35) into the $d\phi/d\xi = y$ of (11), and then integrating them yield

$$v(x,t) = \left(\frac{cmh}{\alpha - \beta\Omega_3\sin\left(\arcsin D_1 - \omega_3(x - ct)\right)}\right)^{1/m}, \quad (36)$$

$$v(x,t) = -\left(\frac{cmh}{\alpha + \beta\Omega_3\sin\left(\arcsin D_2 + \omega_3(x - ct)\right)}\right)^{1/m}, \quad (37)$$

where $\Omega_3 = \sqrt{(\alpha^2 + cmh\beta)/\beta^2}$, $\omega_3 = m\sqrt{-h}$, $D_1 = (\alpha^2 + \alpha\beta\Omega_3 + \beta cmh)/\beta\Omega_3(\alpha + \beta\Omega_3)$, $D_2 = (\alpha^2 - \alpha\beta\Omega_3 + \beta cmh)/\beta\Omega_3(-\alpha + \beta\Omega_3)$. Respectively, substituting (36) and (37) into the transformation $u(x,t) = \ln|v(x,t)|$, we obtain two periodic blow-up wave solutions of (1) as follows:

$$u(x,t) = \frac{1}{m}\ln\left|\frac{cmh}{\alpha - \beta\Omega_3\sin\left(\arcsin D_1 - \omega_3(x - ct)\right)}\right|, \quad (38)$$

$$u(x,t) = \frac{1}{m}\ln\left|\frac{cmh}{\alpha + \beta\Omega_3\sin\left(\arcsin D_2 + \omega_3(x - ct)\right)}\right|. \quad (39)$$

(2) When $n = 2m$, $\alpha\beta < 0$, $\beta c > 0$, $h_{B2} < h < 0$ and m is odd, (14) can be reduced to

$$y = \pm\sqrt{\frac{\beta}{cm}}\phi\sqrt{\frac{cmh}{\beta} - \frac{2\alpha}{\beta}\phi^m - (\phi^m)^2}, \quad (40)$$

which is the same as (35), so the obtained solution is the same as solution (39). Similarly, when $n = 2m$, $\alpha\beta < 0$, $\beta c > 0$,

$h_{B1} < h < 0$ and m is odd, the obtained solution is also the same as solution (38).

(3) when $n = 2m$, $\alpha\beta < 0$, $\beta c < 0$, $h = h_{B1} = h_{B2} = -\alpha^2/\beta cm$ and m is even, (14) can be reduced to

$$y = \pm\sqrt{-\frac{1}{cm\beta}\phi\left(\beta\phi^m + \alpha\right)}. \tag{41}$$

Substituting (41) into the $d\phi/d\xi = y$ of (11) and then integrating it and setting the integral constants as zero yield

$$v(x,t) = \left(-\frac{\alpha}{2\beta}\left(1 \pm \tanh\frac{\Omega_4}{2}(x-ct)\right)\right)^{1/m}, \tag{42}$$

where $\Omega_4 = \alpha\sqrt{-m/c\beta}$. Substituting (42) into the transformation $u(x,t) = \ln|v(x,t)|$, we obtain an unboundary wave solution of (1) as follows:

$$u(x,t) = \frac{1}{m}\ln\left|\frac{\alpha}{2\beta}\left[1 \pm \tanh\frac{\Omega_4}{2}(x-ct)\right]\right|. \tag{43}$$

Equation (42) is smooth kink wave solution of (9), but (43) is a broken kink wave solution without boundary of (1) because $u \to -\infty$ as $v \to 0$. In order to visually show dynamical behaviors of solutions v and u of (42) and (43), we plot graphs of their profiles which are shown in Figures 2(a) and 2(b), respectively.

(4) When $n = 2m$, $\alpha\beta > 0$, $\beta c < 0$, $h = h_A$ or $\alpha\beta < 0$, $\beta c < 0$, $h = h_A$ and m is odd, (14) can be reduced to (41). So the obtained solution is also the same as the solution (43).

3.3. Periodic Blow-Up Wave Solutions and Solitary Wave Solutions of Blow-Up Form in the Special Case $m = 2$, $n = -1$. When $n = -1$, (14) can be reduced to

$$y = \pm\sqrt{h\phi^2 - \frac{2\alpha}{cm}\phi^{m+2} + \frac{2\beta}{c}\phi}. \tag{44}$$

(a) Under the conditions $\alpha\beta > 0$, $\alpha c > 0$, $h \in (-\infty, +\infty)$ and $m = 2$, (44) becomes

$$y = \pm\sqrt{\frac{\alpha}{c}}\sqrt{(\vartheta_1 - \phi)(\phi - 0)(\phi - \gamma_1)(\phi - \delta_1)}, \quad \phi \in (0, \vartheta_1], \tag{45}$$

where $\vartheta_1, \gamma_1, \delta_1$ are three roots of equation $2\beta/\alpha + (ch/\alpha)\phi - \phi^3$ and $\vartheta_1 > 0 > \gamma_1 > \delta_1$. These three roots $\vartheta_1, \gamma_1, \delta_1$ can be obtained by Cardano formula as long as the parameters α, β, c, h are fixed (or given) concretely. For example, taking $\alpha = 4.0$, $\beta = 2.0$, $c = 4.0$, $h = 9.0$, the $\vartheta_1 \doteq 3.054084215$, $\gamma_1 \doteq -0.1112641576$, $\delta_1 \doteq -2.942820058$. Taking $(0, \vartheta_1)$ as the initial values of the variables (ξ, ϕ), substituting (45) into the $d\phi/d\xi = y$ of (11), and then integrating it yield

$$v(x,t) = \phi(\xi) = \frac{\vartheta_1(-\delta_1)\,\mathrm{cn}^2(\Omega_1(\xi), k_1)}{(-\delta_1) + \vartheta_1\mathrm{sn}^2(\Omega_1\xi, k_1)}, \quad \xi \neq T_1, \tag{46}$$

where $\Omega_1 = (1/2)\sqrt{-\alpha\delta_1(\vartheta_1 - \delta_1)/c}$, $k_1 = \sqrt{\vartheta_1(\gamma_1 - \delta_1)/-\delta_1(\vartheta_1 - \gamma_1)}$, $T_1 = (1/\Omega_1)\mathrm{sn}^{-1}(1, k_1)$. By using the transformation $u(x,t) = \ln|v(x,t)|$ and (46), we obtain a periodic blow-up wave solution of (1) as follows:

$$u(x,t) = \ln\left|\frac{\vartheta_1(-\delta_1)\,\mathrm{cn}^2(\Omega_1(x-ct), k_1)}{(-\delta_1) + \vartheta_1\mathrm{sn}^2(\Omega_1(x-ct), k_1)}\right|, \quad x - ct \neq T_1. \tag{47}$$

(b) Under the conditions $\alpha\beta > 0$, $\alpha c > 0$, $h \in (h_A, +\infty)$ and $m = 2$, (44) becomes

$$y = \pm\sqrt{\frac{\alpha}{c}}\sqrt{(\vartheta_1 - \phi)(0 - \phi)(\gamma_1 - \phi)(\phi - \delta_1)}, \quad \phi \in [\delta_1, \gamma_1]. \tag{48}$$

Similarly, taking $(0, \delta_1)$ as the initial values of the variables (ξ, ϕ), substituting (48) into the $d\phi/d\xi = y$ of (11), and then integrating it yield

$$v(x,t) = \phi(\xi) = \frac{\gamma_1\delta_1}{\delta_1 + (\gamma_1 - \delta_1)\,\mathrm{sn}^2(\Omega_1\xi, k_1)}. \tag{49}$$

By using the transformation $u(x,t) = \ln|v(x,t)|$ and (49), we obtain a periodic blow-up wave solution of (1) as follows:

$$u(x,t) = \ln\left|\frac{\gamma_1\delta_1}{\delta_1 + (\gamma_1 - \delta_1)\,\mathrm{sn}^2(\Omega_1(x-ct), k_1)}\right|. \tag{50}$$

(c) Under the conditions $\alpha\beta < 0$, $\alpha c > 0$, $h \in (-\infty, +\infty)$ and $m = 2$, (44) becomes

$$y = \pm\sqrt{\frac{\alpha}{c}}\sqrt{(\vartheta_2 - \phi)(\eta_2 - \phi)(0 - \phi)(\phi - \delta_2)}, \quad \phi \in [\delta_2, 0), \tag{51}$$

where $\vartheta_2, \eta_2, 0, \delta_2$ are roots of equation $\phi(2\beta/\alpha + (ch/\alpha)\phi - \phi^3) = 0$ and $\vartheta_2 > \eta_2 > 0 > \delta_2$. As in the above cases, these three roots can be obtained by Cardano formula as long as the parameters α, β, c, h are given concretely, so the similar cases will be not commentated anymore in the following discussions. Similarly, taking $(0, \delta_2)$ as the initial values of the variables (ξ, ϕ), substituting (51) into the $d\phi/d\xi = y$ of (11), and then integrating it yield

