Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 351619, 11 pages
http://dx.doi.org/10.1155/2013/351619

Research Article

The Upwind Finite Volume Element Method for Two-Dimensional Burgers Equation

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Received 15 October 2012; Accepted 8 January 2013

Academic Editor: Xiaodi Li

A finite volume element method for approximating the solution to two-dimensional Burgers equation is presented. Upwind technique is applied to handle the nonlinear convection term. We present the semi-discrete scheme and fully discrete scheme, respectively. We show that the schemes are convergent to order one in space in L^2-norm. Numerical experiment is presented finally to validate the theoretical analysis.

1. Introduction

We consider the following two-dimensional Burgers equation [1–3]:

$$\begin{aligned}
&\text{(a)}\ \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x_1}+v\frac{\partial u}{\partial x_2}=\zeta\Delta u, \quad x=(x_1,x_2)\in\Omega,\\
&\qquad\qquad\qquad\qquad\qquad\qquad t\in J=(0,T],\\
&\text{(b)}\ \frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x_1}+v\frac{\partial v}{\partial x_2}=\zeta\Delta v, \quad (x,t)\in\Omega\times J,\\
&\text{(c)}\ u(x,0)=\phi(x), \quad v(x,0)=\psi(x), \quad x\in\Omega,\\
&\text{(d)}\ u=g_1, \quad v=g_2, \quad (x,t)\in\partial\Omega\times J,
\end{aligned} \tag{1}$$

for the unknown functions u and v in a bounded spatial domain $\Omega\subset\mathbb{R}^2$, over a time interval $[0,T]$. The coefficient ζ is a positive number.

Burgers equation is the simplest nonlinear convection-diffusion model [1]. It is often used in modeling such physical phenomena as turbulence, shocks, and so forth. The study of Burgers equation has been a very active area because of its importance.

It is well known that strictly parabolic discretization schemes applied to Burgers equation do not work well when it is advection dominated. Effective discretization schemes recognize to some extent the hyperbolic nature of the equation.

The finite volume element method (FVEM) [4–12] is an important discretization technique for partial differential equations, especially those that arise from physical conservation laws. FVEM has ability to be faithful to the physics in general and conservation in particular, to produce simple stencils, and to treat effectively Neumann boundary conditions and nonuniform grids, and so forth.

Liang [11, 12] combined the upwind technique and the FVEM to handle the linear convection-dominated problems. In this paper, we will consider upwind finite volume element method for the approximation of (1). Upwind approximation is applied to handle the nonlinear convection term. The semi-discrete and fully discrete schemes are defined, respectively. We prove that they are both convergent to order one in space. Numerical experiments are presented finally to validate the theoretical analysis.

In this paper, we use the following Sobolev spaces and the norms associated with these spaces:

$$\begin{aligned}
L^2(\Omega)&=\left\{f:\int_\Omega|f|^2dx<\infty\right\}, \quad \|f\|=\left[\int_\Omega|f|^2dx\right]^{1/2},\\
L^\infty(\Omega)&=\left\{f:\operatorname*{ess\,sup}_\Omega|f|<\infty\right\}, \quad \|f\|_\infty=\operatorname*{ess\,sup}_\Omega|f|,
\end{aligned}$$

$$H^m(\Omega) = \left\{ f : \frac{\partial^{|\alpha|} f}{\partial x^\alpha} \in L^2(\Omega), |\alpha| \le m \right\},$$

$$m \ge 0,$$

$$\|f\|_m = \left[\sum_{|\alpha|\le m} \left\| \frac{\partial^{|\alpha|} f}{\partial x^\alpha} \right\|^2 \right]^{1/2},$$

$$W_\infty^m(\Omega) = \left\{ f : \frac{\partial^{|\alpha|} f}{\partial x^\alpha} \in L^\infty(\Omega), |\alpha| \le m \right\},$$

$$m \ge 0.$$

$$\|f\|_{m,\infty} = \max_{|\alpha|\le m} \left\| \frac{\partial^{|\alpha|} f}{\partial x^\alpha} \right\|_{L^\infty}, \tag{2}$$

In particular, $H^0(\Omega) = L^2(\Omega)$, $W_\infty^0(\Omega) = L^\infty(\Omega)$. Let $[a,b] \subset [0,T]$ and let X be any of the spaces just defined. If $f(x,t)$ represents functions on $\Omega \times [a,b]$, we set

$$H^m(a,b;X) = \left\{ f : \int_a^b \left\| \frac{\partial^\alpha f}{\partial t^\alpha}(\cdot,t) \right\|_X^2 dt < \infty, \alpha \le m \right\},$$

$$\|f\|_{H^m(a,b;X)} = \left[\sum_{\alpha=0}^m \int_a^b \left\| \frac{\partial^\alpha f}{\partial t^\alpha}(\cdot,t) \right\|_X^2 dt \right]^{1/2}, \quad m \ge 0,$$

$$W_\infty^m(a,b;X) = \left\{ f : \operatorname{ess\,sup}_{[a,b]} \left\| \frac{\partial^\alpha f}{\partial t^\alpha}(\cdot,t) \right\|_X < \infty, \alpha \le m \right\},$$

$$\|f\|_{W_\infty^m(a,b;X)} = \max_{0\le\alpha\le m} \operatorname{ess\,sup}_{[a,b]} \left\| \frac{\partial^\alpha f}{\partial t^\alpha}(\cdot,t) \right\|_X, \quad m \ge 0,$$

$$L^2(a,b;X) = H^0(a,b;X),$$

$$L^\infty(a,b;X) = W_\infty^0(a,b;X). \tag{3}$$

If $[a,b] = [0,T]$, we drop it from the notation. We also drop Ω; thus, we write $L^\infty(W_\infty^1)$ for $L^\infty(0,T;W_\infty^1(\Omega))$.

If $w = (w_1, w_2)$ is a vector function, we say that $w \in X^2$ if $w_1 \in X$ and $w_2 \in X$.

An outline of the paper follows. In the next section we define the upwind finite volume element schemes for (1). Some lemmas are presented in Section 3. We derive the L^2-norm error estimates for the semi-discrete scheme and the fully discrete scheme in Sections 4 and 5, respectively. Finally in Section 6, we give some numerical experiments.

Throughout the paper we will denote by C and C_i ($i = 1,2,\ldots$) generic constants independent of the mesh parameters, which may take different values in different occurrences.

2. The Approximation Schemes

In order to rewrite (1) as the vector form we define some vector notations. The gradient of a vector function $w = (w_1, w_2) : \mathbb{R}^2 \to \mathbb{R}^2$ is a matrix, and the divergence of a matrix function $A = (a_{ij})_{1\le i,j\le 2} : \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$ is a vector

$$\nabla w = \left(\frac{\partial w_i}{\partial x_j} \right)_{1\le i,j\le 2},$$
$$\nabla \cdot A = \left(\sum_{j=1}^2 \frac{\partial a_{1j}}{\partial x_j}, \sum_{j=1}^2 \frac{\partial a_{2j}}{\partial x_j} \right). \tag{4}$$

Consequently, we have for a vector function $w = (w_1, w_2)$

$$\Delta w = \nabla \cdot \nabla w = (\Delta w_1, \Delta w_2). \tag{5}$$

Let $\theta = (u, v)$, $\theta_0(x) = (u_0(x), v_0(x))$, and let $g = (g_1, g_2)$; then the system (1) can be written as the following vector form:

$$\begin{aligned} &\text{(a)}\ \frac{\partial \theta}{\partial t} + \theta \cdot \nabla\theta - \zeta\Delta\theta = 0, \quad (x,t) \in \Omega \times J, \\ &\text{(b)}\ \theta(x,0) = \theta_0(x), \quad x \in \Omega, \\ &\text{(c)}\ \theta(x,t) = g, \quad (x,t) \in \partial\Omega \times J, \end{aligned} \tag{6}$$

where

$$\theta \cdot \nabla\theta = u\frac{\partial\theta}{\partial x_1} + v\frac{\partial\theta}{\partial x_2}. \tag{7}$$

Let $\mathcal{T}_h = \{K\}$ be a triangulation of the domain Ω, and as usual, we assume the triangles K to be shape regular. Denote by $\overline{\Omega}_h = \{P_i\}$ the set of the vertices of all the triangles K, and let $\Omega_h = \overline{\Omega}_h \setminus \partial\Omega$. For a given triangulation $\mathcal{T}_h$, we construct a dual mesh $\mathcal{T}_h^*$ whose elements are called control volumes. Each triangle $K \in \mathcal{T}_h$ can be divided into three subdomains by connecting an inner point of the triangle to the midpoints of the three edges. Around each $P_i \in \Omega_h$, we associate a control volume $K_i^* = K_{P_i}^*$, which consists of the union of subregions having P_i as a vertex. For a vertex $P_i \in \partial\Omega$, we can define its control volume in a similar way. Then we define the dual partition $\mathcal{T}_h^* = \{K_{P_i}^*, P_i \in \overline{\Omega}_h\}$ to be the union of all the control volumes. Usually we can choose the inner points as the barycenters or the circum centers, and in the later case we assume that all the inner angles of each triangle are not larger than $\pi/2$. We will use the barycenters duel mesh in this paper, while, with some trivial changes, our analysis can be also applied to the case when the circum centers are used.

