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Research Article

Characterizations of Semihyperrings by Their $(\in_\gamma, \in_\gamma \vee q_\delta)$-Fuzzy Hyperideals

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The concepts of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-hyperideals and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-hyperideals of a semihyperring are introduced, and some related properties of such $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy hyperideals are investigated. In particular, the notions of hyperregular semihyperrings and left duo semihyperrings are given, and their characterizations in terms of hyperideals and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy hyperideals are studied.

1. Introduction

The concept of the fuzzy set, initiated by Zadeh in his pioneer paper [1] of 1965, is an important tool for modeling uncertainties in many complicated problems in engineering, economics, environment, medical science, and social science, due to information incompleteness, randomness, limitations of measuring instruments, and so forth. Many classical mathematics is extended to fuzzy mathematics, and various properties of them in the context of fuzzy sets are established. The idea of quasi-coincidence of a fuzzy point with a fuzzy set, which is mentioned in [2], played a prominent role to generalize some basic concepts of fuzzy algebraic systems. Bhakat and Das [3, 4] gave the concepts of (α, β)-fuzzy subgroups by using the "belongs to" relation $(\in)$ and "quasi-coincident with" relation (q) between a fuzzy point and a fuzzy subgroup and introduced the concept of $(\in, \in \vee q)$-fuzzy subgroup. They also defined and investigated the $(\in, \in \vee q)$-fuzzy subrings and ideals of a ring in [5]. Later on, Dudek et al. [6] introduced the notion of $(\in, \in \vee q)$-fuzzy ideal and $(\in, \in \vee q)$-fuzzy h-ideal in hemirings. Davvaz et al. [7] considered the concept of interval-valued $(\in, \in \vee q)$-fuzzy $H\nu$-submodules of $H\nu$-modules. Recently, Yin and Zhan [8] further generalized the above relations $(\in)$ and (q) by using a pair of thresholds γ and δ $(\gamma < \delta)$. They gave some new relations $\in_\gamma$ and q_δ between a fuzzy point and a fuzzy set and studied (α, β)-fuzzy filters in BL-algebras where α, β are two of $\{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta, \in_\gamma \wedge q_\delta\}$ with $\alpha \neq \in_\gamma \wedge q_\delta$. Afterwards, this direction was continued by Ma and Zhan, and others (for instance, [9, 10]).

Algebraic hyperstructure was introduced in 1934 by a French mathematician, Marty [11], at the 8th Congress of Scandinavian Mathematicians. Later on, people have observed that hyperstructures have many applications in both pure and applied sciences. A comprehensive review of the theory of hyperstructures can be found in [12–14]. In a recent book of Corsini and Leoreanu [15], the authors have collected numerous applications of algebraic hyperstructures, especially those from the last fifteen years to the following subjects: geometry, hypergraphs, binaryrelations, lattices, fuzzy sets and roughsets, automata, cryptography, codes, median algebras, relation algebras, artificial intelligence, and probabilities. The study of fuzzy hyperstructures is also an interesting research topic of hyperstructure. Many fuzzy theorems in hyperstructures have been discussed by several authors, for example, Corsini, Cristea, Davvaz, Kazanci, Leoreanu, Yin, and Zhan (see, e.g., [7, 8, 13, 16–24]). Hyperstructure, in particular semihyperring, is a generalization of classical algebra in which the ordinary operations are replaced by hyperoperations which map a pair of elements into a subset. In [25], Ameri and Hedayati gave the notions of semihyperrings and studied the k-hyperideals of them. Davvaz [26] gave the concepts of ternary semihyperrings and investigated their fuzzy hyperideals. Dehkordi and Davvaz introduced the notions of Γ-semihyperrings and discussed roughness and a kind of strong regular (equivalence) relations on Γ-semihyperrings (see, [27, 28]). However, one can see that these semihyperrings are based on a hyperoperation and an ordinary operation (or Γ-operation). In this paper, we consider another kind of semihyperring in which addition and multiplication are both hyperoperations. We define $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy hyperideals, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-hyperideals, and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-hyperideals in a semihyperring and investigate some related properties of them. In the rest, we give the concepts of hyperregular semihyperrings and left duo semihyperrings and address their characterizations in terms of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy hyperideals, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-hyperideals, and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-hyperideals.

2. Preliminaries

Let H be a set and $P(H)$ the family of all nonempty subsets of H and $\circ$ a hyperoperation or join operation; that is, $\circ$ is a map from $H \times H$ to $P(H)$. If $(x, y) \in H \times H$, its image under $\circ$ is denoted by $x \circ y$. If $A, B \subseteq H$, then $A \circ B$ is given by $A \circ B = \bigcup\{x \circ y \mid x \in A, y \in B\}$. $x \circ A$ is used for $\{x\} \circ A$ and $A \circ x$ for $A \circ \{x\}$. Generally, the singleton x is identified by its element x.

H together with a hyperoperation $\circ$ is called a semihypergroup if $(x \circ y) \circ z = x \circ (y \circ z)$ for all $x, y, z \in H$. A semihypergroup H is said to be commutative if $a \circ b = b \circ a$ for all $a, b \in H$. The concept of semihyperrings was introduced by Ameri and Hedayati in 2007 [25]. We formulate it as follows.

Definition 1 (see [25]). A semihyperring is an algebraic hyperstructure $(H, +, \cdot)$ satisfying the following axioms:

(1) $(H, +)$ is a commutative semihypergroup with a zero element 0 satisfying $x + 0 = 0 + x = \{x\}$;

(2) $(H, \cdot)$ is semigroup; that is, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all $x, y, z \in H$;

(3) the multiplication $\cdot$ is distributive over the hyperoperation $+$; that is, for any $x, y, z \in H$, we have $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(y + z) \cdot x = y \cdot x + z \cdot x$;

(4) the element 0 is an absorbing element; that is, $x \cdot 0 = 0 \cdot x = 0$ for all $x \in H$.

In Definition 1, if the multiplication is replaced by a hyperoperation, then we have the following definition.

Definition 2. A semihyperring is an algebraic hyperstructure $(H, +, \circ)$ consisting of a nonempty set H together with two binary hyperoperations on H satisfying the following axioms:

(1) $(H, +)$ is a commutative semihypergroup;

(2) $(H, \circ)$ is semihypergroup;

(3) the hyperoperation $\circ$ is distributive over the hyperoperation $+$; that is, for any $x, y, z \in H$, we have $x \circ (y + z) = x \circ y + x \circ z$ and $(y + z) \circ x = y \circ x + z \circ x$;

(4) there exists an element $0 \in H$ such that $x + 0 = 0 + x = \{x\}$ and $x \circ 0 = 0 \circ x = \{0\}$, which is called the zero of H.

Let $(H, +, \circ)$ be a semihyperring; for any $x_1, x_2, \ldots, x_n \in H$, we write $\sum_{i=1}^{n} x_i = x_1 + x_2 + \cdots + x_n$.

Example 3. Let $H = \{0, a, b\}$ be a set with two hyperoperations $(+)$ and $(\circ)$ as follows:

+	0	a	b
0	$\{0\}$	$\{a\}$	$\{b\}$
a	$\{a\}$	$\{0, a\}$	$\{0, a, b\}$
b	$\{b\}$	$\{0, a, b\}$	$\{0, b\}$

∘	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{0\}$	$\{0\}$	$\{0\}$
b	$\{0\}$	$\{0\}$	$\{0, a\}$

(1)

Then H is a semihyperring with a zero.