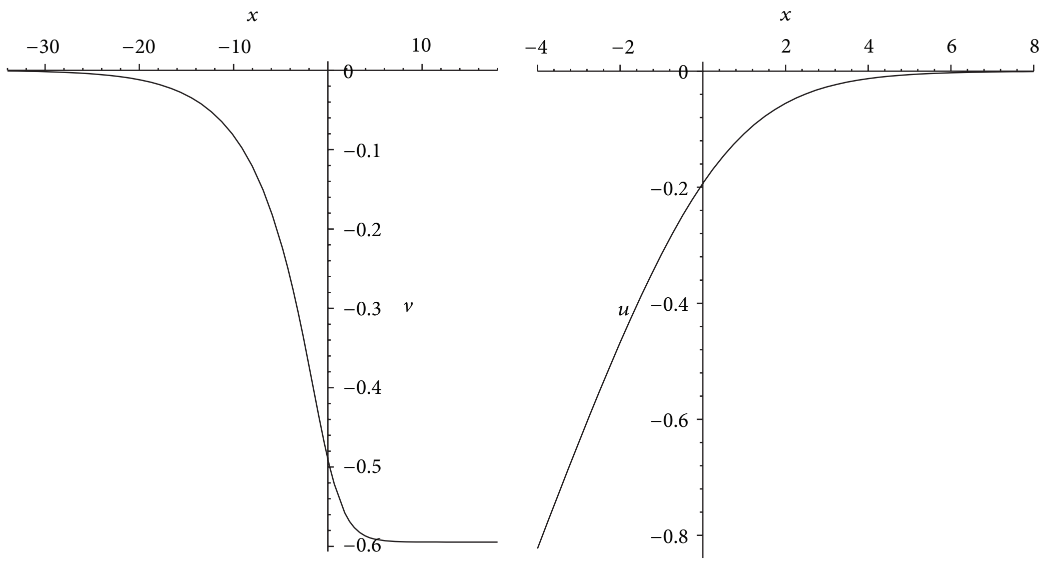
$$v(x,t) = \phi(\xi) = \frac{\eta_2\delta_2\mathrm{sn}^2(\Omega_2(x-ct), k_2)}{(\eta_2 - \delta_2) + \delta_2\mathrm{sn}^2(\Omega_2\xi, k_2)}, \quad \xi \neq T_2, \tag{52}$$

where $\Omega_2 = (1/2)\sqrt{\alpha\vartheta_2(\eta_2 - \delta_2)/c}$, $k_2 = \sqrt{-\delta_2(\vartheta_2 - \eta_2)/\vartheta_2(\eta_2 - \delta_2)}$, $T_2 = (1/\Omega_2)\mathrm{sn}^{-1}(0, k_2)$. By using the transformation $u(x,t) = \ln|v(x,t)|$ and (52), we obtain a periodic blow-up wave solution of (1) as follows:

$$u(x,t) = \ln\left|\frac{\eta_2\delta_2\mathrm{sn}^2(\Omega_2(x-ct), k_2)}{(\eta_2 - \delta_2) + \delta_2\mathrm{sn}^2(\Omega_2\xi, k_2)}\right|, \quad x - ct \neq T_2. \tag{53}$$

(d) Under the conditions $\alpha\beta < 0$, $\alpha c > 0$, $h \in (h_A, +\infty)$ and $m = 2$, (44) becomes

$$y = \pm\sqrt{\frac{\alpha}{c}}\sqrt{(\vartheta_2 - \phi)(\phi - \eta_2)(\phi - 0)(\phi - \delta_2)}. \tag{54}$$



(a) Smooth kink wave

(b) Unboundary wave

FIGURE 2: The phase portraits of solutions (42) and (43) for $m = 4$, $\alpha = 0.3$, $\beta = -0.3$, $c = 2$, $t = 0.1$.

Taking the $(0, \vartheta_2)$ as the initial values of the variables (ξ, ϕ), substituting (54) into the $d\phi/d\xi = y$ of (11), and then integrating it yield

$$v(x,t) = \phi(\xi) = \frac{\vartheta_2(\eta_2 - \delta_2)\,\mathrm{dn}^2(\Omega_2(x-ct), k_2)}{(\eta_2-\delta_2) + (\vartheta_2 - \eta_2)\,\mathrm{sn}^2(\Omega_2\xi, k_2)}. \quad (55)$$

By using the transformations $u(x,t) = \ln|v(x,t)|$ and (55), we obtain a periodic blow-up wave solution of (1) as follows:

$$u(x,t) = \ln\left|\frac{\vartheta_2(\eta_2 - \delta_2)\,\mathrm{dn}^2(\Omega_2(x-ct), k_2)}{(\eta_2-\delta_2) + (\vartheta_2 - \eta_2)\,\mathrm{sn}^2(\Omega_2(x-ct), k_2)}\right|. \quad (56)$$

(e) Under the conditions $\alpha\beta < 0$, $\alpha c < 0$, $h \in (-\infty, h_A)$ and $m = 2$, (44) becomes

$$y = \pm\sqrt{-\frac{\alpha}{c}}\sqrt{(\vartheta_3 - \phi)(\eta_3 - \phi)(\phi - 0)(\phi - \delta_3)}, \quad \phi \in (0, \eta_3], \quad (57)$$

where $\vartheta_3, \eta_3, 0, \delta_3$ are roots of equation $\phi(-2\beta/\alpha - (ch/\alpha)\phi + \phi^3) = 0$ and $\vartheta_3 > \eta_3 > 0 > \delta_3$. Taking the $(0, \eta_3)$ as the initial values of the variables (ξ, ϕ), substituting (57) into the $d\phi/d\xi = y$ of (11), and then integrating it yield

$$v(x,t) = \phi(\xi) = \frac{\vartheta_3\eta_3\mathrm{cn}^2(\Omega_3\xi, k_3)}{\delta_3 - \eta_3\mathrm{sn}^2(\Omega_3\xi, k_3)}, \quad \xi \neq T_3, \quad (58)$$

where $\Omega_3 = (1/2)\sqrt{-\alpha\vartheta_3(\eta_3-\delta_3)/c}$, $k_3 = \sqrt{\eta_3(\vartheta_3-\delta_3)/\vartheta_3(\eta_3-\delta_3)}$, $T_3 = (1/\Omega_3)\mathrm{sn}^{-1}(1, k_3)$. By using the transformation $u(x,t) = \ln|v(x,t)|$ and (58), we obtain a periodic blow-up wave solution of (1) as follows:

$$u(x,t) = \ln\left|\frac{\vartheta_3\eta_3\mathrm{cn}^2(\Omega_3(x-ct), k_3)}{\delta_3 - \eta_3\mathrm{sn}^2(\Omega_3(x-ct), k_3)}\right|, \quad x - ct \neq T_3. \quad (59)$$

(f) Under the conditions $\alpha\beta < 0$, $\alpha c < 0$, $h \in (-\infty, h_A)$ and $m = 2$, (44) becomes

$$y = \pm\sqrt{-\frac{\alpha}{c}}\sqrt{(\vartheta_4 - \phi)(0 - \phi)(\phi - \gamma_4)(\phi - \delta_4)}, \quad \phi \in [\gamma_4, 0), \quad (60)$$

where $\vartheta_4, 0, \gamma_4, \delta_4$ are roots of equation $\phi(-2\beta/\alpha - (ch/\alpha)\phi + \phi^3) = 0$ and $\vartheta_4 > 0 > \gamma_4 > \delta_4$. Taking the $(0, \gamma_4)$ as the initial values of the variables (ξ, ϕ), substituting (60) into the $d\phi/d\xi = y$ of (11), and then integrating it yield

$$v(x,t) = \phi(\xi) = \frac{\delta_4\gamma_4\mathrm{cn}^2(\Omega_4\xi, k_4)}{\delta_4 - \gamma_4\mathrm{cn}^2(\Omega_4\xi, k_4)}, \quad \xi \neq T_4, \quad (61)$$

where $\Omega_4 = (1/2)\sqrt{\alpha\delta_4(\vartheta_4-\gamma_4)/c}$, $k_4 = \sqrt{\gamma_4(\vartheta_4-\delta_4)/\vartheta_4(\vartheta_4-\gamma_4)}$, $T_4 = (1/\Omega_4)\mathrm{sn}^{-1}(1, k_4)$. By using the transformation $u(x,t) = \ln|v(x,t)|$ and (61), we obtain a periodic below-up wave solutions of (1) as follows:

$$u(x,t) = \ln\left|\frac{\delta_4\gamma_4\mathrm{cn}^2(\Omega_4(x-ct), k_4)}{\delta_4 - \gamma_4\mathrm{cn}^2(\Omega_4(x-ct), k_4)}\right|, \quad x - ct \neq T_4. \quad (62)$$