We now characterize the finite-dimensional spaces which will be employed in approximating (6). For the sake of simplicity, we will assume that $g_1 = g_2 = 0$. We define the following finite dimensional spaces:

$$P_h = \left\{ \omega_h \in H_0^1(\Omega), \omega_h\big|_K \in \mathcal{P}_1(K), K \in T_h \right\},$$

$$\begin{aligned} Y_h = \Big\{ &\varphi_h \in L^2(\Omega), \varphi_h\big|_{K_i^*} \in \mathcal{P}_0(K^*), \\ &P_i \in \Omega_h; \varphi_h\big|_{K_i^*} = 0, P_i \in \partial\Omega \Big\}, \end{aligned} \tag{8}$$

$$U_h = P_h^2, \qquad V_h = Y_h^2,$$

where $\mathscr{P}_l(V)$ $(l = 0, 1)$ denotes the set of polynomials on V with a degree of not more than l.

Multiplying (6a) by test function $z \in V_h$ and integrating by parts yield

$$\left(\frac{\partial\theta}{\partial t}, z\right) + B_1(\theta;\theta,z) + B_2(\theta;\theta,z) + A(\theta,z) = 0, \quad \forall z \in V_h, \tag{9}$$

where

$$B_1(\varphi; w, z) = -\sum_{P_i\in\Omega_h} z(P_i)\cdot\int_{K_i^*}(\nabla\cdot w)\varphi\,dx,$$
$$B_2(\varphi; w, z) = \sum_{P_i\in\Omega_h} z(P_i)\cdot\int_{\partial K_i^*}(\varphi\cdot\nu)w\,ds, \tag{10}$$
$$A(w,z) = \sum_{P_i\in\Omega_h} z(P_i)\cdot\int_{\partial K_i^*}\zeta(\nu\cdot\nabla w)\,ds,$$

here ν is the unit outward normal vector of ∂K_i^*.

Now we approximate $B_2(\varphi; w, z)$ by using the upwind technique.

Let $\Lambda_i = \{j : P_j$ is adjoint with $P_i\}$. Assuming that $j \in \Lambda_i$, let $\Gamma_{ij} = \partial K_i^* \cap \partial K_j^*$ and γ_{ij} is the length of Γ_{ij}. Denote by ν_{ij} the unit outward normal vector of Γ_{ij} when Γ_{ij} is regarded as the boundary of K_i^*. Define

$$\beta_{ij}(\varphi) = \int_{\Gamma_{ij}}\varphi\cdot\nu_{ij}ds. \tag{11}$$

Let

$$\beta_{ij}^+(\varphi) = \max\left(\beta_{ij}(\varphi), 0\right), \quad \beta_{ij}^-(\varphi) = \max\left(-\beta_{ij}(\varphi), 0\right),$$
$$\int_{\partial K_i^*}(\varphi\cdot\nu)w\,ds \approx \sum_{j\in\Lambda_i}\left\{\beta_{ij}^+(\varphi)w(P_i) - \beta_{ij}^-(\varphi)w(P_j)\right\}. \tag{12}$$

The upwind discretization of the nonlinear term $B_2(\varphi; w, z)$ is defined by the form

$$B_{2h}(\varphi; w, z) = \sum_{P_i\in\Omega_h}\sum_{j\in\Lambda_i}\left\{\beta_{ij}^+(\varphi)w(P_i) - \beta_{ij}^-(\varphi)w(P_j)\right\}\cdot z(P_i). \tag{13}$$

Using the heaviside function

$$H(r) = \begin{cases} 1, & r \geq 0, \\ 0, & r < 0, \end{cases} \tag{14}$$

we can write $B_{2h}(\varphi; w, z)$ as

$$B_{2h}(\varphi; w, z) = \sum_{P_i\in\Omega_h}\sum_{j\in\Lambda_i}\beta_{ij}(\varphi) \times\left\{H\left(\beta_{ij}(\varphi)\right)w(P_i) + \left(1 - H\left(\beta_{ij}(\varphi)\right)\right)w\left(P_j\right)\right\} \cdot z(P_i). \tag{15}$$

Introduce the interpolation operators $\Pi_h : H_0^1(\Omega) \to P_h$ and $\Pi_h^* : P_h \to Y_h$, respectively. For $w = (w_1, w_2)$, define $\Pi_h w = (\Pi_h w_1, \Pi_h w_2)$ and $\Pi_h^* w = (\Pi_h^* w_1, \Pi_h^* w_2)$. Assuming that $w \in H^2(\Omega)^2$, we can easily get the following interpolation estimates:

$$\|w - \Pi_h w\|_s \leq h^{2-s}\|w\|_2, \quad s = 0, 1. \tag{16}$$

The semi-discrete upwind finite volume scheme of (6) is as follows: find $\theta_h : [0, T] \to U_h$ such that

$$\left(\frac{\partial\theta_h}{\partial t}, \Pi_h^* z_h\right) + B_1(\theta_h;\theta_h,\Pi_h^* z_h) + B_{2h}(\theta_h;\theta_h,\Pi_h^* z_h) + A(\theta_h,\Pi_h^* z_h) = 0, \quad \forall z_h \in U_h, \quad \theta_h(x,0) = \theta_{0h}(x), \tag{17}$$

where $\theta_{0h}(x)$ is the interpolation projection of θ_0, that is, $\theta_{0h}(x) = \Pi_h\theta_0$.

Partition $[0, T]$ into $0 = t^0 < t^1 < \cdots < t^N = T$, with $\tau^n = t^n - t^{n-1}$. Our analysis is valid for variable time steps, but we drop the superscript from τ for convenience. For functions f on $\Omega \times J$, we write $f^n(x)$ for $f(x, t^n)$. By approximating $\partial\theta_h/\partial t$ at the time $t = t_n$ with the backward difference $\partial_t\theta_h^n = (\theta_h^n - \theta_h^{n-1})/\tau$, we define the fully discrete upwind finite volume scheme for (6) as follows: find $\theta_h^n \in U_h$, such that

$$\left(\partial_t\theta_h^n, \Pi_h^* z_h\right) + B_{1h}\left(\theta_h^{n-1};\theta_h^n,\Pi_h^* z_h\right) + B_{2h}\left(\theta_h^{n-1};\theta_h^n,\Pi_h^* z_h\right) + A\left(\theta_h^n,\Pi_h^* z_h\right) = 0, \quad n \geq 1, \ \forall z_h \in U_h, \quad \theta_h^0 = \theta_{0h}. \tag{18}$$

3. Some Lemmas

Now we present several Lemmas. Let $w_h = (w_1, w_2) \in U_h$, $\widetilde{w}_h = (\widetilde{w}_1, \widetilde{w}_2) \in U_h$.