A nonempty subset L (resp., R) of a semihyperring H is called a left (resp., right) hyperideal of H if it satisfies $L + L \subseteq L$ and $H \circ L \subseteq L$ (resp., $R \circ H \subseteq R$). A nonempty subset B of a semihyperring H is called a bi-hyperideal of H if it satisfies $B + B \subseteq B$, $B \circ B \subseteq B$ and $B \circ H \circ B \subseteq B$. A nonempty subset Q of a semihyperring H is called a quasi-hyperideal of H if it satisfies $Q + Q \subseteq Q$ and $Q \circ H \cap H \circ Q \subseteq Q$.

A fuzzy subset of a semihyperring H, by definition, is an arbitrary mapping $\mu : H \to [0, 1]$, where $[0, 1]$ is the usual interval of real numbers. We denote by $F(H)$ the set of all fuzzy subsets of H.

In what follows let $\gamma, \delta \in [0, 1]$ be such that $\gamma < \delta$ and H a semihyperring with a zero 0. For any $A \subseteq H$, we define $\kappa_A^{\gamma,\delta}$ to be the fuzzy subset of H by $\kappa_A^{\gamma,\delta}(x) \geq \delta$ for all $x \in A$ and $\kappa_A^{\gamma,\delta}(x) \leq \gamma$ otherwise. Clearly, $\kappa_A^{\gamma,\delta}$ is the characteristic function of A if $\gamma = 0$ and $\delta = 1$.

A fuzzy subset μ in H defined by

$$\mu(y) = \begin{cases} r\,(\neq 0) & \text{if } y = x, \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

is said to be a fuzzy point with support x and value r and is denoted by x_r.

For a fuzzy point x_r and a fuzzy subset μ of H, we say that

(1) $x_r \in_\gamma \mu$ if $\mu(x) \geq r > \gamma$,

(2) $x_r\ q_\delta\ \mu$ if $\mu(x) + r > 2\delta$,

(3) $x_r \in_\gamma \vee q_\delta \mu$ if $x_r \in_\gamma \mu$ or $x_r\ q_\delta\ \mu$.

In the sequel, unless otherwise stated, $\overline{\alpha}$ means that α does not hold, where $\alpha \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta\}$. For any $\mu, \nu \in F(H)$, by $\mu \subseteq_{(\gamma,\delta)} \nu$, we mean that $x_r \in_\gamma \mu$ implies $x_r \in_\gamma \vee q_\delta \nu$ for all $x \in H$ and $r \in (\gamma, 1]$.

Lemma 4. *Let μ and ν be two fuzzy subsets of H, and then the following conditions are equivalent.*

(1) $\mu \subseteq_{(\gamma,\delta)} \nu$.

(2) $\max\{\nu(x),\gamma\} \geq \min\{\mu(x),\delta\}$ *for all* $x \in H$.

(3) $\mu_r \subseteq \nu_r$ *for all* $r \in [\gamma,\delta]$, *where* $\mu_r = \{x \in H \mid \mu(x) \geq r\}$.

Proof. (1)⇒(2) Assume that (1) holds. Let μ and ν be any fuzzy subsets of H. If $\max\{\nu(x),\gamma\} < \min\{\mu(x),\delta\}$ for some $x \in H$, then there exists r such that $\max\{\nu(x),\gamma\} < r < \min\{\mu(x),\delta\}$; it follows that $x_r \in_\gamma \mu$ but $x_r \overline{\in_\gamma \vee q_\delta}\ \nu$, a contradiction. Hence $\max\{\nu(x),\gamma\} \geq \min\{\mu(x),\delta\}$ for all $x \in H$.

(2)⇒(1) Assume that (2) holds. If $\mu \subseteq_{(\gamma,\delta)} \nu$ does not hold, then there exists $x_r \in_\gamma \mu$ but $x_r \overline{\in_\gamma \vee q_\delta}\ \nu$, which contradicts $\max\{\nu(x),\gamma\} \geq \min\{\mu(x),\delta\}$. Therefore, (1) holds.

It is easy to check that (1)⇔(3). This completes the proof. □

According to Lemma 4, it can be easily seen that $\mu \subseteq_{(\gamma,\delta)} \kappa_H^{\gamma,\delta}$ for all $\mu \in F(H)$. Also, the following result can be easily deduced.

Lemma 5. *Let μ, ν, and ω be any fuzzy subsets of H such that $\mu \subseteq_{(\gamma,\delta)} \nu$ and $\nu \subseteq_{(\gamma,\delta)} \omega$, and then $\mu \subseteq_{(\gamma,\delta)} \omega$.*

We will write $\mu \approx_{(\gamma,\delta)} \nu$ if $\mu \subseteq_{(\gamma,\delta)} \nu$ and $\nu \subseteq_{(\gamma,\delta)} \mu$. Let $\mu \approx_{(\gamma,\delta)} \nu$ be the relation which is defined in the above two fuzzy sets of a hypersemigorup H. Then it satisfies reflexivity, symmetry, and transitivity. That is, it is an equivalence relation on the $F(H)$.

For two fuzzy subsets μ and ν of a semihyperring H; the sum $\mu \oplus \nu$ is defined by

$$\mu \oplus \nu(x) = \sup_{x \in a+b} \min\{\mu(a), \nu(b)\} \tag{3}$$

and $\mu \oplus \nu(x) = 0$ if x cannot be expressible as an element in $a+b$ for all $x \in H$.

Definition 6. Let μ and ν be fuzzy subsets of a semihyperring H, the intrinsic product $\mu \odot \nu$ is defined by:

$$\mu \odot \nu(x) = \sup_{x \in \sum_{i=1}^{n} a_i \circ b_i} \min\{\mu(a_i), \nu(b_i)\} \tag{4}$$

and $\mu \odot \nu(x) = 0$ if x cannot be expressible as an element in $\sum_{i=1}^{n} a_i \circ b_i$ for all $x \in H$.

Lemma 7. *Let μ_1, μ_2, ν_1, and ν_2 be any fuzzy subsets of H such that $\mu_1 \subseteq_{(\gamma,\delta)} \mu_2$ and $\nu_1 \subseteq_{(\gamma,\delta)} \nu_2$, and then*

(1) $\mu_1 \odot \nu_1 \subseteq_{(\gamma,\delta)} \mu_2 \odot \nu_2$,

(2) $\mu_1 \cap \nu_1 \subseteq_{(\gamma,\delta)} \mu_2 \cap \nu_2$.

Proof. (1) For any $x \in H$, by Lemma 4, we have

$$\begin{aligned}
&\max\{(\mu_2 \odot \nu_2)(x), \gamma\} \\
&\quad = \max\left\{\sup_{x \in \sum_{i=1}^{n} a_i \circ b_i} \min\{\mu_2(a_i), \nu_2(b_i)\}, \gamma\right\} \\
&\quad = \sup_{x \in \sum_{i=1}^{n} a_i \circ b_i} \min\{\max\{\mu_2(a_i), \gamma\}, \max\{\nu_2(b_i), \gamma\}\} \\
&\quad \geq \sup_{x \in \sum_{i=1}^{n} a_i \circ b_i} \min\{\min\{\mu_1(a_i), \delta\}, \min\{\nu_1(b_i), \delta\}\} \\
&\quad = \sup_{x \in \sum_{i=1}^{n} a_i \circ b_i} \min\{\min\{\mu_1(a_i), \nu_1(b_i)\}, \delta\} \\
&\quad = \min\left\{\sup_{x \in \sum_{i=1}^{n} a_i \circ b_i} \min\{\mu_1(a_i), \nu_1(b_i)\}, \delta\right\} \\
&\quad = \min\{(\mu_1 \odot \nu_1)(x), \delta\}.
\end{aligned} \tag{5}$$

This implies that $\mu_1 \odot \nu_1 \subseteq_{(\gamma,\delta)} \mu_2 \odot \nu_2$.