(g) Under the conditions $m = 2$, $\alpha c < 0$, $h = h_A = -\alpha^2/2c\beta$, (44) becomes

$$y = \pm\sqrt{-\frac{\alpha}{c}}\,|\phi_1 - \phi|\sqrt{\phi(\phi + 2\phi_1)}, \quad \phi \neq 0, \quad (63)$$

where $\phi_1 = (-\beta/\alpha)^{1/3}$. Taking $(0, (1/2)\phi_1)$ as the initial values of the variables (ξ, ϕ), substituting (63) into the $d\phi/d\xi = y$ of (11), and then integrating it yield

$$v(x,t) = \phi(\xi) = \phi_1\left[1 - \frac{3}{\cosh(\Omega_5\xi) + 2}\right], \quad \xi \neq T_5, \quad (64)$$

where $\Omega_5 = \phi_1\sqrt{-3\alpha/c}$, $T_5 = (1/\Omega_5)\cosh^{-1}(1)$. Clearly, we have $\phi(T_5) = 0$, $\phi(\pm\infty) = \phi_1$. By using the transformation $u(x,t) = \ln|v(x,t)|$ and (64), we obtain a solitary wave solution of blow-up form of (1) as follows:

$$u(x,t) = \ln\left|\phi_1\left[1 - \frac{3}{\cosh(\Omega_5(x-ct)) + 2}\right]\right|, \quad x - ct \neq T_5. \tag{65}$$

Equation (64) is smooth solitary wave solution of (9), but (65) is a solitary wave solution of blow-up form for (1) because $u \to -\infty$ as $v \to 0$. In order to visually show dynamical behaviors of solutions v and u of (64) and (65), we plot graphs of their profiles which are shown in Figures 3(a) and 3(b), respectively.

3.4. Different Kinds of Periodic Wave, Broken Solitary Wave Solutions in the Special Case $m = 1$, $n = -1$. (1) Under the conditions $h \in (h_A, +\infty)$, $\alpha\beta > 0$, $\beta c > 0$ (or $\alpha\beta < 0$, $\beta c > 0$), $\alpha c > 0$ and $m = 1$, (44) becomes

$$y = \pm\sqrt{\frac{2\alpha}{c}}\sqrt{(\phi_M - \phi)(\phi - \phi_m)(\phi - 0)}, \quad \phi \in [\phi_m, \phi_M], \tag{66}$$

where $\phi_{M,m} = (hc \pm \sqrt{h^2c^2 + 16\alpha\beta})/4\alpha$ and 0 are three roots of equation $h\phi^2 - (2\alpha/c)\phi^3 + (2\beta/c)\phi = 0$ and $\phi_M > \phi_m > 0$. Taking $(0, \phi_m)$ as the initial values of the variables (ξ, ϕ), substituting (66) into the $d\phi/d\xi = y$ of (11), and then integrating it yield

$$v(x,t) = \phi(\xi) = \phi_m - (\phi_M - \phi_m)\,\mathrm{sn}^2(\Omega_6\xi, k_6), \tag{67}$$

where $\Omega_6 = \sqrt{\alpha\phi_M/2c}$, $k_6 = \sqrt{(\phi_M - \phi_m)/\phi_M}$. By using the transformation $u(x,t) = \ln|v(x,t)|$ and (67), we obtain a double periodic wave solution of (1) as follows:

$$u(x,t) = \ln\left|\phi_m - (\phi_M - \phi_m)\,\mathrm{sn}^2(\Omega_6(x-ct), k_6)\right|. \tag{68}$$

Equation (67) is smooth periodic wave solution of (9), but (68) is a double periodic wave solution of blow-up form of (1) because $u \to -\infty$ as $v \to 0$. In order to visually show dynamical behaviors of solutions v and u of (67) and (68), we plot graphs of their profiles which are shown in Figures 4(a) and 4(b), respectively.

(2) Under the condition $h \in (-\infty, h_A)$, $\alpha\beta < 0$, $\beta c > 0$ (or $\alpha\beta < 0$, $\beta c < 0$),$\alpha c > 0$ and $m = 1$, (44) becomes

$$y = \pm\sqrt{\frac{2\alpha}{c}}\sqrt{(0 - \phi)(\phi - \phi_M)(\phi - \phi_m)}, \quad \phi \in [\phi_M, 0), \tag{69}$$

where ϕ_M, ϕ_m, 0 are roots of equation $h\phi^2 - (2\alpha/c)\phi^3 + (2\beta/c)\phi = 0$ and $\phi_m < \phi_M < 0$. Taking $(0, \phi_M)$ as the initial values of the variables (ξ, ϕ), substituting (69) into the $d\phi/d\xi = y$ of (11), and then integrating it yield

$$v(x,t) = \phi(\xi) = \phi_M\mathrm{sn}^2(\Omega_7\xi, k_7), \quad \xi \neq T_7, \tag{70}$$

where $\Omega_7 = \sqrt{-\alpha\phi_M/2c}$, $k_7 = \sqrt{\phi_M/\phi_m}$, $T_7 = (1/\Omega_7)\mathrm{sn}^{-1}(0, k_7)$. By using the transformation $u(x,t) = \ln|v(x,t)|$ and (70), we obtain a periodic blow-up wave solution of (1) as follows:

$$u(x,t) = \ln\left|\phi_M\mathrm{sn}^2(\Omega_7(x-ct), k_7)\right|, \quad x - ct \neq T_7. \tag{71}$$

(3) Under the conditions $h \in (h_A, +\infty)$, $\alpha\beta < 0$, $\beta c > 0$ (or $\alpha\beta > 0$, $\beta c > 0$) and $m = 1$, (44) becomes

$$y = \pm\sqrt{-\frac{2\alpha}{c}}\sqrt{(0 - \phi)(\phi_M - \phi)(\phi - \phi_m)}, \quad \phi \in [\phi_m, \phi_M], \tag{72}$$

where ϕ_M, ϕ_m, 0 are roots of equation $h\phi^2 - (2\alpha/c)\phi^3 + (2\beta/c)\phi = 0$ and $0 > \phi_M > \phi_m$. Taking $(0, \phi_M)$ as the initial values of the variables (ξ, ϕ), substituting (72) into the $d\phi/d\xi = y$ of (11), and then integrating it yield

$$v(x,t) = \phi(\xi) = \frac{\phi_M}{\mathrm{dn}^2(\Omega_8\xi, k_8)}, \tag{73}$$

where $\Omega_8 = \sqrt{\alpha\phi_m/2c}$, $k_8 = \sqrt{(\phi_M - \phi_m)/-\phi_m}$. By using the transformation $u(x,t) = \ln|v(x,t)|$ and (73), we obtain a periodic blow-up wave solutions of (1) as follows:

$$u(x,t) = \ln\left|\frac{\phi_M}{\mathrm{dn}^2(\Omega_8(x-ct), k_8)}\right|. \tag{74}$$

(4) Under the conditions $h \in (-\infty, h_A)$, $\alpha\beta < 0$, $\beta c > 0$ (or $\alpha\beta < 0$, $\beta c < 0$) and $m = 1$, (44) becomes

$$y = \pm\sqrt{-\frac{2\alpha}{c}}\sqrt{(\phi_M - \phi)(\phi_m - \phi)(\phi - 0)}, \quad \phi \in (0, \phi_m], \tag{75}$$

where ϕ_M, ϕ_m, 0 are roots of equation $h\phi^2 - (2\alpha/c)\phi^3 + (2\beta/c)\phi = 0$ and $\phi_M > \phi_m > 0$. Taking $(0, \phi_m)$ as the initial values of the variables (ξ, ϕ), substituting (75) into the $d\phi/d\xi = y$ of (11), and then integrating it yield

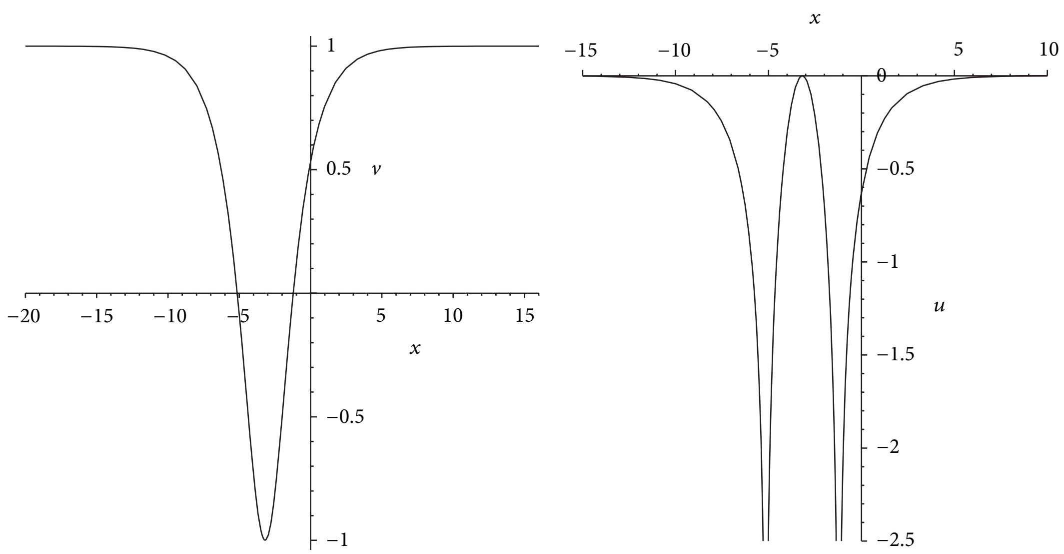
$$v(x,t) = \phi(\xi) = \frac{\phi_m\mathrm{cn}^2(\Omega_9\xi, k_9)}{\mathrm{dn}^2(\Omega_9\xi, k_9)}, \quad \xi \neq T_9, \tag{76}$$