Lemma 1. (i) Π_h^* *is a self-adjoint operator, that is,*

$$\left(w_h, \Pi_h^*\widetilde{w}_h\right) = \left(\widetilde{w}_h, \Pi_h^* w_h\right), \quad \forall w_h, \widetilde{w}_h \in U_h. \tag{19}$$

(ii) *Let* $|||w_h||| = (w_h, \Pi_h^* w_h)^{1/2}$. *Then, for some positive constants* C_1 *and* C_2 *that are independent of* h,

$$C_1\|w_h\| \leq |||w_h||| \leq C_2\|w_h\|, \quad \forall w_h \in U_h. \tag{20}$$

Proof. It is easy to know that

$$\left(w_h, \Pi_h^*\widetilde{w}_h\right) = \left(w_1, \Pi_h^*\widetilde{w}_1\right) + \left(w_2, \Pi_h^*\widetilde{w}_2\right),$$
$$|||w_h|||^2 = \left(w_h, \Pi_h^* w_h\right) = \left(w_1, \Pi_h^* w_1\right) + \left(w_2, \Pi_h^* w_2\right). \tag{21}$$

From [4] we know that for $w_1, w_2, \widetilde{w}_1, \widetilde{w}_2 \in P_h$,

$$\left(w_1, \Pi_h^*\widetilde{w}_1\right) = \left(\widetilde{w}_1, \Pi_h^* w_1\right), \qquad \left(w_2, \Pi_h^*\widetilde{w}_2\right) = \left(\widetilde{w}_2, \Pi_h^* w_2\right),$$
$$C_1^2\|w_1\|^2 \leq \left(w_1, \Pi_h^* w_1\right) \leq C_2^2\|w_1\|^2,$$
$$C_1^2\|w_2\|^2 \leq \left(w_2, \Pi_h^* w_2\right) \leq C_2^2\|w_2\|^2, \tag{22}$$

where C_1 and C_2 are some positive constants that are independent of h. Thus we obtain (19) and (20) immediately. □

Lemma 2. *For the bilinear form $A(\cdot,\Pi_h^*\cdot)$, one has the following conclusions:*

(i) *For $w_h, \widetilde{w}_h \in U_h$, one has*

$$A\left(w_h,\Pi_h^*\widetilde{w}_h\right)=A\left(\widetilde{w}_h,\Pi_h^*w_h\right). \quad (23)$$

(ii) *There exists a positive constant C such that*

$$\left|A\left(w-\Pi_h w,\Pi_h^*\widetilde{w}_h\right)\right| \leq Ch\|w\|_2\left\|\widetilde{w}_h\right\|_1, \quad \forall w\in H^2(\Omega)^2,\ \widetilde{w}_h\in U_h. \quad (24)$$

(iii) *There exists a positive constant α such that*

$$A\left(w_h,\Pi_h^*w_h\right)\geq \alpha\left\|w_h\right\|_1^2, \quad \forall w_h\in U_h. \quad (25)$$

Proof. For $\phi,\psi\in P_h$, define the bilinear form

$$a\left(\phi,\Pi_h^*\psi\right)=-\sum_{P_i\in\Omega_h}\psi\left(P_i\right)\int_{\partial K_i^*}\zeta\nabla\phi\cdot\nu\,ds. \quad (26)$$

Then, we get

$$\begin{aligned}
A\left(w_h,\Pi_h^*\widetilde{w}_h\right)&=-\sum_{P_i\in\Omega_h}\widetilde{w}_h\left(P_i\right)\cdot\int_{\partial K_i^*}\zeta\left(\nu\cdot\nabla w_h\right)ds\\
&=-\sum_{P_i\in\Omega_h}\widetilde{w}_1\left(P_i\right)\int_{\partial K^*}\zeta\nabla w_1\cdot\nu\,ds\\
&\quad-\sum_{P_i\in\Omega_h}\widetilde{w}_2\left(P_i\right)\int_{\partial K_i^*}\zeta\nabla w_2\cdot\nu\,ds\\
&=a\left(w_1,\Pi_h^*\widetilde{w}_1\right)+a\left(w_2,\Pi_h^*\widetilde{w}_2\right),\\
A\left(w-\Pi_h w,\Pi_h^*\widetilde{w}_h\right)&=a\left(w_1-\Pi_h w_1,\Pi_h^*\widetilde{w}_1\right)\\
&\quad+a\left(w_2-\Pi_h w_2,\Pi_h^*\widetilde{w}_2\right).
\end{aligned} \quad (27)$$

By combining the above results and the corresponding conclusions for $a(\cdot,\Pi_h^*\cdot)$ in [4], we can obtain (23)–(25). □

Lemma 3. *For $\varphi\in(W_\infty^0(\Omega))^2$, $\theta\in(H_0^1(\Omega))^2$, $\varphi_h\in U_h$, and $z_h\in U_h$, one has*

$$\begin{aligned}
&\left|B_2\left(\varphi;\theta,\Pi_h^*z_h\right)-B_{2h}\left(\varphi_h;\Pi_h\theta,\Pi_h^*z_h\right)\right|\\
&\quad\leq\left|z_h\right|_1\left\{h\|\varphi\|_\infty|\theta|_1+\|\theta\|_\infty\left(\left\|\varphi-\varphi_h\right\|+h\left|\varphi-\varphi_h\right|_1\right)\right\}.
\end{aligned} \quad (28)$$

Proof. First we have

$$\begin{aligned}
&B_2\left(\varphi;\theta,\Pi_h^*z_h\right)-B_{2h}\left(\varphi_h;\Pi_h\theta,\Pi_h^*z_h\right)\\
&\quad=B_2\left(\varphi;\theta,\Pi_h^*z_h\right)-B_{2h}\left(\varphi_h;\theta,\Pi_h^*z_h\right)\\
&\qquad+B_{2h}\left(\varphi_h;\theta,\Pi_h^*z_h\right)-B_{2h}\left(\varphi_h;\Pi_h\theta,\Pi_h^*z_h\right).
\end{aligned} \quad (29)$$

Noting that $\theta(P_i)=\Pi_h\theta(P_i)$, $P_i\in\Omega_h$, we can easily deduce

$$B_{2h}\left(\varphi_h;\theta,\Pi_h^*z_h\right)-B_{2h}\left(\varphi_h;\Pi_h\theta,\Pi_h^*z_h\right)=0 \quad (30)$$

by the definition of $B_{2h}(\cdot;\cdot,\Pi_h^*\cdot)$. Now we only need to bound

$$\begin{aligned}
&B_2\left(\varphi;\theta,\Pi_h^*z_h\right)-B_{2h}\left(\varphi_h;\theta,\Pi_h^*z_h\right)\\
&\quad=\sum_{P_i\in\Omega_h}z_h\left(P_i\right)\cdot\sum_{j\in\Lambda_i}\int_{\Gamma_{ij}}\left(\varphi\cdot\nu_{ij}\right)\theta\,ds\\
&\qquad-\sum_{P_i\in\Omega_h}z_h\left(P_i\right)\cdot\sum_{j\in\Lambda_i}\beta_{ij}\left(\varphi_h\right)\\
&\qquad\times\left[H\left(\beta_{ij}\left(\varphi_h\right)\right)\theta\left(P_i\right)+\left(1-H\left(\beta_{ij}\left(\varphi_h\right)\right)\right)\theta\left(P_j\right)\right]\\
&\quad=\sum_{P_i\in\Omega_h}z_h\left(P_i\right)\\
&\qquad\cdot\sum_{j\in\Lambda_i}\int_{\Gamma_{ij}}\left(\varphi\cdot\nu_{ij}\right)\\
&\qquad\qquad\times\left\{\theta-\left[H\left(\beta_{ij}\left(\varphi_h\right)\right)\theta\left(P_i\right)\right.\right.\\
&\qquad\qquad\quad\left.\left.+\left(1-H\left(\beta_{ij}\left(\varphi_h\right)\right)\right)\theta\left(P_j\right)\right]\right\}ds\\
&\qquad+\sum_{P_i\in\Omega_h}z_h\left(P_i\right)\cdot\sum_{j\in\Lambda_i}\int_{\Gamma_{ij}}\left(\varphi-\varphi_h\right)\cdot\nu_{ij}ds\\
&\qquad\times\left[H\left(\beta_{ij}\left(\varphi_h\right)\right)\theta\left(P_i\right)+\left(1-H\left(\beta_{ij}\left(\varphi_h\right)\right)\right)\theta\left(P_j\right)\right].
\end{aligned} \quad (31)$$

We denote the last two terms on the right-hand side of (31) by I_1 and I_2, respectively. We now turn to analyze the two terms. Noting that $\beta_{ij}=-\beta_{ji}$, we rewrite I_1 as

$$\begin{aligned}
I_1&=\frac{1}{2}\sum_{K\in\mathcal{T}_h}\sum_{i,j\in\Lambda_K}\left[z_h\left(P_i\right)-z_h\left(P_j\right)\right]\cdot\\
&\quad\int_{\Gamma_{ij}\cap K}\left(\varphi\cdot\nu_{ij}\right)\left[H\left(\beta_{ij}\left(\varphi_h\right)\right)\left(\theta-\theta\left(P_i\right)\right)\right.\\
&\qquad\left.+\left(1-H\left(\beta_{ij}\left(\varphi_h\right)\right)\right)\left(\theta-\theta\left(P_j\right)\right)\right]ds,
\end{aligned} \quad (32)$$

here Λ_K is the set of vertex of K. From the Taylor's Formula and the linear property of $z_h=(z_1,z_2)$, we obtain that

$$\left|z_h\left(P_i\right)-z_h\left(P_j\right)\right|^2\leq h^2\left(\left|\nabla z_1\right|^2+\left|\nabla z_2\right|^2\right)=h^2|z|_{1,K}^2. \quad (33)$$