(2) It is straightforward by Lemma 4. □

Lemma 8. *Let μ, ν, and ω be any fuzzy subsets of H. Then*

(1) $(\mu \odot \nu) \odot \omega = \mu \odot (\nu \odot \omega)$,

(2) $\mu \odot (\nu \cup \omega) = \mu \odot \omega \cup \nu \odot \omega$, $(\mu \cup \nu) \odot \omega = \mu \odot \omega \cup \nu \odot \omega$,

(3) $\mu \odot (\nu \cap \omega) \subseteq_{(\gamma,\delta)} \mu \odot \omega \cap \nu \odot \omega$, $(\mu \cap \nu) \odot \omega \subseteq_{(\gamma,\delta)} \mu \odot \omega \cap \nu \odot \omega$.

Proof. It is straightforward. □

Lemma 9. *Let A, B be any subsets in H. Then one has*

(1) $A \subseteq B$ *if and only if* $\kappa_A^{\gamma,\delta} \subseteq_{(\gamma,\delta)} \kappa_B^{\gamma,\delta}$,

(2) $\kappa_A^{\gamma,\delta} \cap \kappa_B^{\gamma,\delta} \approx_{(\gamma,\delta)} \kappa_{A \cap B}^{\gamma,\delta}$,

(3) $\kappa_A^{\gamma,\delta} \odot \kappa_B^{\gamma,\delta} \approx_{(\gamma,\delta)} \kappa_{A \circ B}^{\gamma,\delta}$.

Proof. (1) Assume that $\kappa_A^{\gamma,\delta} \subseteq_{(\gamma,\delta)} \kappa_B^{\gamma,\delta}$. If $A \nsubseteq B$, then there exists $x \in A$ but $x \notin B$. This implies that $x_1 \in_\gamma \kappa_A^{\gamma,\delta}$ and $x_1 \overline{\in_\gamma \vee q_\delta}\ \kappa_B^{\gamma,\delta}$, which contradicts $\kappa_A^{\gamma,\delta} \subseteq_{(\gamma,\delta)} \kappa_B^{\gamma,\delta}$. Conversely, assume that $A \subseteq B$. Let $x \in H$ and $r \in (\gamma, 1]$ be such that $x_r \in_\gamma \kappa_A^{\gamma,\delta}$. Then $\kappa_A^{\gamma,\delta}(x) \geq r > \gamma$, and so $x \in A$. Thus $x \in B$ and $\kappa_B^{\gamma,\delta}(x) \geq \delta$. This gives that $x_r \in_\gamma \vee q_\delta \kappa_B^{\gamma,\delta}$. Hence $\kappa_A^{\gamma,\delta} \subseteq_{(\gamma,\delta)} \kappa_B^{\gamma,\delta}$.

(2) It is straightforward.

(3) Let $x \in H$. If $x \in A \circ B$, then $\kappa_{A \circ B}^{\gamma,\delta}(x) \geq \delta$ and $x \in y \circ z$ for some $y \in A$ and $z \in B$. Thus we have

$$\begin{aligned}
\left(\kappa_A^{\gamma,\delta} \odot \kappa_B^{\gamma,\delta}\right)(x) &= \sup_{x \in \sum_{i=1}^{n} a_i \circ b_i} \min\left\{\kappa_A^{\gamma,\delta}(a_i), \kappa_B^{\gamma,\delta}(b_i)\right\} \\
&\geq \min\left\{\kappa_A^{\gamma,\delta}(y), \kappa_B^{\gamma,\delta}(z)\right\} \geq \delta.
\end{aligned} \tag{6}$$

This implies that $\kappa_A^{\gamma,\delta} \odot \kappa_B^{\gamma,\delta} \approx_{(\gamma,\delta)} \kappa_{A \circ B}^{\gamma,\delta}$.

If $x \notin A \circ B$, then $(\kappa_{A \circ B}^{\gamma,\delta})(x) \leq \gamma$ and x cannot be expressible as a element in $y \circ z$ for any $y \in A$ and $z \in B$.

Thus $(\kappa_A^{\gamma,\delta} \odot \kappa_B^{\gamma,\delta})(x) = 0 \leq \gamma$. This implies that $\kappa_A^{\gamma,\delta} \odot \kappa_B^{\gamma,\delta} \approx_{(\gamma,\delta)} \kappa_{A\circ B}^{\gamma,\delta}$. □

Definition 10. A fuzzy subset μ in a semihyperring H is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (resp., right) hyperideal if it satisfies

(F1a) $\mu \oplus \mu \subseteq_{(\gamma,\delta)} \mu$,

(F2a) $\kappa_H^{\gamma,\delta} \odot \mu \subseteq_{(\gamma,\delta)} \mu$ (resp., $\mu \odot \kappa_H^{\gamma,\delta} \subseteq_{(\gamma,\delta)} \mu$).

A fuzzy subset of a semihyperring H is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy hyperideal if it is both an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left hyperideal and an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right hyperideal.

Definition 11. A fuzzy subset μ in a semihyperring H is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-hyperideal if it satisfies conditions (F1a) and

(F3a) $\mu \odot \mu \subseteq_{(\gamma,\delta)} \mu$,

(F4a) $\mu \odot \kappa_H^{\gamma,\delta} \odot \mu \subseteq_{(\gamma,\delta)} \mu$.

Definition 12. A fuzzy subset μ in a semihyperring H is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-hyperideal if it satisfies conditions (F1a) and

(F5a) $\mu \odot \kappa_H^{\gamma,\delta} \cap \kappa_H^{\gamma,\delta} \odot \mu \subseteq_{(\gamma,\delta)} \mu$.

Theorem 13. *Let μ be a fuzzy subset of a semihyperring H. Then μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (resp., right) hyperideal if and only if for any $x, y \in H$, one has*

(F1b) $\max\{\inf_{z\in x+y}\mu(z), \gamma\} \geq \min\{\mu(x), \mu(y), \delta\}$ *for all $x, y \in H$.*

(F2b) $\max\{\inf_{z\in x\circ y}\mu(z), \gamma\} \geq \min\{\mu(y), \delta\}$ *(resp., $\max\{\inf_{z\in x\circ y}\mu(z), \gamma\} \geq \min\{\mu(x), \delta\}$).*

Proof. Assume that (F1a) holds. For any $x, y \in H$, if possible, let $\max\{\inf_{z\in x+y}\mu(z), \gamma\} < \min\{\mu(x), \mu(y), \delta\}$. Choose r such that $\max\{\inf_{z\in x+y}\mu(z), \gamma\} < r < \min\{\mu(x), \mu(y), \delta\}$. Then there exists $z \in x + y$ such that $\mu(z) < r < \min\{\mu(x), \mu(y), \delta\}$; that is, $z_r \overline{\in_\gamma \vee q_\delta} \mu$. Then $(\mu \oplus \mu)(z) = \sup_{z\in a+b} \min\{\mu(a), \mu(b)\} \geq \min\{\mu(x), \mu(y), \delta\} \geq r$; hence $z_r \in_\gamma \mu \oplus \mu$, which contradicts (F1a). Therefore $\max\{\inf_{z\in x+y}\mu(z), \gamma\} \geq \min\{\mu(x), \mu(y), \delta\}$. That is, (F1b) holds.