where $\Omega_9 = \sqrt{-\alpha\phi_M/2c}$, $k_9 = \sqrt{\phi_m/\phi_M}$, $T_9 = (1/\Omega_9)\mathrm{sn}^{-1}(0, k_9)$. By using the transformation $u(x,t) = \ln|v(x,t)|$ and (76), we obtain periodic blow-up wave solutions of (1) as follows:

$$u(x,t) = \ln\left|\frac{\phi_m\mathrm{cn}^2(\Omega_9(x-ct), k_9)}{\mathrm{dn}^2(\Omega_9(x-ct), k_9)}\right|, \quad x - ct \neq T_9. \tag{77}$$

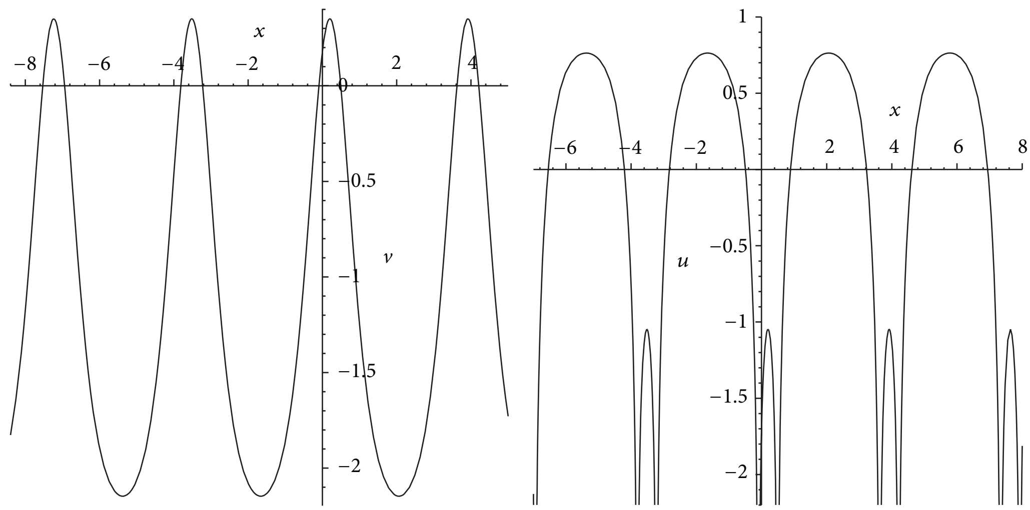
(5) Under the conditions $h = h_A = -4\alpha\phi_1/c$, $\alpha\beta < 0$, $\beta c > 0$ and $m = 1$, (44) becomes

$$y = \pm\sqrt{\frac{2\alpha}{c}}\,|\phi_1 + \phi|\sqrt{-\phi}, \quad \phi \in [-\phi_1, 0), \tag{78}$$



(a) Smooth solitary wave (b) Solitary wave of blow-up form

Figure 3: The phase portraits of solutions (64) and (65) for $m = 2$, $\alpha = 0.3$, $\beta = -0.3$, $c = -2$, $t = 0.1$.



(a) Smooth periodic wave (b) Double periodic wave of blow-up type

Figure 4: The phase portraits of solutions (67) and (68) for $h = 8$, $\alpha = 2.5$, $\beta = -2.5$, $c = 2$, $t = 0.1$.

where $\phi_1 = (-\beta/\alpha)^{1/2}$. Taking $(0, -\phi_1)$ as the initial values of the variables (ξ, ϕ), substituting (78) into the $d\phi/d\xi = y$ of (11), and then integrating it yield

$$v(x,t) = \phi(\xi) = -\phi_1 \tanh^2(0.5\Omega_{10}\xi), \quad \xi \neq 0, \tag{79}$$

where $\Omega_{10} = \sqrt{2\alpha\phi_1/c}$. By using the transformation $u(x,t) = \ln|v(x,t)|$ and (79), we obtain a solitary wave solution of blow-up form of (1) as follows:

$$u(x,t) = \ln\left|\phi_1 \tanh^2(0.5\Omega_{10}(x-ct))\right|, \quad x - ct \neq 0. \tag{80}$$

(6) Under the conditions $h = h_A = 4\alpha\phi_1/c$, $\alpha\beta < 0$, $\beta c > 0$ and $m = 1$, (44) becomes

$$y = \pm\sqrt{-\frac{2\alpha}{c}}\left|\phi - \phi_1\right|\sqrt{\phi}, \quad \phi \in (0, \phi_1], \tag{81}$$

where $\phi_1 = (-\beta/\alpha)^{1/2}$. Taking $(0, \phi_1)$ as the initial values of the variables (ξ, ϕ), substituting (81) into the $d\phi/d\xi = y$ of (11), and then integrating it yield

$$v(x,t) = \phi(\xi) = \phi_1 \tanh^2(0.5\Omega_{11}\xi), \quad \xi \neq 0, \tag{82}$$

where $\Omega_{11} = \sqrt{-2\alpha\phi_1/c}$. By using the transformation $u(x,t) = \ln|v(x,t)|$ and (82), we obtain a broken solitary wave solution of (1) as follows:

$$u(x,t) = \ln\left|\phi_1 \tanh^2(0.5\Omega_{11}(x-ct))\right|, \quad x - ct \neq 0. \tag{83}$$

(7) Under the conditions $h \in (-\infty, 0]$, $\alpha\beta > 0$, $\beta c < 0$ and $m = 1$, (44) becomes

$$y = \pm\sqrt{-\frac{2\alpha}{c}}\sqrt{(\phi_M - \phi)(0 - \phi)(\phi - \phi_m)}, \quad \phi \in [\phi_m, 0), \tag{84}$$

where ϕ_M, ϕ_m, 0 are roots of equation $\phi^2 h - (2\alpha/c)\phi^3 + (2\beta/c)\phi = 0$ and $\phi_M > 0 > \phi_m$. Taking $(0, \phi_M)$ as the initial values of the variables (ξ, ϕ), substituting (84) into the $d\phi/d\xi = y$ of (11), and then integrating it yield

$$v(x,t) = \phi(\xi) = \frac{-\phi_M k_{12}^2 \mathrm{sn}^2(\Omega_{12}\xi, k_{12})}{\mathrm{dn}^2(\Omega_{12}\xi, k_{12})}, \quad \xi \neq T_{12}, \tag{85}$$

where $\Omega_{12} = \sqrt{-\alpha(\phi_M - \phi_m)/2c}$, $k_{12} = \sqrt{-\phi_m/(\phi_M - \phi_m)}$, $T_{12} = (1/\Omega_{12})\mathrm{sn}^{-1}(0, k_{12})$. By using the transformation $u(x,t) = \ln|v(x,t)|$ and (85), we obtain a periodic blow-up wave solution of (1) as follows:

$$u(x,t) = \ln\left|\frac{-\phi_M k_{12}^2 \mathrm{sn}^2(\Omega_{12}(x-ct), k_{12})}{\mathrm{dn}^2(\Omega_{12}(x-ct), k_{12})}\right|, \quad x - ct \neq T_{12}. \tag{86}$$

(8) Under the conditions $h \in (-\infty, 0]$, $\alpha\beta > 0$, $\beta c > 0$ and $m = 1$, (44) becomes

$$y = \pm\sqrt{\frac{2\alpha}{c}}\sqrt{(\phi_M - \phi)(\phi - 0)(\phi - \phi_m)}, \quad \phi \in (0, \phi_M], \tag{87}$$

where ϕ_M, ϕ_m, 0 are roots of equation $h\phi^2 - (2\alpha/c)\phi^3 + (2\beta/c)\phi = 0$ and $\phi_M > 0 > \phi_m$. Taking $(0, \phi_M)$ as the initial values of the variables (ξ, ϕ), substituting (87) into the $d\phi/d\xi = y$ of (11), and then integrating it yield

$$v(x,t) = \phi_M \mathrm{cn}^2(\Omega_{13}\xi, k_{13}), \quad \xi \neq T_{13}, \tag{88}$$

where $\Omega_{13} = \sqrt{\alpha(\phi_M - \phi_m)/2c}$, $k_{13} = \sqrt{\phi_M/(\phi_M - \phi_m)}$, $T_{13} = (1/\Omega_{13})\mathrm{sn}^{-1}(0, k_{13})$. By using the transformation $u(x,t) = \ln|v(x,t)|$ and (88), we obtain a periodic blow-up wave solution of (1) as follows:

$$u(x,t) = \ln\left|\phi_M \mathrm{cn}^2(\Omega_{13}(x-ct), k_{13})\right|, \quad x - ct \neq T_{13}. \tag{89}$$

3.5. Periodic Blow-Up Wave, Broken Solitary Wave, Broken Kink, and Antikink Wave Solutions in the Special Cases $m = 2$ or $m = 1$ and $n = -2$. When $m = 2$ is even and $n = -2$, (14) can be reduced to

$$y = \pm\sqrt{h\phi^2 - \frac{2\alpha}{cm}\phi^{m+2} + \frac{\beta}{c}}. \tag{90}$$

Because the singular straight line $\phi = 0$, this implies that (1) has broken kink and antikink wave solutions: see the following discussions.