Applying the trace inequality, we get

$$\begin{aligned}
&\int_{\Gamma_{ij}\cap K} |\theta - \theta(P_i)|\, ds \\
&\quad \le Ch^{1/2}\left\{\int_{\Gamma_{ij}\cap K} |\theta - \theta(P_i)|^2 ds\right\}^{1/2} \\
&\quad \le Ch^{1/2}\left\{\int_{\Gamma_{ij}\cap K} \left[|u - u(P_i)|^2 + |v - v(P_i)|^2\right] ds\right\}^{1/2} \\
&\quad \le Ch\left\{\left(h^{-1}|u - u(P_i)|_{0,K} + |u - u(P_i)|_{1,K}\right)^2\right. \\
&\qquad\qquad \left. + \left(h^{-1}|v - v(P_i)|_{0,K} + |v - v(P_i)|_{1,K}\right)^2\right\}^{1/2} \\
&\quad \le Ch\left(|u|_{1,K}^2 + |v|_{1,K}^2\right)^{1/2} = Ch|\theta|_{1,K}.
\end{aligned} \tag{34}$$

Similarly, we can deduce that

$$\int_{\Gamma_{ij}\cap K} \left|\theta - \theta(P_j)\right| ds \le Ch|\theta|_{1,K}. \tag{35}$$

We conclude that

$$|I_1| \le Ch\|\varphi\|_\infty |z_h|_1 |\theta|_1. \tag{36}$$

The similar argument yields the estimate

$$\begin{aligned}
|I_2| = &\left|\frac{1}{2}\sum_{K\in\mathcal{T}_h}\sum_{i,j\in\Lambda_K}\int_{\Gamma_{ij}\cap K}[(\varphi - \varphi_h)\cdot n]\, ds\right. \\
&\times\left[H\left(\beta_{ij}(\varphi_h)\right)\theta(P_i) + \left(1 - H\left(\beta_{ij}(\varphi_h)\right)\right)\theta\left(P_j\right)\right] \\
&\left.\cdot\left[z_h(P_i) - z_h\left(P_j\right)\right]\right| \\
\le\ & C\|\theta\|_\infty |z_h|_1\left(\|\varphi - \varphi_h\| + h|\varphi - \varphi_h|_1\right).
\end{aligned} \tag{37}$$

Substituting the estimates (36) and (37) into (31), we obtain

$$\begin{aligned}
&\left|B_2(\varphi;\theta,\Pi_h^* z_h) - B_{2h}(\varphi_h;\theta,\Pi_h^* z_h)\right| \\
&\quad \le C|z_h|_1\left\{h\|\varphi\|_\infty|\theta|_1 + \|\theta\|_\infty\right. \\
&\qquad\qquad \left. \times\left(\|\varphi - \varphi_h\| + h|\varphi - \varphi_h|_1\right)\right\}.
\end{aligned} \tag{38}$$

This yields the desired result immediately. □

4. Error Bounds for Semi-Discrete Scheme

Theorem 4. *Assume that θ and θ_h are solutions to* (6) *and* (17)*, respectively. also assumes that θ is regular enough. Then there exists a positive constant C such that*

$$\|\theta - \theta_h\| \le Ch, \tag{39}$$

where C depends on principally $\|\theta_0\|_2$, $\|\theta\|_{L^\infty((W_\infty^1)^2)}$, and $\|\theta\|_{H^1((H^2)^2)}$.

Proof. We derive the following error equation from (6) and (17):

$$\begin{aligned}
&\left(\frac{\partial\theta}{\partial t} - \frac{\partial\theta_h}{\partial t}, \Pi_h^* z_h\right) + B_1(\theta;\theta,\Pi_h^* z_h) - B_1(\theta_h;\theta_h,\Pi_h^* z_h) \\
&\quad + B_2(\theta;\theta,\Pi_h^* z_h) - B_{2h}(\theta_h;\theta_h,\Pi_h^* z_h) \\
&\quad + A(\theta - \theta_h, \Pi_h^* z_h) = 0.
\end{aligned} \tag{40}$$

Let $\rho = \theta - \Pi_h\theta$, $\xi = \Pi_h\theta - \theta_h$. We rewrite the previously mentioned equation as

$$\begin{aligned}
&\left(\frac{\partial\xi}{\partial t}, \Pi_h^* z_h\right) + B_1(\theta;\theta,\Pi_h^* z_h) - B_1(\theta_h;\theta_h,\Pi_h^* z_h) \\
&\qquad + B_2(\theta;\theta,\Pi_h^* z_h) - B_{2h}(\theta_h;\theta_h,\Pi_h^* z_h) + A(\xi,\Pi_h^* z_h) \\
&\quad = -\left(\frac{\partial\rho}{\partial t}, \Pi_h^* z_h\right) - A(\rho,\Pi_h^* z_h).
\end{aligned} \tag{41}$$

We choose $z_h = \xi$ in (41) to get

$$\begin{aligned}
&\left(\frac{\partial\xi}{\partial t}, \Pi_h^*\xi\right) + A(\xi,\Pi_h^*\xi) \\
&\quad = -\left(\frac{\partial\rho}{\partial t}, \Pi_h^*\xi\right) - A(\rho,\Pi_h^*\xi) \\
&\qquad - \left[B_1(\theta;\theta,\Pi_h^*\xi) - B_1(\theta_h;\theta_h,\Pi_h^*\xi)\right] \\
&\qquad - \left[B_2(\theta;\theta,\Pi_h^*\xi) - B_{2h}(\theta_h;\theta_h,\Pi_h^*\xi)\right].
\end{aligned} \tag{42}$$

Using Lemmas 1, 2 and Young's inequality, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\!|\!|\xi|\!|\!|^2 + \alpha\|\xi\|_1^2 \\
&\quad \le C\left(\left\|\frac{\partial\rho}{\partial t}\right\|^2 + \|\xi\|^2 + h^2\|\theta\|_2^2\right) + \varepsilon\|\xi\|_1^2 \\
&\qquad + \left|B_1(\theta;\theta,\Pi_h^*\xi) - B_1(\theta_h;\theta_h,\Pi_h^*\xi)\right| \\
&\qquad + \left|B_2(\theta;\theta,\Pi_h^*\xi) - B_{2h}(\theta_h;\theta_h,\Pi_h^*\xi)\right|.
\end{aligned} \tag{43}$$

Now we bound the last two terms on the right-hand side of (43). We need the following induction hypothesis:

$$\left(\log\frac{1}{h}\right)^{1/2}\|\xi\|(s) \longrightarrow 0, \qquad h \longrightarrow 0,\ 0 \le s < t,\ 0 < t \le T. \tag{44}$$

We know from [13] that

$$\|\phi\|_\infty \le C\left(\log\frac{1}{h}\right)^{1/2}\|\phi\|_1, \quad \forall\phi \in P_h. \tag{45}$$

This implies that

$$\|\varphi\|_\infty \le C\left(\log\frac{1}{h}\right)^{1/2}\|\varphi\|_1, \quad \forall\varphi \in U_h. \tag{46}$$

Also we have the following inverse inequality:

$$\|\varphi\|_1 \le Ch^{-1}\|\varphi\|, \quad \forall \varphi \in U_h. \tag{47}$$

Using (46), we get

$$\begin{aligned}
&\left|B_1(\theta;\theta,\Pi_h^*\xi) - B_1(\theta_h;\theta_h,\Pi_h^*\xi)\right| \\
&\quad \le \sum_{P_i\in\Omega_h} |\xi(P_i)| \int_{K_i^*} \left|(\nabla\cdot\theta)\theta - (\nabla\cdot\theta_h)\theta_h\right| dx \\
&\quad \le \sum_{P_i\in\Omega_h} |\xi(P_i)| \int_{K_i^*} \{|(\nabla\cdot\theta)|\,|\theta-\theta_h| + |\nabla\cdot\theta - \nabla\cdot\theta_h| \\
&\qquad\qquad \times (|\xi| + |\Pi_h\theta|)\}\, dx \\
&\quad \le C\{\|\nabla\cdot\theta\|_\infty \|\theta-\theta_h\|\,\|\xi\| + \|\nabla\cdot(\theta-\theta_h)\| \\
&\qquad \times \|\xi\|(\|\xi\|_\infty + \|\theta\|_\infty)\} \\
&\quad \le C\left\{\|\rho\|_1 + \|\xi\|_1 + \left(\log\frac{1}{h}\right)^{1/2}\|\xi\|_1\right. \\
&\qquad \left.\times(\|\rho\|_1 + \|\xi\|_1)\right\}\|\xi\| \\
&\quad \le C\left\{\|\rho\|_1^2 + \|\xi\|^2 + \|\xi\|\left(\log\frac{1}{h}\right)^{1/2}\|\xi\|_1^2\right\} + \varepsilon\|\xi\|_1^2.
\end{aligned} \tag{48}$$