Conversely, assume that (F1b) holds. Let $x \in H$ and $r \in (\gamma, 1]$ be such that $z_r \in_\gamma \mu \oplus \mu$, if possible; let $z_r \overline{\in_\gamma \vee q_\delta} \mu$. Then $\mu(x) < r$ and $\mu(x) + r \leq 2\delta$. Hence $\mu(x) < \delta$. Now, $r \leq (\mu\oplus\mu)(x) = \sup_{x\in y+z} \min\{\mu(y), \mu(z)\} \leq \sup_{x\in y+z}\mu(z) \leq \mu(x)$ (since $x \in y + z$ and the assumption imply that $\delta > \mu(x) \geq \min\{\mu(z), \delta\} = \mu(y)$), a contradiction. Hence, $x_r \in \vee q \ \mu$. This implies that $\mu \oplus \mu \subseteq_{(\gamma,\delta)} \mu$. Therefore (F1a) holds.

In a similar way, we can prove that (F2a)⇔(F2b). □

Theorem 14. *Let μ be a fuzzy subset of a semihyperring H. Then μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-hyperideal if and only if it satisfies (F1b) and for any $x, y, z \in H$, one has*

(F3b) $\max\{\inf_{w\in x\circ y}\mu(w), \gamma\} \geq \min\{\mu(x), \mu(y), \delta\}$,

(F4b) $\max\{\inf_{w\in x\circ y\circ z}\mu(w), \gamma\} \geq \min\{\mu(x), \mu(z), \delta\}$.

Proof. The proof is analogous to that of Theorem 13. □

Remark 15. Let μ be any fuzzy subsets of H. If μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (resp., right) hyperideal of H, then $a \in x \circ y$ implies that $\max\{\mu(a), \gamma\} \geq \min\{\mu(y), \delta\}$ (resp., $\max\{\mu(a), \gamma\} \geq \min\{\mu(x), \delta\}$). If μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-hyperideal of H, then $a \in x \circ y \circ z$ implies that $\max\{\mu(a), \gamma\} \geq \min\{\mu(x), \mu(z), \delta\}$.

Theorem 16. *(1) $\mu \cap \nu$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-hyperideal of H for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left hyperideal μ and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right hyperideal ν of H.*

(2) Any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-hyperideal of H is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-hyperideal.

Proof. (1) Let μ and ν be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left hyperideal and any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left hyperideal of H, respectively. Then $\kappa_H^{\gamma,\delta} \odot \mu \subseteq_{(\gamma,\delta)} \mu$ and $\nu \odot \kappa_H^{\gamma,\delta} \subseteq_{(\gamma,\delta)} \nu$. By Lemma 7, we have $(\mu \cap \nu) \odot \kappa_H^{\gamma,\delta} \cap \kappa_H^{\gamma,\delta} \odot (\mu \cap \nu) \subseteq_{(\gamma,\delta)} \nu \odot \kappa_H^{\gamma,\delta} \cap \kappa_H^{\gamma,\delta} \odot \mu \subseteq_{(\gamma,\delta)} \mu \cap \nu$, and so $\mu \cap \nu$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-hyperideal of H.

(2) Let μ be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-hyperideal of H. Then, $\mu \odot \mu \subseteq_{(\gamma,\delta)} \kappa_H^{\gamma,\delta} \odot \mu$, $\mu \odot \mu \subseteq_{(\gamma,\delta)} \mu \odot \kappa_H^{\gamma,\delta}$, $\mu \odot \kappa_H^{\gamma,\delta} \odot \mu \subseteq_{(\gamma,\delta)} \kappa_H^{\gamma,\delta} \odot \mu$ and $\mu \odot \kappa_H^{\gamma,\delta} \odot \mu \subseteq_{(\gamma,\delta)} \mu \odot \kappa_H^{\gamma,\delta}$, and so $\mu \odot \mu \subseteq_{(\gamma,\delta)} \kappa_H^{\gamma,\delta} \odot \mu \cap \mu \odot \kappa_H^{\gamma,\delta} \subseteq_{(\gamma,\delta)} \mu$ and $\mu \odot \kappa_H^{\gamma,\delta} \odot \mu \subseteq_{(\gamma,\delta)} \kappa_H^{\gamma,\delta} \odot \mu \cap \mu \odot \kappa_H^{\gamma,\delta} \subseteq_{(\gamma,\delta)} \mu$. Hence μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-hyperideal of H. □

One may easily see that any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left hyperideal and any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right hyperideal of a semihyperring H are an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-hyperideal of H. However, the converse of the property and Theorem 16 do not hold in general as shown in the following examples.

Example 17. Let $H = \{0, a, b, c\}$ be a set with two hyperoperations $(+)$ and $(\circ)$ as follows:

+	0	a	b	c
0	{0}	{a}	{b}	{c}
a	{a}	{a}	{a}	{a}
b	{b}	{a}	{0, b}	{0, b, c}
c	{c}	{a}	{0, b, c}	{0, c}

∘	0	a	b	c
0	{0}	{0}	{0}	{0}
a	{0}	{a}	{0, b}	{0}
b	{0}	{0}	{0}	{0}
c	{0}	{0, c}	{0}	{0}

(7)

Then H is a semihyperring and $Q = \{0, a\}$ is a quasi-hyperideal of H, and it is not a left (right) hyperideal of H. Define the fuzzy set μ of H as follows:

$$\mu(0) = \mu(a) = 0.6, \qquad \mu(b) = \mu(c) = 0.3. \tag{8}$$

Then μ is an $(\in_{0.4}, \in_{0.4} \vee q_{0.6})$-fuzzy quasi-hyperideal of H, and it is not an $(\in_{0.4}, \in_{0.4} \vee q_{0.6})$-fuzzy left (right) hyperideal of H.

Example 18. Let $H = \{0, a, b, c\}$ be a set with two hyperoperations $(+)$ and $(\circ)$ as follows:

$$\begin{array}{c|cccc} + & 0 & a & b & c \\ \hline 0 & \{0\} & \{a\} & \{b\} & \{c\} \\ a & \{a\} & \{a\} & \{0,a,b\} & \{0,a,c\} \\ b & \{b\} & \{0,a,b\} & \{0,b\} & \{0,b,c\} \\ c & \{c\} & \{0,a,c\} & \{0,b,c\} & \{0,c\} \end{array}$$
$$\begin{array}{c|cccc} \circ & 0 & a & b & c \\ \hline 0 & \{0\} & \{0\} & \{0\} & \{0\} \\ a & \{0\} & \{0\} & \{0\} & \{0\} \\ b & \{0\} & \{0\} & \{0\} & \{0,a\} \\ c & \{0\} & \{0\} & \{0,a\} & \{0,b\} \end{array} \quad (9)$$

Then $B = \{0, b\}$ is a bi-hyperideal of H, and it is not a quasi-hyperideal of H. Define the fuzzy set μ of H as follows:

$$\mu(0) = \mu(b) = 0.6, \qquad \mu(a) = \mu(c) = 0.4. \quad (10)$$

Then μ is an $(\in_{0.4}, \in_{0.4} \vee q_{0.6})$-fuzzy bi-hyperideal of H, and it is not an $(\in_{0.4}, \in_{0.4} \vee q_{0.6})$-fuzzy quasi-hyperideal of H.