(i) Under the conditions $m = 2$, $n = -2$, $h = \sqrt{-4\alpha\beta/c^2}$, $\alpha\beta < 0$, $\alpha c < 0$, (90) becomes

$$y = \pm\sqrt{-\frac{\alpha}{c}}\left|\phi^2 - \phi_1^2\right|, \quad (\phi \neq 0), \tag{91}$$

where $\phi_1 = \sqrt{-\beta/\alpha}$. Taking $(0, \phi_1)$ as the initial values of the variables (ξ, ϕ), substituting (91) into the $d\phi/d\xi = y$ of (11), and then integrating it yield

$$v(x,t) = \phi(\xi) = \pm\phi_1 \tanh[\Omega_{14}\xi], \tag{92}$$

where $\Omega_{14} = \phi_1\sqrt{-\alpha/c}$. By using the transformations $u(x,t) = \ln|v(x,t)|$ and (92), we obtain a broken kink wave solutions of (1) as follows:

$$u(x,t) = \ln\left|\phi_1 \tanh[\Omega_{14}(x-ct)]\right|, \tag{93}$$

where $(x - ct) \in (-\infty, 0)$ and $(x - ct) \in (0, +\infty)$.

(ii) Under the conditions $h \in (h_{B1}, +\infty)$, $\alpha\beta > 0$, $\alpha c > 0$ and $m = 2$, $n = -2$, (90) becomes

$$y = \pm\sqrt{\frac{\alpha}{c}}\sqrt{(a^2 - \phi^2)(\phi^2 - b^2)}, \tag{94}$$

where $a^2 = (1/2\alpha)(ch + \sqrt{c^2h^2 + 4\alpha\beta})$, $b^2 = (1/2\alpha)(ch - \sqrt{c^2h^2 + 4\alpha\beta})$. Taking $(0, a)$ as the initial values of the variables (ξ, ϕ), substituting (94) into the $d\phi/d\xi = y$ of (11), and then integrating it yield

$$v(x,t) = \phi(\xi) = \pm\sqrt{a^2 - (a^2 - b^2)\mathrm{sn}^2(\Omega_{15}\xi, k_{15})}, \tag{95}$$

where $a^2 = (1/2\alpha)(ch + \sqrt{c^2h^2 + 4\alpha\beta})$, $b^2 = (1/2\alpha)(ch - \sqrt{c^2h^2 + 4\alpha\beta})$, $\Omega_{15} = a\sqrt{\alpha/c}$, $k_{15} = \sqrt{(a^2 - b^2)/a^2}$. By using the transformations $u(x,t) = \ln|v(x,t)|$ and (95), we obtain a periodic blow-up wave solution of (1) as follows:

$$u(x,t) = 0.5\ln\left|a^2 - \left(a^2 - b^2\right)\mathrm{sn}^2(\Omega_{15}\xi, k_{15})\right|. \tag{96}$$

(iii) Under the conditions $m = 1$, $n = -2$, $h = h_{B1} = 3\alpha\phi_2/c$, $\alpha\beta > 0$, $\alpha c > 0$, (90) becomes

$$y = \pm\sqrt{\frac{\alpha}{c}}\left|-\phi_2 + \phi\right|\sqrt{-\phi_2 - 2\phi}, \quad (\phi \neq 0), \tag{97}$$

where $\phi_2 = (-\beta/\alpha)^{1/3}$. Taking $(0, -0.5\phi_2)$ as the initial values of the variables (ξ, ϕ), substituting (97) into the $d\phi/d\xi = y$ of (11), and then integrating it yield

$$v(x,t) = \phi(\xi) = -\phi_2\left[\frac{3}{\cosh(\Omega_{16}\xi) + 1} - 1\right], \quad \xi \neq T_{16}, \tag{98}$$

where $\Omega_{16} = \sqrt{-3\alpha\phi_2/c}$, $T_{16} = (1/\Omega_{16})\cosh^{-1}(2)$. By using the transformations $u(x,t) = \ln|v(x,t)|$ and (98), when

$(x-ct) \in (-\infty, T_{16}) \bigcup (T_{16}, +\infty)$, we obtain a broken solitary wave solution of (1) as follows:

$$u(x,t) = \ln\left|\frac{3\phi_2}{\cosh(\Omega_{16}(x-ct))+1} - \phi_2\right|. \quad (99)$$

(iv) Under the conditions $h = h_{B1} = 3\alpha\phi_2/c$, $\alpha\beta < 0$, $\alpha c < 0$ and $m = 1$, $n = -2$, (90) becomes

$$y = \pm\sqrt{-\frac{\alpha}{c}}\,|\phi_2 - \phi|\,\sqrt{\phi_2 + 2\phi}, \quad (\phi \neq 0), \quad (100)$$

where $\phi_2 = (-\beta/\alpha)^{1/3}$. Taking $(0, -0.5\phi_2)$ as the initial values of the variables (ξ, ϕ), substituting (100) into the $d\phi/d\xi = y$ of (11), and then integrating it yield

$$v(x,t) = \phi(\xi) = \phi_2\left[1 - \frac{3}{\cosh(\Omega_{17}\xi)+1}\right], \quad \xi \neq T_{17}, \quad (101)$$

where $\Omega_{17} = \sqrt{3\alpha\phi_1/c}$, $T_{17} = (1/\Omega_{17})\cosh^{-1}(2)$. By using the transformations $u(x,t) = \ln|v(x,t)|$ and (101), when $(x-ct) \in (-\infty, T_{17}) \bigcup (T_{17}, +\infty)$, we obtain a broken solitary wave solution of (1) as follows:

$$u(x,t) = \ln\left|\phi_2 - \frac{3\phi_2}{\cosh(\Omega_{17}(x-ct))+1}\right|. \quad (102)$$

(v) Under the conditions $h \in (h_{B1}, +\infty)$, $\alpha\beta < 0$, $\beta c < 0$ and $m = 1$, $n = -2$, (90) becomes

$$y = \pm\sqrt{\frac{2\alpha}{c}}\sqrt{(\vartheta_5 - \phi)(\phi - \eta_5)(\phi - \gamma_5)}, \quad \phi \in [\eta_5, \vartheta_5], \quad (103)$$

where $\vartheta_5, \eta_5, \gamma_5$ are roots of equation $\beta/2\alpha + (ch/2\alpha)\phi^2 - \phi^3$ and $\gamma_5 < 0 < \eta_5 < \vartheta_5$. Taking $(0, \vartheta_5)$ as the initial values of the variables (ξ, ϕ), substituting (103) into the $d\phi/d\xi = y$ of (11), and then integrating it yield

$$v(x,t) = \phi(\xi) = \vartheta_5 - (\vartheta_5 - \eta_5)\,\mathrm{sn}^2(\Omega_{18}\xi, k_{18}), \quad (104)$$

where $\Omega_{18} = \sqrt{\alpha(\vartheta_5 - \gamma_5)/2c}$, $k_{18} = \sqrt{(\vartheta_5 - \eta_5)/(\vartheta_5 - \gamma_5)}$. By using the transformations $u(x,t) = \ln|v(x,t)|$ and (104), we obtain a periodic blow-up wave solution of (1) as follows:

$$u(x,t) = \ln\left|\vartheta_5 - (\vartheta_5 - \eta_5)\,\mathrm{sn}^2(\Omega_{18}(x-ct), k_{18})\right|. \quad (105)$$