Next, we write

$$\begin{aligned}
&\left|B_2(\theta;\theta,\Pi_h^*\xi) - B_{2h}(\theta_h;\theta_h,\Pi_h^*\xi)\right| \\
&\quad \le \left|B_2(\theta;\theta,\Pi_h^*\xi) - B_{2h}(\theta_h;\Pi_h\theta,\Pi_h^*\xi)\right| \\
&\qquad + \left|B_{2h}(\theta_h;\xi,\Pi_h^*\xi)\right| = D_1 + D_2.
\end{aligned} \tag{49}$$

By Choosing $\varphi = \theta$, $\varphi_h = \theta_h$, and $z_h = \xi$ in Lemma 3, using (47) and the Young's inequality, we can obtain

$$\begin{aligned}
D_1 &\le C|\xi|_1\{h\|\theta\|_\infty|\theta|_1 + \|\theta\|_\infty \\
&\qquad \times(\|\theta-\theta_h\| + h|\theta-\theta_h|_1)\} \\
&\le C\{h^2|\theta|_1^2 + \|\rho\|_1^2 + \|\xi\|^2\} + \varepsilon\|\xi\|_1^2.
\end{aligned} \tag{50}$$

By an argument like (36) and then by (46) and (47), we have

$$\begin{aligned}
D_2 &= \left|\sum_{P_i\in\Omega_h}\xi(P_i)\cdot\sum_{j\in\Lambda_i}\int_{\Gamma_{ij}}(\theta_h\cdot\nu_{ij})\,ds\right. \\
&\qquad \left.\times\left[H(\beta_{ij})\xi(P_i) + (1-H(\beta_{ij}))\xi(P_j)\right]\right| \\
&\le \frac{1}{2}\sum_{K\in T_h}\sum_{i,j\in\Lambda_K}\left|\xi(P_i)-\xi(P_j)\right| \\
&\quad \times\left|H(\beta_{ij})\xi(P_i) + (1-H(\beta_{ij}))\xi(P_j)\right| \\
&\quad \times\int_{\Gamma_{ij}\cap K}\left|\xi\cdot\nu_{ij}\right| ds \\
&\quad + \frac{1}{2}\sum_{K\in T_h}\sum_{i,j\in\Lambda_K}\left|\xi(P_i)-\xi(P_j)\right| \\
&\quad \times\left|H(\beta_{ij})\xi(P_i) + (1-H(\beta_{ij}))\xi(P_j)\right| \\
&\quad \times\int_{\Gamma_{ij}\cap K}\left|\Pi_h\theta\cdot\nu_{ij}\right| ds \\
&\le C\{\|\xi\|_1(\|\xi\| + h|\xi|_1)\|\xi\|_\infty + \|\Pi_h\theta\|_\infty\|\xi\|_1\|\xi\|\} \\
&\le C\left\{\|\xi\|^2 + \|\xi\|\left(\log\frac{1}{h}\right)^{1/2}\|\xi\|_1^2\right\} + \varepsilon\|\xi\|_1^2.
\end{aligned} \tag{51}$$

Substituting (50) and (51) into (49), we get

$$\begin{aligned}
&\left|B_2(\theta;\theta,\Pi_h^*\xi) - B_{2h}(\theta_h;\theta_h,\Pi_h^*\xi)\right| \\
&\quad \le C\{h^2|\theta|_1^2 + \|\rho\|_1^2 + \|\xi\|^2\} \\
&\qquad + C\|\xi\|\left(\log\frac{1}{h}\right)^{1/2}\|\xi\|_1^2 + 2\varepsilon\|\xi\|_1^2.
\end{aligned} \tag{52}$$

Make (43), (48), and (52) together to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|||\xi|||^2 + \alpha\|\xi\|_1^2 \\
&\quad \le C\left\{h^2\|\theta\|_2^2 + \left\|\frac{\partial\rho}{\partial t}\right\|^2 + \|\rho\|_1^2 + \|\xi\|^2\right\} \\
&\qquad + C\|\xi\|\left(\log\frac{1}{h}\right)^{1/2}\|\xi\|_1^2 + 4\varepsilon\|\xi\|_1^2.
\end{aligned} \tag{53}$$

Integrating the previously mentioned equation from 0 to t and noting (44), we obtain that

$$\begin{aligned}
&\frac{1}{2}\{|||\xi|||^2(t) - |||\xi|||^2(0)\} + \frac{\alpha}{2}\int_0^t\|\xi\|_1^2 d\tau \\
&\quad \le \left\{h^2\|\theta\|_2^2 + \int_0^t\left\|\frac{\partial\rho}{\partial t}\right\|^2 d\tau + \int_0^t\|\rho\|_1^2 d\tau\right. \\
&\qquad \left.+ \int_0^t\|\xi\|^2 d\tau\right\}
\end{aligned} \tag{54}$$

for sufficiently small h and ε. By using Lemma 2(ii) and the Gronwall's inequality, we have that

$$\begin{aligned}
&\|\xi\|^2(t) + \int_0^t\|\xi\|_1^2 d\tau \\
&\quad \le C\left\{h^2\|\theta\|_2^2 + \int_0^t\left\|\frac{\partial\rho}{\partial t}\right\|^2 d\tau + \int_0^t\|\rho\|_1^2 d\tau\right\}.
\end{aligned} \tag{55}$$

It follows from the interpolation estimates that

$$\|\xi\|^2(t)+\int_0^t\|\xi\|_1^2d\tau \le C\left\{h^2\left(\|\theta_0\|_2^2+\int_0^T\left\|\frac{\partial\theta}{\partial t}\right\|_2^2d\tau\right) +h^4\int_0^T\left\|\frac{\partial\theta}{\partial t}\right\|_2^2d\tau+h^2\int_0^T\|\theta\|_2^2d\tau\right\}. \quad (56)$$

Now we prove the induction hypothesis (44). Noting that $\|\xi\|(0)=0$, we know that (44) holds obviously for $t=0$. It follows from (56) that

$$\left(\log\frac{1}{h}\right)^{1/2}\|\xi\|(t)\le Ch\left(\log\frac{1}{h}\right)^{1/2}\longrightarrow 0,\quad h\longrightarrow 0. \quad (57)$$

Then (44) holds for any $t\in[0,T]$.

By (16), we have

$$\|\rho\|\le Ch^2\|\theta\|_2\le Ch^2\left\{\|\theta_0\|_2+\int_0^t\|\theta\|_2d\tau\right\}. \quad (58)$$

From the triangle inequality, we obtain

$$\|\theta-\theta_h\|\le Ch, \quad (59)$$

where C depends on $\|\theta_0\|_2$, $\|\theta\|_{H^1((W^1_\infty)^2)}$, and $\|\theta\|_{H^1((H^2)^2)}$. We now complete the proof of the theorem. □

5. Error Bound for the Fully Discrete Scheme

Theorem 5. *Assume that θ satisfies the necessary regularities and the discretization parameters obey the relation $\tau=O(h)$. Then the error of the approximation* (18) *of* (6) *satisfies*

$$\max_{0\le n\le T/\tau}\|\theta^n-\theta_h^n\|\le C\{h+\tau\}, \quad (60)$$

where C depends on $\|\theta_0\|_2$, $\|\theta\|_{L^\infty((W^1_\infty)^2)}$, $\|\theta\|_{H^1((H^2)^2)}$, and $\|\theta\|_{H^2((L^2)^2)}$.