Lemma 19. *Let A be any subset in H. Then the following conditions hold.*

(1) *A is a left (resp., right) hyperideal of H if and only if $\kappa_A^{\gamma,\delta}$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (resp., right) hyperideal of H.*

(2) *A is a bi-hyperideal of H if and only if $\kappa_A^{\gamma,\delta}$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-hyperideal of H.*

(3) *A is a quasi-hyperideal of H if and only if $\kappa_A^{\gamma,\delta}$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-hyperideal of H.*

Proof. Let A be any subset in H. According to Lemma 9, $H \circ A \subseteq A$ if and only if $\kappa_H^{\gamma,\delta} \odot \kappa_A^{\gamma,\delta} \approx_{(\gamma,\delta)} \kappa_{H\circ A}^{\gamma,\delta} \subseteq_{(\gamma,\delta)} \kappa_A^{\gamma,\delta}$. Hence A is a left hyperideal of H if and only if $\kappa_A^{\gamma,\delta}$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left hyperideal of H.

The case for bi-hyperdeals and quasi-hyperideals can be similarly proven. □

3. Characterizations of Hyperregular Semihyperrings

Definition 20. A semihyperring H is said to be hyperregular if for each $x \in H$, there exists $a \in H$ such that $x \in x \circ a \circ x$. Equivalent definitions: (1) $x \in x \circ H \circ x$ for all $x \in H$; (2) $A \subseteq A \circ H \circ A$ for all $A \subseteq H$.

Example 21. Let $H = \{0, a, b\}$ be a set with two hyperoperations $(+)$ and $(\circ)$ as follows:

$$\begin{array}{c|ccc} + & 0 & a & b \\ \hline 0 & \{0\} & \{a\} & \{b\} \\ a & \{a\} & \{0,a\} & \{0,a,b\} \\ b & \{b\} & \{0,a,b\} & \{0,b\} \end{array}$$
$$\begin{array}{c|ccc} \circ & 0 & a & b \\ \hline 0 & \{0\} & \{0\} & \{0\} \\ a & \{0\} & \{0,a\} & \{0,b\} \\ b & \{0\} & \{0,b\} & \{0,a\} \end{array} \quad (11)$$

Then H is a hyperregular semihyperring. Since $0 \in 0 \circ 0 \circ 0$, $a \in a \circ a \circ a$ and $b \in b \circ b \circ b$.

Remark 22. If H is a hyperregular semihyperring, then $H \circ H = H$ and $\kappa_H^{\gamma,\delta} \odot \kappa_H^{\gamma,\delta} \approx_{(\gamma,\delta)} \kappa_H^{\gamma,\delta}$.

Theorem 23. *Let H be a semihyperring. Then the following conditions are equivalent.*

(1) *H is hyperregular.*

(2) *$\mu \cap \nu \approx_{(\gamma,\delta)} \mu \odot \nu$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right hyperideal μ and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left hyperideal ν of H.*

(3) *$R \cap L = R \circ L$ for every right hyperideal R and every left hyperideal L of H.*

Proof. (1)⇒(2) Let H be a hyperregular semihyperring, with μ being any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right hyperideal and ν any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left hyperideal of H, respectively. Then $\mu \odot \nu \subseteq_{(\gamma,\delta)} \mu \odot \kappa_H^{\gamma,\delta} \subseteq_{(\gamma,\delta)} \mu$ and $\mu \odot \nu \subseteq_{(\gamma,\delta)} \kappa_H^{\gamma,\delta} \odot \nu \subseteq_{(\gamma,\delta)} \nu$. Hence $\mu \odot \nu \subseteq_{(\gamma,\delta)} \mu \cap \nu$. Let x be any element of H. Then, since H is hyperregular, there exists $a \in H$ such that $x \in x \circ a \circ x$, and then $x \in x \circ b$ for some $b \in x \circ a$. Now, since μ is $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right hyperideal of H, by Remark 15, we have $\max\{\mu(b), \gamma\} \geq \min\{\mu(x), \delta\}$. Hence,

$$\begin{aligned} \max\{\mu \odot \nu(x), \gamma\} &= \max\left\{\sup_{x \in \sum_{i=1}^{n} a_i \circ b_i} \min\{\mu(a_i), \nu(b_i)\}, \gamma\right\} \\ &\geq \max\{\min\{\mu(b), \nu(x)\}, \gamma\} \\ &= \min\{\max\{\mu(b), \gamma\}, \max\{\nu(x), \gamma\}\} \\ &\geq \min\{\min\{\mu(x), \delta\}, \max\{\nu(x), \gamma\}\} \\ &= \min\{\mu \cap \nu(x), \delta\}. \end{aligned} \quad (12)$$

This implies that $\mu \cap \nu \subseteq_{(\gamma,\delta)} \mu \odot \nu$. Hence (2) holds.

(2)⇒(3) It is straightforward by Lemmas 9 and 19.

(3)⇒(1) Let x be any element of H. Then $x \circ H + Mx$ and $H \circ x + Nx$, where $M = \{1, 2, \ldots\}$ and $N = \{1, 2, \ldots\}$ are the principal right hyperideal and principal left hyperideal generated by x, respectively. By the assumption, we have

$$\begin{aligned} x \in\ & (x \circ 0 + x) \cap (0 \circ x + x) \\ \subseteq\ & (x \circ H + Mx) \cap (H \circ x + Nx) \\ =\ & (x \circ H + Mx) \circ (H \circ x + Nx) \\ =\ & x \circ H \circ x + (x \circ H) \circ (Nx) \\ & + (Mx) \circ (H \circ x) + (Mx) \circ (Nx). \end{aligned} \quad (13)$$

This implies that $x \in x \circ a \circ x$ for some $a \in H$. Hence H is hyperregular. □

Corollary 24. *Let H be semihyperring. Then the following conditions are equivalent.*

(1) *H is hyperregular.*

(2) *$\mu \cap \nu \subseteq_{(\gamma,\delta)} \mu \odot \nu$ for every fuzzy subset μ and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left hyperideal ν of H.*

(3) *$\mu \cap \nu \subseteq_{(\gamma,\delta)} \mu \odot \nu$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right hyperideal μ and every fuzzy subset ν of H.*

Corollary 25. *Let H be a hyperregular semihyperring. Then every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-hyperideal is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-hyperideal of H.*

Proof. Let μ be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-hyperideal of H. Evidently, $\mu \odot \kappa_H^{\gamma,\delta}$ and $\kappa_H^{\gamma,\delta} \odot \mu$ are an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right hyperideal and an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left hyperideal of H, respectively. Now, by Theorem 23 and Remark 22, we have

$$\begin{aligned} &\mu \odot \kappa_H^{\gamma,\delta} \cap \kappa_H^{\gamma,\delta} \odot \mu \\ &\quad \approx_{(\gamma,\delta)} \mu \odot \kappa_H^{\gamma,\delta} \odot \kappa_H^{\gamma,\delta} \odot \mu \\ &\quad \approx_{(\gamma,\delta)} \mu \odot \kappa_H^{\gamma,\delta} \odot \mu \subseteq_{(\gamma,\delta)} \mu. \end{aligned} \tag{14}$$

This implies that μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-hyperideal of H. □

Lemma 26. *Let H be a semihyperring. Then the following conditions are equivalent.*

(1) *H is hyperregular.*

(2) *$B = B \circ H \circ B$ for every bi-hyperideal B of H.*

(3) *$Q = Q \circ H \circ Q$ for every quasi-hyperideal Q of H.*

Proof. (1)⇒(2) Assume that (1) holds. Let B be any bi-hyperideal of H and x any element of B. Then there exists $a \in H$ such that $x \in x \circ a \circ x$. It is easy to see that $x \circ a \circ x \subseteq B \circ H \circ B$ and so $x \in B \circ H \circ B$. Hence, $B \subseteq B \circ H \circ B$. On the other hand, since B is an bi-hyperideal of H, we have $B \circ H \circ B \subseteq B$. Therefore, $B = B \circ H \circ B$.