(vi) Under the conditions $h \in (h_{B1}, +\infty)$, $\alpha\beta > 0$, $\beta c < 0$ and $m = 1$, $n = -2$, (90) becomes

$$y = \pm\sqrt{\frac{2\alpha}{c}}\sqrt{(\vartheta_5 - \phi)(\eta_5 - \phi)(\phi - \gamma_5)}, \quad \phi \in [\gamma_5, \eta_5], \quad (106)$$

where $\gamma_5 < \eta_5 < 0 < \vartheta_5$. Taking $(0, \vartheta_5)$ as the initial values of the variables (ξ, ϕ), substituting (106) into the $d\phi/d\xi = y$ of (11), and then integrating it yield

$$v(x,t) = \phi(\xi) = \gamma_5 + (\eta_5 - \gamma_5)\,\mathrm{sn}^2(\Omega_{19}\xi, k_{19}), \quad (107)$$

where $\gamma_5 < \eta_5 < 0 < \vartheta_5$, $\Omega_{19} = \sqrt{-\alpha(\vartheta_5 - \gamma_5)/2c}$, $k_{19} = \sqrt{(\vartheta_5 - \eta_5)/(\vartheta_5 - \gamma_5)}$. By using the transformations $u(x,t) = \ln|v(x,t)|$ and (113), we obtain a periodic blow-up wave solution of (1) as follows:

$$u(x,t) = \ln\left|\gamma_5 + (\eta_5 - \gamma_5)\,\mathrm{sn}^2(\Omega_{19}(x-ct), k_{19})\right|. \quad (108)$$

3.6. Periodic Wave, Broken Solitary Wave, Broken Kink, and Antikink Wave Solutions in the Special Cases $n=-m$ or $n=-2m$ and $m \in \mathbf{Z}^+$. (a) When $n = -m$, $m \in \mathbf{Z}^+$ and $\alpha\beta < 0$, $\beta c < 0$, $\alpha c > 0$, $h > 4\sqrt{-\alpha\beta}/cm$, (14) can be reduced to

$$\begin{aligned} y &= \pm\frac{\sqrt{(2\beta/cm)\phi^m + h(\phi^m)^2 - (2\alpha/cm)(\phi^m)^3}}{\phi^{m-1}} \\ &= \pm\frac{\sqrt{2\alpha/cm}\sqrt{(\phi_g^m - \phi^m)(\phi^m - \phi_l^m)(\phi^m - 0)}}{\phi^{m-1}}, \end{aligned} \quad (109)$$

where $\phi_{g,l} = [(1/4\alpha)(cmh \pm \sqrt{c^2m^2h^2 + 16\alpha\beta})]^{1/m}$. Taking $(0, \phi_g)$ as the initial values of the variables (ξ, ϕ), substituting (109) into the $d\phi/d\xi = y$ of (11), and then integrating it yields

$$v(x,t) = \phi(\xi) = \pm\left[\phi_g - (\phi_g - \phi_l)\,\mathrm{sn}^2(\Omega_{19}\xi, k_{19})\right]^{1/m}, \quad (110)$$

where $\Omega_{19} = m\sqrt{\alpha\phi_g/2cm}$, $k_{19} = \sqrt{(\phi_g - \phi_l)/\phi_g}$. By using the transformations $u(x,t) = \ln|v(x,t)|$ and (110), we obtain a periodic wave solution of (1) as follows:

$$u(x,t) = \frac{1}{m}\ln\left|\left[\phi_g - (\phi_g - \phi_l)\,\mathrm{sn}^2(\Omega_{19}(x-ct), k_{19})\right]\right|. \quad (111)$$

(b) When $n = -2m$, $m \in \mathbf{Z}^+$ and $\alpha\beta < 0$, $\beta c < 0$, $\alpha c > 0$, $h > h_{B1}$, (14) can be reduced to

$$\begin{aligned} y &= \pm\frac{\sqrt{\beta/cm + h(\phi^m)^2 - (2\alpha/cm)(\phi^m)^3}}{\phi^{m-1}} \\ &= \pm\frac{\sqrt{2\alpha/cm}\sqrt{(\vartheta^m - \phi^m)(\phi^m - \eta^m)(\phi^m - \gamma^m)}}{\phi^{m-1}}, \end{aligned} \quad (112)$$

where ϑ^m, η^m, γ^m are roots of equation $\beta/cm + h(\phi^m)^2 - (2\alpha/cm)(\phi^m)^3 = 0$ and $\vartheta^m > \eta^m > 0 > \gamma^m$. Taking $(0, \vartheta)$ as the initial values of the variables (ξ, ϕ), substituting (112) into the $d\phi/d\xi = y$ of (11), and then integrating it yield

$$v(x,t) = \phi(\xi) = \left[\vartheta^m - (\vartheta^m - \eta^m)\,\mathrm{sn}^2(\Omega_{20}\xi, k_{20})\right]^{1/m}, \quad (113)$$

where $\Omega_{20} = m\sqrt{\alpha(\vartheta^m - \gamma^m)/2cm}$, $k_{20} = \sqrt{(\vartheta^m - \eta^m)/(\vartheta^m - \gamma^m)}$. By using the transformations $u(x,t) = \ln|v(x,t)|$ and (113), we obtain periodic wave solutions of (1) as follows:

$$u(x,t) = \frac{1}{m}\ln\left|\vartheta^m - (\vartheta^m - \eta^m)\,\mathrm{sn}^2(\Omega_{20}(x-ct), k_{20})\right|. \quad (114)$$

(c) When $n = -2m$, $m \in \mathbf{Z}^+$ and $\alpha\beta > 0$, $\beta c < 0$, $\alpha c > 0$, $h > h_{B2}$, (14) can be reduced to

$$y = \pm \frac{\sqrt{\beta/cm + h(\phi^m)^2 - (2\alpha/cm)(\phi^m)^3}}{\phi^{m-1}} = \pm \frac{\sqrt{-2\alpha/cm}\sqrt{(\vartheta^m - \phi^m)(\eta^m - \phi^m)(\phi^m - \gamma^m)}}{\phi^{m-1}}, \tag{115}$$

where $\vartheta^m > 0 > \eta^m > \gamma^m$. Taking $(0, \eta)$ as the initial values of the variables (ξ, ϕ), substituting (115) into the $d\phi/d\xi = y$ of (11), and then integrating it yield

$$v(x,t) = \left[\frac{\eta^m - \vartheta^m k_{21}^2 \operatorname{sn}^2(\Omega_{21}\xi, k_{21})}{\operatorname{dn}^2(\Omega_{21}\xi, k_{21})}\right]^{1/m}, \tag{116}$$

where $\Omega_{21} = m\sqrt{-\alpha(\vartheta^m - \gamma^m)/2cm}$, $k_{21} = \sqrt{(\eta^m - \eta^m)/(\vartheta^m - \gamma^m)}$. By using the transformations $u(x,t) = \ln|v(x,t)|$ and (116), we obtain a periodic wave solution of (1) as follows:

$$u(x,t) = \frac{1}{m}\ln\left|\frac{\eta^m - \vartheta^m k_{21}^2 \operatorname{sn}^2(\Omega_{21}\xi, k_{21})}{\operatorname{dn}^2(\Omega_{21}\xi, k_{21})}\right|. \tag{117}$$

4. Exact Traveling Wave Solutions of (1) and Their Properties under the Conditions $m \in \mathbf{Z}^-, n \in \mathbf{Z}^-$

In this sections, we consider the case $m \in \mathbf{Z}^-$, $n \in \mathbf{Z}^-$ of (1); that is, we will investigate different kinds of exact traveling wave solutions of (1) under the conditions $m \in \mathbf{Z}^-$, $n \in \mathbf{Z}^-$. Letting $m = -q$, $n = -p$, $p \in \mathbf{Z}^+$, $q \in \mathbf{Z}^+$, (11) becomes

$$\frac{d\phi}{d\xi} = y, \qquad \frac{dy}{d\xi} = \frac{cy^2 - \alpha\phi^{2-q} - \beta\phi^{2-p}}{c\phi}. \tag{118}$$

First, we consider a special case $p = q = 1$. When $p = q = 1$, under the transformation $d\xi = c\phi d\tau$, system (118) can be reduced to the following regular system:

$$\frac{d\phi}{d\tau} = c\phi y, \qquad \frac{dy}{d\tau} = cy^2 - (\alpha + \beta)\phi, \tag{119}$$

which is an integrable system and it has the following first integral:

$$y^2 = \phi^2\widetilde{h} + \frac{2(\alpha+\beta)}{c}\phi, \tag{120}$$

where $\widetilde{h}$ is integral constant. When $\phi \neq 0$, we define

$$H(\phi, y) = \frac{y^2}{\phi^2} - \frac{2(\alpha+\beta)}{c\phi} = \widetilde{h}. \tag{121}$$

Obviously, when $\alpha \neq -\beta$, system (119) has only one equilibrium point $O(0,0)$ in the ϕ-axis.