Proof. Subtract (18) from (9) to obtain that

$$\begin{aligned}&\left(\frac{\partial\theta^n}{\partial t}-\partial_t\theta_h^n,\Pi_h^*z_h\right)+B_1\left(\theta^n;\theta^n,\Pi_h^*z_h\right)\\&\quad-B_1\left(\theta_h^{n-1};\theta_h^n,\Pi_h^*z_h\right)+B_2\left(\theta^n;\theta^n,\Pi_h^*z_h\right)\\&\quad-B_{2h}\left(\theta_h^{n-1};\theta_h^n,\Pi_h^*z_h\right)+A\left(\theta^n-\theta_h^n,\Pi_h^*z_h\right)=0.\end{aligned} \quad (61)$$

Choose $z_h=\xi^n$ to obtain that

$$\begin{aligned}&\left(\partial_t\xi^n,\Pi_h^*\xi^n\right)+A\left(\xi^n,\Pi_h^*\xi^n\right)\\&=-\left(\frac{\partial\theta^n}{\partial t}-\partial_t\theta^n,\Pi_h^*\xi^n\right)-\left(\partial_t\rho^n,\Pi_h^*\xi^n\right)-A\left(\rho^n,\Pi_h^*\xi^n\right)\\&\quad-\left[B_1\left(\theta^n;\theta^n,\Pi_h^*\xi^n\right)-B_1\left(\theta_h^{n-1};\theta_h^n,\Pi_h^*\xi^n\right)\right]\\&\quad-\left[B_2\left(\theta^n;\theta^n,\Pi_h^*\xi^n\right)-B_{2h}\left(\theta_h^{n-1};\theta_h^n,\Pi_h^*\xi^n\right)\right].\end{aligned} \quad (62)$$

For the left-hand side of (62), from Lemmas 1 and 2, we have

$$\begin{aligned}\left(\partial_t\xi^n,\Pi_h^*\xi^n\right)&=\frac{1}{\tau}\left(\xi^n-\xi^{n-1},\Pi_h^*\xi^n\right)\\&=\frac{1}{2\tau}\left(\xi^n-\xi^{n-1},\Pi_h^*\left(\xi^n+\xi^{n-1}\right)\right)\\&\quad+\frac{1}{2\tau}\left(\xi^n-\xi^{n-1},\Pi_h^*\left(\xi^n-\xi^{n-1}\right)\right)\\&\ge\frac{1}{2\tau}\left(|||\xi^n|||^2-|||\xi^{n-1}|||^2\right),\\A\left(\xi^n,\Pi_h^*\xi^n\right)&\ge\alpha\|\xi^n\|_1^2.\end{aligned} \quad (63)$$

We denote terms on the right-hand side of (62) by $T_1,\ldots,T_5$. Then, (62) can be rewritten as

$$\frac{1}{2\tau}\left(|||\xi^n|||^2-|||\xi^{n-1}|||^2\right)+\alpha\|\xi^n\|_1^2\le T_1+\cdots+T_5. \quad (64)$$

Now we estimates the terms $T_1,\ldots,T_5$ one by one. From the Taylor's formula, we have

$$\frac{\partial\theta^n}{\partial t}-\partial_t\theta^n=\frac{1}{\tau}\int_{t^{n-1}}^{t^n}\left(t-t^{n-1}\right)\frac{\partial^2\theta}{\partial t^2}dt. \quad (65)$$

It follows that

$$\begin{aligned}\left|T_1\right|&\le C\left\|\frac{\partial\theta^n}{\partial t}-\partial_t\theta^n\right\|\|\xi^n\|\\&\le C\left\{\tau\int_{t^{n-1}}^{t^n}\left\|\frac{\partial^2\theta}{\partial t^2}\right\|^2dt+\|\xi^n\|^2\right\}.\end{aligned} \quad (66)$$

For the next two terms, we have

$$\begin{aligned}\left|T_2\right|&\le C\left\|\partial_t\rho^n\right\|\|\xi^n\|\\&\le C\left\{\tau^{-1}\int_{t^{n-1}}^{t^n}\left\|\frac{\partial\rho}{\partial t}\right\|^2dt+\|\xi^n\|^2\right\},\\\left|T_3\right|&\le Ch\|\theta^n\|_2\|\xi^n\|_1\\&\le Ch^2\|\theta^n\|_2^2+\varepsilon\|\xi^n\|_1^2.\end{aligned} \quad (67)$$

We make the following induction hypothesis:

$$\left\|\xi^{n-1}\right\|\left(\log\frac{1}{h}\right)^{1/2}\longrightarrow 0,\quad h\longrightarrow 0,\ 1\le n\le L. \quad (68)$$

For T_4, using the similar argument as (48) and noting (68), we deduce that

$$\begin{aligned}
|T_4| &= \left|B_1(\theta^n;\theta^n,\Pi_h^*\xi^n) - B_1\left(\theta_h^{n-1};\theta_h^n,\Pi_h^*\xi^n\right)\right| \\
&\le \sum_{P_i\in\Omega_h} |\xi^n(P_i)| \int_{K_i^*} \left|(\nabla\cdot\theta^n)\theta^n - (\nabla\cdot\theta_h^n)\theta_h^{n-1}\right| dx \\
&\le \sum_{P_i\in\Omega_h} |\xi^n(P_i)| \int_{K_i^*} \left\{|\nabla\cdot\theta^n|\left|\theta^n-\theta_h^{n-1}\right| + |\nabla\cdot\theta^n - \nabla\cdot\theta_h^n|\right. \\
&\qquad\qquad \left.\times\left(\left|\xi^{n-1}\right| + \left|\Pi_h\theta^{n-1}\right|\right)\right\} dx \\
&\le C\left\{\|\nabla\cdot\theta^n\|_\infty \left\|\theta^n-\theta_h^{n-1}\right\| \|\xi^n\| + \|\nabla\cdot(\theta^n-\theta_h^n)\|\right. \\
&\qquad \left.\times\left(\|\xi^n\|_\infty \left\|\xi^{n-1}\right\| + \left\|\Pi_h\theta^{n-1}\right\|_\infty \|\xi^n\|\right)\right\} \\
&\le C\left\{\left[\left(\tau\int_{t^{n-1}}^{t^n}\left\|\frac{\partial\theta}{\partial t}\right\|^2 dt\right)^{1/2} + \left\|\rho^{n-1}\right\| + \left\|\xi^{n-1}\right\|\right]\right. \\
&\qquad \times\|\nabla\cdot\theta^n\|_\infty\|\xi^n\| + \left(\|\rho^n\|_1 + \|\xi^n\|_1\right) \\
&\qquad \left.\times\left[\left(\log\frac{1}{h}\right)^{1/2}\left\|\xi^{n-1}\right\|\|\xi^n\|_1 + \left\|\theta^{n-1}\right\|_\infty\|\xi^n\|\right]\right\} \\
&\le C\left\{\tau\int_{t^{n-1}}^{t^n}\left\|\frac{\partial\theta}{\partial t}\right\|^2 dt + \left\|\rho^{n-1}\right\|^2 + \|\rho^n\|_1^2\right. \\
&\qquad \left. + \left\|\xi^{n-1}\right\|^2 + \|\xi^n\|^2\right\} \\
&\quad + C\left(\log\frac{1}{h}\right)^{1/2}\left\|\xi^{n-1}\right\|\|\xi^n\|_1^2 + \varepsilon\|\xi^n\|_1^2.
\end{aligned} \tag{69}$$

Now, we write

$$\begin{aligned}
|T_5| &= \left|B_2(\theta^n;\theta^n,\Pi_h^*\xi^n) - B_{2h}\left(\theta_h^{n-1};\theta_h^n,\Pi_h^*\xi^n\right)\right| \\
&\le \left|B_2(\theta^n;\theta^n,\Pi_h^*\xi^n) - B_{2h}\left(\theta_h^{n-1};\Pi_h\theta^n,\Pi_h^*\xi^n\right)\right| \\
&\quad + \left|B_{2h}\left(\theta_h^{n-1};\xi^n,\Pi_h^*\xi^n\right)\right| \\
&= E_1 + E_2.
\end{aligned} \tag{70}$$