(2)⇒(3) Evidently, every quasi-hyperideal of H is a bi-hyperideal of H. Then by the assumption, we have $Q = Q \circ H \circ Q$ for every quasi-hyperideal Q of H.

(3)⇒(1) Assume that (3) holds. Let R and L be any right hyperideal and any left hyperideal of H, respectively. Then we have $(R \cap L) \circ S \cap S \circ (R \cap L) \subseteq R \circ S \cap S \circ L \subseteq R \cap L$, and so it is easy to see that $R \cap L$ is a quasi-hyperideal of H. By the assumption and Theorem 23, we have $R \cap L = (R \cap L) \circ S \circ (R \cap L) \subseteq R \circ S \circ L \subseteq R \circ L \subseteq R \cap L$. Hence, $R \circ L = R \cap L$ and so S is hyperregular. □

Theorem 27. *Let H be a semihyperring. Then the following conditions are equivalent.*

(1) *H is hyperregular.*

(2) *$\mu \approx_{(\gamma,\delta)} \mu \odot \kappa_H^{\gamma,\delta} \odot \mu$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-hyperideal μ of H.*

(3) *$\mu \approx_{(\gamma,\delta)} \mu \odot \kappa_H^{\gamma,\delta} \odot \mu$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-hyperideal μ of H.*

Proof. (1)⇒(2) Assume that (1) holds. Let μ be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-hyperideal of H and x any element of H. Then $\mu \odot \kappa_H^{\gamma,\delta} \odot \mu \subseteq_{(\gamma,\delta)} \mu$. On the other hand, since H is hyperregular, there exists $a \in H$ such that $x \in x \circ a \circ x$, and then $x \in b \circ x$ for some $b \in x \circ a$. Thus, by Remark 15, we have

$$\begin{aligned} &\max \left\{ \left(\mu \odot \kappa_H^{\gamma,\delta} \odot \mu \right)(x), \gamma \right\} \\ &\quad = \max \left\{ \sup_{x \in \sum_{i=1}^{n} a_i \circ b_i} \min \left\{ \left(\mu \odot \kappa_H^{\gamma,\delta} \right)(a_i), \mu(b_i) \right\}, \gamma \right\} \\ &\quad \geq \max \left\{ \min \left\{ \left(\mu \odot \kappa_H^{\gamma,\delta} \right)(b), \mu(x) \right\}, \gamma \right\} \\ &\quad = \max \left\{ \min \left\{ \sup_{b \in \sum_{j=1}^{m} c_j \circ d_j} \min \left\{ \mu(c_j), \kappa_H^{\gamma,\delta}(d_j) \right\}, \mu(x) \right\}, \gamma \right\} \\ &\quad \geq \max \left\{ \min \left\{ \mu(x), \kappa_H^{\gamma,\delta}(a), \mu(x) \right\}, \gamma \right\} \\ &\quad \geq \max \left\{ \min \left\{ \mu(x), \delta, \mu(x) \right\}, \gamma \right\} \\ &\quad = \min \left\{ \mu(x), \delta \right\}. \end{aligned} \tag{15}$$

This implies that $\mu \subseteq_{(\gamma,\delta)} \mu \odot \kappa_H^{\gamma,\delta} \odot \mu$. Hence we have $\mu \approx_{(\gamma,\delta)} \mu \odot \kappa_H^{\gamma,\delta} \odot \mu$.

(2)⇒(3) It is straightforward by Theorem 16.

(3)⇒(1) Assume that (3) holds. Let Q be any quasi-hyperideal of H. Then by Lemma 19, $\kappa_Q^{\gamma,\delta}$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-hyperideal of H. Then, by the assumption and Lemma 9, we have

$$\kappa_Q^{\gamma,\delta} \subseteq_{(\gamma,\delta)} \kappa_Q^{\gamma,\delta} \odot \kappa_H^{\gamma,\delta} \odot \kappa_Q^{\gamma,\delta} \approx_{(\gamma,\delta)} \kappa_{Q \circ H \circ Q}^{\gamma,\delta}. \tag{16}$$

It follows from Lemma 9 that $Q \subseteq Q \circ H \circ Q$. Now, since Q is a quasi-hyperideal of H, we have $Q \circ H \circ Q = Q \circ H \cap H \circ Q \subseteq Q$, and so $Q \circ H \circ Q = Q$. Therefore H is hyperregular by Lemma 26. □

Corollary 28. *Let H be a semihyperring. Then the following conditions are equivalent.*

(1) *H is hyperregular.*

(2) *$\mu \subseteq_{(\gamma,\delta)} \mu \odot \kappa_H^{\gamma,\delta} \odot \mu$ for every fuzzy subset μ of H.*

Theorem 29. *Let H be a semihyperring. Then the following conditions are equivalent.*

(1) *H is hyperregular.*

(2) *$\mu \cap \nu \approx_{(\gamma,\delta)} \mu \odot \nu \odot \mu$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-hyperideal μ and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy hyperideal ν of H.*

(3) *$\mu \cap \nu \approx_{(\gamma,\delta)} \mu \odot \nu \odot \mu$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-hyperideal μ and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy hyperideal ν of H.*

Proof. (1)⇒(2) Assume that (1) holds. Let μ and ν be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-hyperideal and any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy hyperideal of H, respectively. Then it is clear that $\mu \odot \nu \odot \mu \subseteq_{(\gamma,\delta)} \mu \cap \nu$. Now let x be any element of H. Then since H is hyperregular, there exists $a \in H$ such that $x \in x \circ a \circ x \subseteq (x \circ a \circ x) \circ a \circ x = x \circ (a \circ x \circ a) \circ x$. Then there exist $b \in a \circ x \circ a$, $c \in x \circ (a \circ x \circ a)$ such that $c \in x \circ b$ and $x \in c \circ x$. Now, since ν is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy hyperideal of H, by Remark 15, we have $\max\{\nu(b), \gamma\} \geq \min\{\nu(x), \delta\}$. Thus,

$$\begin{aligned}
&\max\{(\mu \odot \nu \odot \mu)(x), \gamma\} \\
&\quad = \max\left\{\sup_{x \in \sum_{i=1}^{n} a_i \circ b_i} \min\{(\mu \odot \nu)(a_i), \mu(b_i)\}, \gamma\right\} \\
&\quad \geq \max\{\min\{(\mu \odot \nu)(c), \mu(x)\}, \gamma\} \\
&\quad = \max\left\{\min\left\{\sup_{c \in \sum_{j=1}^{m} c_j \circ d_j} \min\{\mu(c_j), \nu(d_j)\}, \mu(x)\right\}, \gamma\right\} \\
&\quad \geq \max\{\min\{\mu(x), \nu(b), \mu(x)\}, \gamma\} \\
&\quad = \min\{\max\{\mu(x), \gamma\}, \max\{\nu(b), \gamma\}, \max\{\mu(x), \gamma\}\} \\
&\quad \geq \min\{\max\{\mu(x), \gamma\}, \min\{\nu(x), \delta\}, \max\{\mu(x), \gamma\}\} \\
&\quad = \min\{(\mu \cap \nu)(x), \delta\}.
\end{aligned} \tag{17}$$

This implies that $\mu \cap \nu \subseteq_{(\gamma,\delta)} \mu \odot \nu \odot \mu$. Hence, (2) holds.