Second, we consider the special case $p = 2$, $q = 1$, and another case $p = 1$, $q = 2$ is very similar to the front case, so we only discuss the front case here. When $p = 2$, $q = 1$, under the transformation $d\xi = c\phi d\tau$, system (118) can be reduced to the following regular system:

$$\frac{d\phi}{d\tau} = c\phi y, \qquad \frac{dy}{d\tau} = cy^2 - \alpha\phi - \beta, \tag{122}$$

which is an integrable system and it has the following first integral:

$$y^2 = \widetilde{h}\phi^2 + \frac{2\alpha}{c}\phi + \frac{\beta}{c}, \tag{123}$$

also we define

$$H(\phi, y) = \frac{y^2}{\phi^2} - \frac{2\alpha}{c\phi} - \frac{\beta}{c\phi^2} = \widetilde{h}. \tag{124}$$

Indeed, system (122) has only one equilibrium point $\widetilde{A}(\phi_0, 0)$ in the ϕ-axis, where $\phi_0 = -\beta/\alpha$. When $c\beta > 0$, system (122) has two equilibrium points $S_\pm(0, Y_\pm)$ in the singular line $\phi = 0$, where $Y_\pm = \pm\sqrt{\beta/c}$.

Finally, we consider the general case $p > 2$. When $p > 2$, under the transformation $d\xi = c\phi^{p-1}d\tau$, system (118) can be reduced to the following regular system:

$$\frac{d\phi}{d\tau} = c\phi^{p-1}y, \qquad \frac{dy}{d\tau} = c\phi^{p-2}y^2 - \alpha\phi^{p-q} - \beta \tag{125}$$

which is an integrable system and it has the following first integral:

$$y^2 = \phi^2\widetilde{h} + \frac{2\alpha}{cq}\phi^{2-q} + \frac{2\beta}{cp}\phi^{2-p}. \tag{126}$$

When $\phi \neq 0$, we define

$$H(\phi, y) = \frac{y^2}{\phi^2} - \frac{2\alpha}{cq}\phi^{-q} - \frac{2\beta}{cp}\phi^{-p} = \widetilde{h}. \tag{127}$$

Obviously, when $p = q$, system (125) has not any equilibrium point in the ϕ-axis. When $p - q$ is even number and $\alpha\beta < 0$, system (125) has two equilibrium points $B_{1,2}(\pm\phi_+, 0)$ in the ϕ-axis, where $\phi_+ = (-\beta/\alpha)^{1/(p-q)}$. When $p - q$ is odd number, system (125) has only one equilibrium point at $B_1(\phi_+, 0)$ in the ϕ-axis.

Respectively substituting the above equilibrium points into (121), (124), and (127), we have

$$\widetilde{h}_{\widetilde{A}} = H(\phi_0, 0) = -\frac{\alpha^2}{c\beta}, \tag{128}$$

$$\widetilde{h}_1 = H(\phi_+, 0) = -\frac{2\alpha}{cq}\phi_+^{-q} - \frac{2\beta}{cp}\phi_+^{-p}, \tag{129}$$

$$\widetilde{h}_2 = H(-\phi_+, 0) = (-1)^{q+1}\frac{2\alpha}{cq}\phi_+^{-q} + (-1)^{p+1}\frac{2\beta}{cp}\phi_+^{-p}. \tag{130}$$

4.1. Nonsmooth Peakon Solutions and Periodic Blow-Up Wave Solutions under Two Special Cases $p=q=1$ and $p=2$, $q=1$. (1) When $p=q=1$ and $\alpha c>0$, $\widetilde{h}<0$, (120) can be reduced to

$$y=\pm\sqrt{\widetilde{h}\phi^2-\frac{2(\alpha+\beta)}{c}\phi}. \tag{131}$$

Substituting (131) into the first equation of (118), taking $(0, c\widetilde{h}/(2(\alpha+\beta)))$ as initial value conditions, and integrating it, we obtain

$$v(x,t)=\phi(\xi)=\frac{\alpha+\beta}{ch}\left(\cos\sqrt{-\widetilde{h}}\xi-1\right). \tag{132}$$

From the transformation $u(x,t)=\ln|v(x,t)|$ and (132), we obtain a periodic blow-up wave solution of (1) as follows:

$$u(x,t)=\ln\left|\frac{\alpha+\beta}{ch}\left[\cos\left(\sqrt{-\widetilde{h}}(x-ct)\right)-1\right]\right|, \tag{133}$$

where $(x-ct)\neq 2K\pi/\sqrt{-\widetilde{h}}$, $K\in\mathbf{Z}$.

(2) When $p=2$, $q=1$ and $\alpha\beta>0$, $\beta c>0$ (or $\alpha\beta<0$, $\beta c>0$), $\widetilde{h}=\widetilde{h}_{\widetilde{A}}=\alpha^2/\beta c$, (123) can be reduced to

$$y=\pm\frac{\alpha}{\sqrt{\beta c}}\left|\phi+\frac{\beta}{\alpha}\right|. \tag{134}$$

Similarly, substituting (134) into the first equation of (118) and then integrating it, and setting the integral constant as zero, we have

$$v(x,t)=\phi(\xi)=-\frac{\beta}{\alpha}\left[1-\exp\left(-\frac{|\alpha\xi|}{\sqrt{\beta c}}\right)\right]. \tag{135}$$

From the transformation $u(x,t)=\ln|v(x,t)|$ and (135), we obtain a broken solitary cusp wave solution of (1) as follows:

$$u(x,t)=\ln\left|-\frac{\beta}{\alpha}\left[1-\exp\left(-\frac{|\alpha(x-ct)|}{\sqrt{\beta c}}\right)\right]\right|,\quad (x-ct)\neq 0. \tag{136}$$

(3) When $p=2$, $q=1$ and $\alpha\beta>0$, $\beta c>0$ (or $\alpha\beta<0$, $\beta c>0$), $\widetilde{h}<0$, (123) can be reduced to

$$\begin{aligned} y&=\pm\sqrt{-\widetilde{h}}\sqrt{-\frac{\beta}{c\widetilde{h}}-\frac{2\alpha}{c\widetilde{h}}\phi-\phi^2}\\ &=\pm\sqrt{-\widetilde{h}}\sqrt{(\phi_M-\phi)(\phi-\phi_m)}, \end{aligned} \tag{137}$$

where $\phi_M=(-\alpha+\sqrt{\alpha^2-c\widetilde{h}\beta})/c\widetilde{h}$, $\phi_m=(-\alpha-\sqrt{\alpha^2-c\widetilde{h}\beta})/c\widetilde{h}$ are roots of equation $-\beta/c\widetilde{h}-(2\alpha/c\widetilde{h})\phi-\phi^2=0$. Substituting (137) into the first equation of (118), taking $(0,\phi_M)$ as initial value conditions, and then integrating it, we have

$$v(x,t)=\phi(\xi)=\frac{1}{2}\left[(\phi_M+\phi_m)-(\phi_M-\phi_m)\cos\left(\sqrt{-h}\xi\right)\right],\quad \xi\in(-T_1,T_1), \tag{138}$$

where $T_1=(1/\sqrt{-\widetilde{h}})\arccos((\phi_M+\phi_m)/(\phi_M-\phi_m))$. From the transformation $u(x,t)=\ln|v(x,t)|$ and (138), we obtain a periodic blow-up wave solution of (1) as follows:

$$u(x,t)=\ln\left|\frac{1}{2}\left[(\phi_M+\phi_m)-(\phi_M-\phi_m)\cos\left(\sqrt{-\widetilde{h}}(x-ct)\right)\right]\right|, \tag{139}$$

where $x-ct\neq T_1+K\pi$, $K\in\mathbf{Z}$.

(4) When $p=2$, $q=1$ and $\alpha\beta>0$, $\beta c<0$ (or $\alpha\beta<0$, $\beta c<0$), $\widetilde{h}>-\alpha^2/c\beta$, (123) can be reduced to

$$y=\pm\sqrt{\widetilde{h}\phi^2+\frac{2\alpha}{c}\phi+\frac{\alpha}{c}}. \tag{140}$$

Similarly, we have

$$v(x,t)=\phi(\xi)=-\frac{\alpha}{c\widetilde{h}}-\frac{\sqrt{\alpha^2-\beta c\widetilde{h}}}{c\widetilde{h}}\cos\left(\sqrt{-\widetilde{h}}\xi\right). \tag{141}$$

From the transformation $u(x,t)=\ln|v(x,t)|$ and (141), we obtain a periodic blow-up wave solution of (1) as follows:

$$u(x,t)=\ln\left|\frac{\alpha}{c\widetilde{h}}+\frac{\sqrt{\alpha^2-\beta c\widetilde{h}}}{c\widetilde{h}}\cos\sqrt{-\widetilde{h}}(x-ct)\right|. \tag{142}$$

4.2. Periodic Blow-Up Wave Solutions in the Special Case $p=2q>1$. When $p=2q>2$ and $\alpha\beta<0$, $\alpha c>0$ (or $\alpha\beta>0$, $\alpha c<0$), $\widetilde{h}\in(\widetilde{h}_1,0)$, (118) becomes

$$\begin{aligned} y&=\pm\sqrt{\phi^2\left(\widetilde{h}+\frac{2\alpha}{cq}\phi^{-q}+\frac{\beta}{cq}\phi^{-2q}\right)}\\ &=\pm\frac{\sqrt{-\widetilde{h}}\sqrt{-\beta/cq\widetilde{h}-(2\alpha/cq\widetilde{h})\phi^q-(\phi^q)^2}}{\phi^{q-1}}. \end{aligned} \tag{143}$$