E_1 and E_2 can be handled as D_1 and D_2 in Theorem 4. Thus, we have

$$\begin{aligned}
E_1 &= \left|B_2(\theta^n;\theta^n,\Pi_h^*\xi^n) - B_{2h}\left(\theta_h^{n-1};\Pi_h\theta^n,\Pi_h^*\xi^n\right)\right| \\
&\le C\|\theta^n\|_\infty|\xi^n|_1\left[h|\theta^n|_1 + \left(\left\|\theta^n-\theta_h^{n-1}\right\| + h\left|\theta^n-\theta_h^{n-1}\right|_1\right)\right] \\
&\le C\|\theta^n\|_\infty|\xi^n|_1 \\
&\quad\times\left\{h|\theta^n|_1 + \left\|\rho^{n-1}\right\| + \left\|\xi^{n-1}\right\| + \left(\tau\int_{t^{n-1}}^{t^n}\left\|\frac{\partial\theta}{\partial t}\right\|^2 dt\right)^{1/2}\right. \\
&\qquad \left. + h\left[\left(\tau\int_{t^{n-1}}^{t^n}\left\|\frac{\partial\theta}{\partial t}\right\|_1^2 dt\right)^{1/2} + \left\|\rho^{n-1}\right\|_1 + \left\|\xi^{n-1}\right\|_1\right]\right\} \\
&\le C\left\{\left\|\rho^{n-1}\right\|^2 + \left\|\xi^{n-1}\right\|^2 + \|\xi^n\|^2 + \tau\int_{t^{n-1}}^{t^n}\left\|\frac{\partial\theta}{\partial t}\right\|_1^2 dt\right\} \\
&\quad + \varepsilon\left\|\xi^{n-1}\right\|_1^2 + \varepsilon\|\xi^n\|_1^2, \\
E_2 &= \left|B_{2h}\left(\theta_h^{n-1};\xi^n,\Pi_h^*\xi^n\right)\right| \\
&= \left|\sum_{P_i\in\Omega_h}\xi^n(P_i)\cdot\sum_{j\in\Lambda_i}\int_{\Gamma_{ij}}\left(\theta_h^{n-1}\cdot\nu_{ij}\right)ds\right. \\
&\qquad\times\left[H\left(\beta_{ij}\left(\theta_h^{n-1}\right)\right)\xi^n(P_i)\right. \\
&\qquad\qquad \left.\left. + \left(1-H\left(\beta_{ij}\left(\theta_h^{n-1}\right)\right)\right)\xi^n(P_j)\right]\right| \\
&\le \frac{1}{2}\sum_{K\in T_h}\sum_{i,j\in\Lambda_K}\left|\xi^n(P_i)-\xi^n(P_j)\right|\int_{\Gamma_{ij}\cap K}\left|\xi^{n-1}\cdot\nu_{ij}\right| ds \\
&\quad\times\left|H\left(\beta_{ij}\left(\theta_h^{n-1}\right)\right)\xi^n(P_i) + \left(1-H\left(\beta_{ij}\left(\theta_h^{n-1}\right)\right)\right)\xi^n(P_j)\right| \\
&\quad + \frac{1}{2}\sum_{K\in T_h}\sum_{i,j\in\Lambda_K}\left|\xi^n(P_i)-\xi^n(P_j)\right|\int_{\Gamma_{ij}\cap K}\left|\Pi_h\theta^{n-1}\cdot\nu_{ij}\right| ds \\
&\quad\times\left|H\left(\beta_{ij}\left(\theta_h^{n-1}\right)\right)\xi(P_i) + \left(1-H\left(\beta_{ij}\left(\theta_h^{n-1}\right)\right)\right)\xi(P_j)\right| \\
&\le C\left\{\|\xi^n\|_1\left(\left\|\xi^{n-1}\right\| + h\left|\xi^{n-1}\right|_1\right)\|\xi^n\|_\infty\right. \\
&\qquad \left. + \left\|\Pi_h\theta^{n-1}\right\|_\infty\|\xi^n\|_1\|\xi^n\|\right\} \\
&\le C\left\{\|\xi^n\|^2 + \left\|\xi^{n-1}\right\|\left(\log\frac{1}{h}\right)^{1/2}\|\xi^n\|_1^2\right\} + \varepsilon\|\xi^n\|_1^2.
\end{aligned} \tag{71}$$

Substituting the previously mentioned estimates into (64), we get

$$\begin{aligned}
&\frac{1}{2\tau}\left(|||\xi^n|||^2 - \left|\left|\left|\xi^{n-1}\right|\right|\right|^2\right) + \alpha\|\xi^n\|_1^2 \\
&\quad\le C\left\{h^2\|\theta^n\|_2^2 + \|\rho^n\|_1^2 + \left\|\rho^{n-1}\right\|_1^2 + \|\xi^n\|^2 + \left\|\xi^{n-1}\right\|^2\right\} \\
&\qquad + C\tau\int_{t^{n-1}}^{t^n}\left(\left\|\frac{\partial^2\theta}{\partial t^2}\right\|^2 + \left\|\frac{\partial\theta}{\partial t}\right\|_1^2\right)dt + C\tau^{-1}\int_{t^{n-1}}^{t^n}\left\|\frac{\partial\rho}{\partial t}\right\|^2 dt \\
&\qquad + C\left(\log\frac{1}{h}\right)^{1/2}\left\|\xi^{n-1}\right\|\|\xi^n\|_1^2 + \varepsilon\left\|\xi^{n-1}\right\|_1^2 + 4\varepsilon\|\xi^n\|_1^2.
\end{aligned} \tag{72}$$

Multiplying (72) by 2τ and summing over $1\le n\le L$, we have

$$\begin{aligned}
&\left|\left|\left|\xi^L\right|\right|\right|^2 - \left|\left|\left|\xi^0\right|\right|\right|^2 + 2\tau\alpha\sum_{n=1}^{L}\|\xi^n\|_1^2 \\
&\quad\le C\left\{h^2\|\theta^n\|_2^2 + \tau\sum_{n=0}^{L}\|\rho^n\|_1^2 + \tau\sum_{n=1}^{L}\|\xi^n\|^2\right\}
\end{aligned}$$

Table 1: Numerical results for $\zeta = 1$.

h	1/8	1/16	1/32	1/64
$\|u - u_h\|_h$	$1.18416e-007$	$5.33942e-008$	$2.52582e-008$	$1.22755e-008$
Rate		1.15	1.08	1.05
$\|v - v_h\|_h$	$6.73307e-008$	$2.36565e-008$	$9.49449e-009$	$4.21298e-009$
Rate		1.51	1.32	1.17

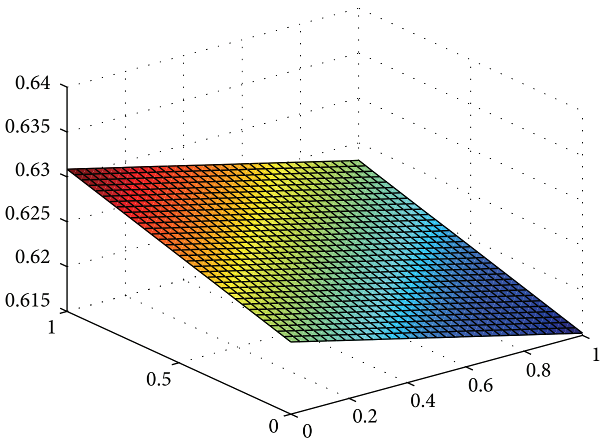

Figure 1: The exact solution u when $\zeta = 1$ at $t = 1.0$.

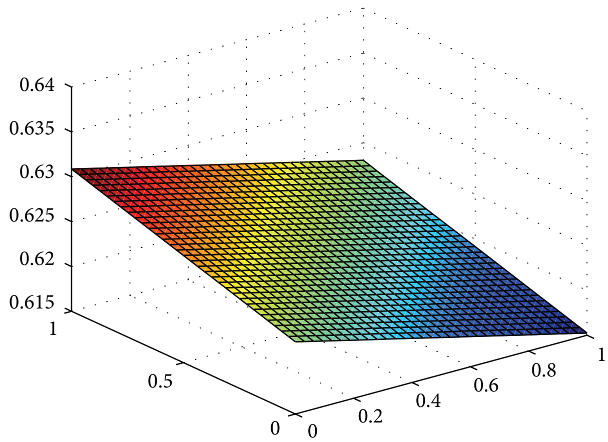

Figure 2: The numerical solution u_h when $\zeta = 1$ at $t = 1.0$, for $h = 1/32$.

$$+ C\tau^2 \int_0^{t^L} \left(\left\| \frac{\partial^2 \theta}{\partial t^2} \right\|^2 + \left\| \frac{\partial \theta}{\partial t} \right\|_1^2 \right) dt + C \int_0^{t^L} \left\| \frac{\partial \rho}{\partial t} \right\|^2 dt$$
$$+ C\tau \sum_{n=1}^{L} \left(\log \frac{1}{h} \right)^{1/2} \left\| \xi^{n-1} \right\| \left\| \xi^n \right\|_1^2 + 5\varepsilon \sum_{n=1}^{L} \left\| \xi^n \right\|_1^2. \tag{73}$$

By choosing h and ε small enough and noting Lemma 2(ii), we have

$$\begin{aligned} &\left\| \xi^L \right\|^2 + \tau \sum_{n=1}^{L} \left\| \xi^n \right\|_1^2 \\ &\quad \leq C \left\{ h^2 \left\| \theta^n \right\|_2^2 + \tau \sum_{n=0}^{L} \left\| \rho^n \right\|_1^2 \right\} \\ &\qquad + \tau \sum_{n=1}^{L} \left\| \xi^n \right\|^2 + C\tau^2 \int_0^{t^L} \left(\left\| \frac{\partial^2 \theta}{\partial t^2} \right\|^2 + \left\| \frac{\partial \theta}{\partial t} \right\|_1^2 \right) dt \\ &\qquad + C \int_0^{t^L} \left\| \frac{\partial \rho}{\partial t} \right\|^2 dt. \end{aligned} \tag{74}$$