(2)⇒(3) This is straightforward by Theorem 16.

(3)⇒(1) Assume that (3) holds. Let μ be any fuzzy quasi-hyperideal of H. Then since $\kappa_H^{\gamma,\delta}$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy hyperideal of H, we have

$$\mu \approx_{(\gamma,\delta)} \mu \cap \kappa_H^{\gamma,\delta} \approx_{(\gamma,\delta)} \mu \odot \kappa_H^{\gamma,\delta} \odot \mu. \tag{18}$$

Then it follows from Theorem 27 that H is hyperregular. □

Corollary 30. *Let H be a semihyperring. Then the following conditions are equivalent.*

(1) *H is hyperregular.*

(2) *$B \cap A = B \circ A \circ B$ for every bi-hyperideal B and every hyperideal A of H.*

(3) *$Q \cap A = Q \circ A \circ Q$ for every quasi-hyperideal Q and every hyperideal A of H.*

Proof. (1)⇒(2) Assume that (1) holds. Let B and A be any bi-hyperideal and any hyperideal of H, respectively. Then by Lemma 19, $\kappa_B^{\gamma,\delta}$ and $\kappa_A^{\gamma,\delta}$ are an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-hyperideal and an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy hyperideal of H, respectively. Thus, by Theorem 29, we have

$$\kappa_{B\cap A}^{\gamma,\delta} \approx_{(\gamma,\delta)} \kappa_B^{\gamma,\delta} \cap \kappa_A^{\gamma,\delta} \approx_{(\gamma,\delta)} \kappa_B \odot \kappa_A^{\gamma,\delta} \odot \kappa_B^{\gamma,\delta} \approx_{(\gamma,\delta)} \kappa_{B\circ A\circ B}^{\gamma,\delta}. \tag{19}$$

Then it follows from Lemma 9 that $B \cap A = B \circ A \circ B$.

(2)⇒(3) It is straightforward.

(3)⇒(1) Assume that (3) holds. Let Q be any quasi-hyperideal of H. Then since H itself is a hyperideal of H, we have

$$Q = Q \cap H = Q \circ H \circ Q. \tag{20}$$

Then it follows from Lemma 26 that H is hyperregular. □

Theorem 31. *Let H be a semihyperring. Then the following conditions are equivalent.*

(1) *H is hyperregular.*

(2) *$\mu \cap \nu \subseteq_{(\gamma,\delta)} \mu \odot \nu$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-hyperideal μ and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left hyperideal ν of H.*

(3) *$\mu \cap \nu \subseteq_{(\gamma,\delta)} \mu \odot \nu$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-hyperideal μ and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left hyperideal ν of H.*

(4) *$\mu \cap \nu \subseteq_{(\gamma,\delta)} \mu \odot \nu$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right hyperideal μ and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-hyperideal ν of H.*

(5) *$\mu \cap \nu \subseteq_{(\gamma,\delta)} \mu \odot \nu$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right hyperideal μ and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-hyperideal ν of H.*

(6) *$\mu \cap \nu \cap \omega \subseteq_{(\gamma,\delta)} \mu \odot \nu \odot \omega$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right hyperideal μ, every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-hyperideal ν, and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left hyperideal ω of H.*

(7) *$\mu \cap \nu \cap \omega \subseteq_{(\gamma,\delta)} \mu \odot \nu \odot \omega$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right hyperideal μ, every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-hyperideal ν, and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left hyperideal ω of H.*

Proof. (1)⇒(2) Assume that (1) holds. Let μ and ν be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-hyperideal and any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left hyperideal of H, respectively. Now let x be any element of H. Since H is hyperregular, there exists $a \in H$ such that $x \in x \circ a \circ x$, and then $x \in x \circ b$ for some $b \in a \circ x$. Since ν is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left hyperideal of H, we have $\max\{\nu(b), \gamma\} \geq \min\{\nu(x), \delta\}$. Thus,

$$\begin{aligned}
&\max\{(\mu \odot \nu)(x), \gamma\} \\
&\quad = \max\left\{\sup_{x \in \sum_{i=1}^{n} a_i \circ b_i} \min\{\mu(a_i), \nu(b_i)\}, \gamma\right\} \\
&\quad \geq \max\{\min\{\mu(x), \nu(b)\}, \gamma\} \\
&\quad = \min\{\max\{\mu(x), \gamma\}, \max\{\nu(b), \gamma\}\} \\
&\quad \geq \min\{\max\{\mu(x), \gamma\}, \min\{\nu(x), \delta\}\} \\
&\quad = \min\{(\mu \cap \nu)(x), \delta\}.
\end{aligned} \tag{21}$$

This implies that $\mu \cap \nu \subseteq_{(\gamma,\delta)} \mu \odot \nu$.

(2)⇒(1) Assume that (2) holds. Let μ and ν be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right hyperideal and any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left hyperideal of H, respectively. Then it is easy to check that μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-hyperideal of H. By the assumption, we have $\mu \cap \nu \subseteq_{(\gamma,\delta)} \mu \odot \nu \subseteq_{(\gamma,\delta)} \mu \odot \kappa_H^{\gamma,\delta} \cap \kappa_H^{\gamma,\delta} \odot \nu \subseteq_{(\gamma,\delta)} \mu \cap \nu$. Hence, $\mu \cap \nu \approx_{(\gamma,\delta)} \mu \odot \nu$. So H is hyperregular by Theorem 23.

Similarly, we can show that (1)⇔(3), (1)⇔(4), and (1)⇔(5).

(1)⇒(6) Assume that (1) holds. Let μ, ν, and ω be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right hyperideal, any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-hyperideal, and any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left hyperideal of H, respectively. Now let x be any element of H. Since H is hyperregular, there exists $a \in H$ such that $x \in x \circ a \circ x \subseteq$

$(x\circ a\circ x)\circ a\circ x$. Then there exist $b\in x\circ a$, $c\in a\circ x$ and $d\in b\circ x$ such that $x\in d\circ c$. Now, since μ is an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy right hyperideal and ω is an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy left hyperideal of H, by Remark 15, we have $\max\{\mu(b),\gamma\}\geq\min\{\mu(x),\delta\}$ and $\max\{\omega(c),\gamma\}\geq\min\{\omega(x),\delta\}$. Thus, we have

$$\begin{aligned}
&\max\{(\mu\odot\nu\odot\omega)(x),\gamma\}\\
&=\max\left\{\sup_{x\in\sum_{i=1}^{n}a_i\circ b_i}\min\{(\mu\odot\nu)(a_i),\omega(b_i)\},\gamma\right\}\\
&\geq\max\{\min\{(\mu\odot\nu)(d),\omega(c)\},\gamma\}\\
&=\max\left\{\min\left\{\sup_{d\in\sum_{j=1}^{m}c_j\circ d_j}\min\{\mu(c_j),\nu(d_j)\},\omega(c)\right\},\gamma\right\}\\
&\geq\max\{\min\{\mu(b),\nu(x),\omega(c)\},\gamma\}\\
&=\min\{\max\{\mu(b),\gamma\},\max\{\nu(x),\gamma\},\max\{\omega(c),\gamma\}\}\\
&\geq\min\{\min\{\mu(x),\delta\},\max\{\nu(x),\gamma\},\min\{\omega(x),\delta\}\}\\
&=\min\{(\mu\cap\nu\cap\omega)(x),\delta\}.
\end{aligned}\tag{22}$$

This implies that $\mu\cap\nu\cap\omega\subseteq_{(\gamma,\delta)}\mu\odot\nu\odot\omega$.