Substituting (143) into the first equation of (118) and then integrating it, we have

$$v(x,t)=\phi(\xi)=\left[-\frac{\alpha}{cq\widetilde{h}}+\sqrt{\frac{\alpha^2-cq\widetilde{h}\beta}{c^2q^2\widetilde{h}^2}}\cos\left(q\sqrt{-\widetilde{h}}\xi\right)\right]^{1/q}. \tag{144}$$

From the transformation $u(x,t)=\ln|v(x,t)|$ and (144), we obtain periodic blow-up wave solutions of (1) as follows:

$$u(x,t)=\frac{1}{q}\ln\left|-\frac{\alpha}{cq\widetilde{h}}+\sqrt{\frac{\alpha^2-cq\widetilde{h}\beta}{c^2q^2\widetilde{h}^2}}\cos\left(q\sqrt{-\widetilde{h}}(x-ct)\right)\right|. \tag{145}$$

5. Conclusion

In this work, by using the integral bifurcation method, a generalized Tzitzéica-Dodd-Bullough-Mikhailov (TDBM) equation is studied. In some special cases, we obtained many exact traveling wave solutions of TDBM equation. These exact solutions include smooth solitary wave solutions, singular periodic wave solutions of blow-up form, singular solitary wave solutions of blow-up form, broken solitary wave solutions, broken kink wave solutions, and some unboundary wave solutions. Comparing with the results in many references (such as [24–27]), these types of solutions in this paper are very different than those types of solutions which are obtained by Exp-function method. Though the types of solutions derived from two methods are different, they are all useful in nonlinear science. The smooth solitary wave solutions and singular solitary wave solutions obtained in this paper especially can be explained phenomenon of soliton surface that the total curvature at each point is proportional to the fourth power of the distance from a fixed point to the tangent plane.

Acknowledgments

The authors would like to thank the reviewers very much for their useful comments and helpful suggestions. This work is supported by the Natural Science Foundation of Yunnan Province (no. 2011FZ193) and the National Natural Science Foundation of China (no. 11161020). The e-mail address “weiguorhhu@aliyun.com” in the author’s former publications (published articles) has been replaced by new e-mail address “weiguorhhu@aliyun.com” because Yahoo Company in China will be closed on August 19, 2013. Please use the new E-mail address “weiguorhhu@aliyun.com” if the readers contact him.

References

[1] G. Tzitzéica, “Geometric infinitesimale-sur une nouvelle classes de surfaces,” *Comptes Rendus de l’Académie des Sciences*, vol. 144, pp. 1257–1259, 1907.

[2] G. Tzitzéica, “Geometric infinitesimale-sur une nouvelle classes de surfaces,” *Comptes Rendus de l’Académie des Sciences*, vol. 150, pp. 955–956, 1910.

[3] G. Tzitzéica, “Sur certaines courbes gauches,” *Annales Scientifiques de l’École Normale Supérieure. Troisième Série*, vol. 28, pp. 9–32, 1911.

[4] R. K. Bullough and R. K. Dodd, “Polynomial conserved densities for the sine-Gordon equations,” *Proceedings of the Royal Society A*, vol. 352, no. 1671, pp. 481–503, 1977.

[5] A. V. Žiber and A. B. Šabat, “The Klein-Gordon equation with nontrivial group,” *Doklady Akademii Nauk SSSR*, vol. 247, no. 5, pp. 1103–1107, 1979.

[6] R. Abazari, “The G'/G-expansion method for Tzitzéica type nonlinear evolution equations,” *Mathematical and Computer Modelling*, vol. 52, no. 9-10, pp. 1834–1845, 2010.

[7] A.-M. Wazwaz, “The variable separated ODE and the tanh methods for solving the combined and the double combined sinh-cosh-Gordon equations,” *Applied Mathematics and Computation*, vol. 177, no. 2, pp. 745–754, 2006.

[8] F. Natali, “On periodic waves for sine- and sinh-Gordon equations,” *Journal of Mathematical Analysis and Applications*, vol. 379, no. 1, pp. 334–350, 2011.

[9] Y. Zeng, “Constructing hierarchy of sinh-Gordon-type equations from soliton hierarchy with self-consistent source,” *Physica A*, vol. 259, no. 3-4, pp. 278–290, 1998.

[10] J. Teschner, “On the spectrum of the sinh-Gordon model in finite volume,” *Nuclear Physics B*, vol. 799, no. 3, pp. 403–429, 2008.

[11] A. H. Salas and J. E. Castillo H., “New exact solutions to sinh-cosh-Gordon equation by using techniques based on projective Riccati equations,” *Computers & Mathematics with Applications*, vol. 61, no. 2, pp. 470–481, 2011.

[12] X. Fan, S. Yang, J. Yin, and L. Tian, “Traveling wave solutions to the $(n+1)$-dimensional sinh-cosh-Gordon equation,” *Computers & Mathematics with Applications*, vol. 61, no. 3, pp. 699–707, 2011.

[13] A.-M. Wazwaz, “The tanh method: solitons and periodic solutions for the Dodd-Bullough-Mikhailov and the Tzitzéica-Dodd-Bullough equations,” *Chaos, Solitons and Fractals*, vol. 25, no. 1, pp. 55–63, 2005.

[14] V. A. Andreev, “Bäcklund transformations of the Bullough-Dodd-Zhiber-Shabat equation and symmetries of integrable equations,” *Akademiya Nauk SSSR. Teoreticheskaya i Matematicheskaya Fizika*, vol. 79, no. 1, pp. 151–154, 1989 (Russian).

[15] I. Yu. Cherdantsev and R. A. Sharipov, “Finite-gap solutions of the Bullough-Dodd-Zhiber-Shabat equation,” *Akademiya Nauk SSSR. Teoreticheskaya i Matematicheskaya Fizika*, vol. 82, no. 1, pp. 155–160, 1990 (Russian).

[16] I. Y. Cherdantzev and R. A. Sharipov, “Solitons on a finite-gap background in Bullough-Dodd-Jiber-Shabat model,” *International Journal of Modern Physics A*, vol. 5, no. 15, pp. 3021–3027, 1990.

[17] Y. V. Brezhnev, “Darboux transformation and some multi-phase solutions of the Dodd-Bullough-Tzitzéica equation: $U_{xt} = e^U - e^{-2U}$,” *Physics Letters A*, vol. 211, no. 2, pp. 94–100, 1996.

[18] W. K. Schief, “Self-dual Einstein spaces via a permutability theorem for the Tzitzéica equation,” *Physics Letters A*, vol. 223, no. 1-2, pp. 55–62, 1996.

[19] S. V. Meleshko and W. K. Schief, “A truncated Painlevé expansion associated with the Tzitzéica equation: consistency and general solution,” *Physics Letters A*, vol. 299, no. 4, pp. 349–352, 2002.

[20] W. Rui, B. He, Y. Long, and C. Chen, “The integral bifurcation method and its application for solving a family of third-order dispersive PDEs,” *Nonlinear Analysis. Theory, Methods & Applications*, vol. 69, no. 4, pp. 1256–1267, 2008.

[21] R. Weiguo, L. Yao, H. Bin, and L. Zhenyang, “Integral bifurcation method combined with computer for solving a higher order wave equation of KdV type,” *International Journal of Computer Mathematics*, vol. 87, no. 1-3, pp. 119–128, 2010.

[22] W. Rui and Y. Long, “Integral bifurcation method together with a translation-dilation transformation for solving an integrable 2-component Camassa-Holm shallow water system,” *Journal of Applied Mathematics*, vol. 2012, Article ID 736765, 21 pages, 2012.

[23] W. Rui, “Different kinds of exact solutions with two-loop character of the two-component short pulse equations of the first kind,” *Communications in Nonlinear Science and Numerical Simulation*, vol. 18, no. 10, pp. 2667–2678, 2013.

[24] F. Khani and S. Hamedi-Nezhad, "Some new exact solutions of the (2 + 1)-dimensional variable coefficient Broer-Kaup system using the Exp-function method," *Computers & Mathematics with Applications*, vol. 58, no. 11-12, pp. 2325–2329, 2009.

[25] F. Khani, "Analytic study on the higher order Ito equations: new solitary wave solutions using the Exp-function method," *Chaos, Solitons and Fractals*, vol. 41, no. 4, pp. 2128–2134, 2009.

[26] F. Khani, F. Samadi, and S. Hamedi-Nezhad, "New exact solutions of the Brusselator reaction diffusion model using the exp-function method," *Mathematical Problems in Engineering*, vol. 2009, Article ID 346461, 9 pages, 2009.

[27] F. Khani, S. Hamedi-Nezhad, and A. Molabahrami, "A reliable treatment for nonlinear Schrödinger equations," *Physics Letters A*, vol. 371, no. 3, pp. 234–240, 2007.