Applying the Gronwall inequality and the interpolation theory, we deduce that

$$\begin{aligned} \left\| \xi^L \right\|^2 \leq C\tau^2 \int_0^T \left(\left\| \frac{\partial^2 \theta}{\partial t^2} \right\|^2 + \left\| \frac{\partial \theta}{\partial t} \right\|_1^2 \right) dt \\ + Ch^2 \left\{ \left\| \theta_0 \right\|_2^2 + \int_0^T \left\| \frac{\partial \theta}{\partial t} \right\|_2^2 dt \right\}. \end{aligned} \tag{75}$$

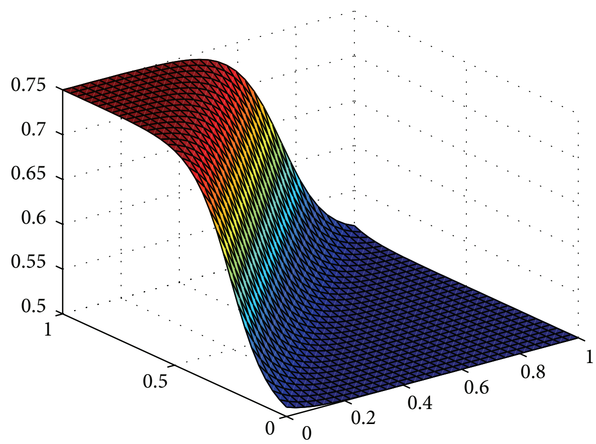

Figure 3: The exact solution u when $\zeta = 0.01$ at $t = 1.0$.

Now we prove the induction hypothesis (68). Noting that $\theta_h^0 = \Pi_h \theta_0$, we know that $\xi^0 = 0$. From (75) and the assumption $\tau = O(h)$, we get that

$$\left(\log \frac{1}{h} \right)^{1/2} \left\| \xi^L \right\| \leq Ch \left(\log \frac{1}{h} \right)^{1/2} \longrightarrow 0, \quad h \longrightarrow 0. \tag{76}$$

Thus we know that (68) holds for any $1 \leq L \leq N$. Using triangular inequality and the interpolation theory completes the proof. $\square$

Table 2: Numerical results for $\zeta = 0.01$.

h	1/8	1/16	1/32	1/64
$\|u - u_h\|_h$	$7.57108e-003$	$3.81036e-003$	$1.92640e-003$	$9.71683e-004$
Rate		0.99	0.98	0.99
$\|v - v_h\|_h$	$7.57108e-003$	$3.81036e-003$	$1.92640e-003$	$9.71683e-004$
Rate		0.99	0.98	0.99

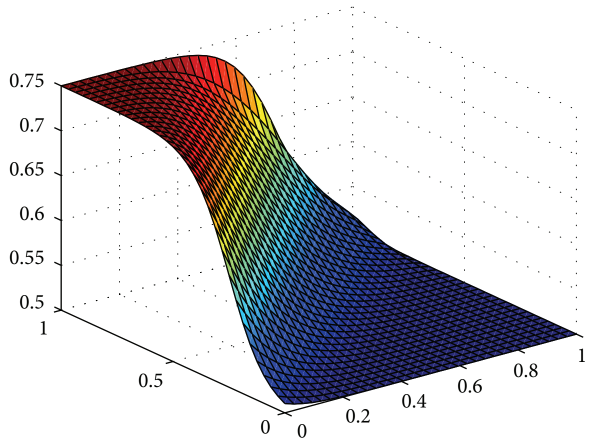

Figure 4: The numerical solution of u_h when $\zeta = 0.01$ at $t = 1.0$, for $h = 1/32$.

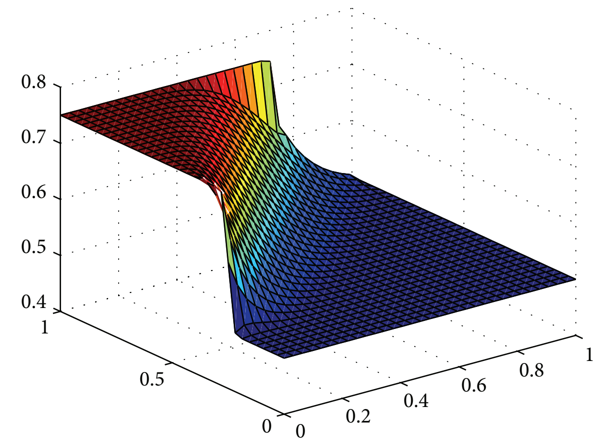

Figure 6: The numerical solution u_h by FVEM with upwinding when $\zeta = 0.001$ at $t = 1$.

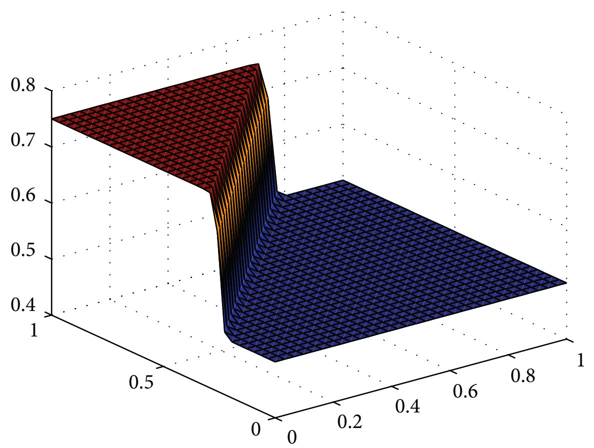

Figure 5: The exact solution u at $\zeta = 0.001$ at $t = 1.0$.

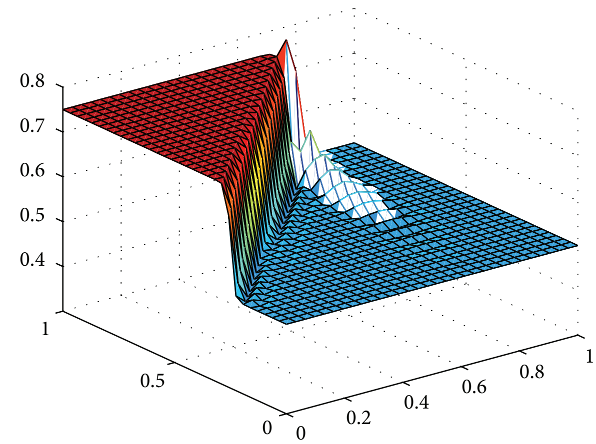

Figure 7: The numerical solution $\tilde{u}_h$ by FVEM without upwinding when $\zeta = 0.001$ at $t = 1$.

6. Numerical Example

In this section, we will show the affectivity of our method by numerical experiments. The exact solutions to problem (1) can be obtained by employing Cole-Hopf transformation. For $\Omega = \{(x_1, x_2) : 0 \le x_1, x_2 \le 1\}$, we consider the following solutions:

$$u = \frac{3}{4} - \frac{1}{4\left(1 + \exp\left(\eta\left(-4x_1 + 4x_2 - t\right)/32\right)\right)},$$
$$v = \frac{3}{4} + \frac{1}{4\left(1 + \exp\left(\eta\left(-4x_1 + 4x_2 - t\right)/32\right)\right)}, \tag{77}$$

where $\eta = 1/\zeta$. We present numerical results for the L^2-norm estimates of $u - u_h$ and $v - v_h$. In Tables 1 and 2, we present the numerical results for $\zeta = 1$ and $\zeta = 0.01$, respectively. In all runs, we use the uniform mesh step $h = \Delta t$ and choose the time $t = 1$. As seen in these tables, in all cases the errors decrease by a factor of about two as h decreases by the factor of two. This indicates that all L^2-norm error estimates are of first-order convergence, which is consistent with our theoretical analysis.

When $\zeta = 1.0$ and $\zeta = 0.01$, the figures of the exact solutions u and the numerical solutions u_h at $t = 1$ for $h = 1/32$ are given in Figures 1, 2, 3, and 4. In order to show that our method keeps stable when ζ is smaller, we also give the comparison figures of exact solution u and numerical solution u_h for $\zeta = 0.001$ in Figures 5 and 6. The comparison figure of numerical solution by using finite volume element method (FVEM) without upwinding is given in Figure 7, which show that the approximation produces unacceptable nonphysical oscillations.

Acknowledgments

This paper is supported by The Natural Science Foundation of China (Grant nos. 10971254 and 11171193) and the Natural Science Foundation of Shandong Province (Grant nos. ZR2009AZ003 and ZR2011AM016).

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