(6)⇒(1) Assume that (6) holds. Let μ and ν be any $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy right hyperideal and any $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy left hyperideal of H, respectively. Since $\kappa_H^{\gamma,\delta}$ is an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy bi-hyperideal of H, by the assumption and Lemma 9, we have

$$\begin{aligned}
\mu\cap\nu&\approx_{(\gamma,\delta)}\mu\cap\kappa_H^{\gamma,\delta}\cap\nu\\
&\subseteq_{(\gamma,\delta)}\mu\odot\nu\subseteq_{(\gamma,\delta)}\mu\odot\kappa_H^{\gamma,\delta}\cap\kappa_H^{\gamma,\delta}\odot\nu\\
&\subseteq_{(\gamma,\delta)}\mu\cap\nu.
\end{aligned}\tag{23}$$

Hence, $\mu\cap\nu\approx_{(\gamma,\delta)}\mu\odot\nu$. Therefore, H is hyperregular by Theorem 23.

Similarly, we can show that (1)⇔(7). This completes the proof. □

Corollary 32. *Let H be a semihyperring. Then the following conditions are equivalent.*

(1) *H is hyperregular.*

(2) *$B\cap L\subseteq B\circ L$ for every bi-hyperideal B and every left hyperideal L of H.*

(3) *$Q\cap L\subseteq Q\circ L$ for every quasi-hyperideal Q and every left hyperideal L of H.*

(4) *$R\cap B\subseteq R\circ B$ for every right hyperideal R and every bi-hyperideal B of H.*

(5) *$R\cap Q\subseteq R\circ Q$ for every right hyperideal R and every quasi-hyperideal Q of H.*

(6) *$R\cap B\cap L\subseteq R\circ B\circ L$ for every right hyperideal R, every bi-hyperideal B, and every left hyperideal L of H.*

(7) *$R\cap Q\cap L\subseteq R\circ Q\circ L$ for every right hyperideal R, every quasi-hyperideal Q, and every left hyperideal L of H.*

Definition 33. A subset A in a semihyperring H is called idempotent if $A=A\circ A$. A fuzzy subset μ in a semihyperring H is called $(\in_\gamma,\in_\gamma\vee q_\delta)$-idempotent if $\mu\approx_{(\gamma,\delta)}\mu\odot\mu$.

Example 34. Consider Example 21. Let $A=\{0,a\}$. Then $A\circ A=A$ and so A is idempotent. Define a fuzzy subset of H by $\mu(0)=0.6$, $\mu(a)=0.5$, and $\mu(b)=\mu(c)=0$. Then $\mu\odot\mu\approx_{(0.4,0.6)}\mu$ and so μ is $(\in_{0.4},\in_{0.4}\vee q_{0.6})$-fuzzy idempotent.

Theorem 35. *A semihyperring H is hyperregular if and only if all $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy right and $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy left hyperideals of H are $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy idempotent and for every $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy right hyperideal μ and every $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy left hyperideal ν of H, the fuzzy subset $\mu\odot\nu$ is an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy quasi-hyperideal of H.*

Proof. Assume that H is hyperregular. Let μ and ν be any $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy right hyperideal and any $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy left hyperideal of H, respectively. Then we have $\mu\odot\mu\subseteq_{(\gamma,\delta)}\mu\odot\kappa_H^{\gamma,\delta}\subseteq_{(\gamma,\delta)}\mu$. Since H is hyperregular, by Theorem 31, we have $\mu\subseteq_{(\gamma,\delta)}\mu\odot\mu$ and so $\mu\approx_{(\gamma,\delta)}\mu\odot\mu$. Hence μ is $(\in_\gamma,\in_\gamma\vee q_\delta)$-idempotent. In a similar way, we may prove that every $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy left hyperideal is $(\in_\gamma,\in_\gamma\vee q_\delta)$-idempotent. Now let μ and ν be any $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy right hyperideal and any $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy left hyperideal of H, respectively. By Theorem 23, we have $\mu\cap\nu\approx_{(\gamma,\delta)}\mu\odot\nu$ and so

$$\begin{aligned}
&\kappa_H^{\gamma,\delta}\odot(\mu\odot\nu)\cap(\mu\odot\nu)\odot\kappa_H^{\gamma,\delta}\\
&\approx_{(\gamma,\delta)}\kappa_H^{\gamma,\delta}\odot(\mu\cap\nu)\cap(\mu\cap\nu)\odot\kappa_H^{\gamma,\delta}\\
&\subseteq_{(\gamma,\delta)}\kappa_H^{\gamma,\delta}\odot\mu\cap\nu\odot\kappa_H^{\gamma,\delta}\subseteq_{(\gamma,\delta)}\mu\cap\nu\\
&\approx_{(\gamma,\delta)}\mu\odot\nu.
\end{aligned}\tag{24}$$

Therefore $\mu\odot\nu$ is an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy quasi-hyperideal of H.

Conversely, assume that the given conditions hold. Let μ be any $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy quasi-hyperideal of H. It is easy to check that $\mu\cup\kappa_H^{\gamma,\delta}\odot\mu$ is an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy left hyperideal of H; then by the assumption, it is $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy idempotent. Thus, we have

$$\begin{aligned}
\mu&\subseteq_{(\gamma,\delta)}\mu\cup\kappa_H^{\gamma,\delta}\odot\mu\\
&\approx_{(\gamma,\delta)}\left(\mu\cup\kappa_H^{\gamma,\delta}\odot\mu\right)\odot\left(\mu\cup\kappa_H^{\gamma,\delta}\odot\mu\right)\\
&\approx_{(\gamma,\delta)}\mu\odot\mu\cup\mu\odot\kappa_H^{\gamma,\delta}\odot\mu\cup\kappa_H^{\gamma,\delta}\odot\mu\odot\mu\\
&\quad\cup\kappa_H^{\gamma,\delta}\odot\mu\odot\kappa_H^{\gamma,\delta}\odot\mu\subseteq_{(\gamma,\delta)}\kappa_H^{\gamma,\delta}\odot\mu.
\end{aligned}\tag{25}$$

That is, $\mu \subseteq_{(\gamma,\delta)} \kappa_H^{\gamma,\delta} \odot \mu$. Similarly, we can show that $\mu \subseteq_{(\gamma,\delta)} \mu \odot \kappa_H^{\gamma,\delta}$. Thus,

$$\mu \subseteq_{(\gamma,\delta)} \kappa_H^{\gamma,\delta} \odot \mu \cap \mu \odot \kappa_H^{\gamma,\delta} \subseteq_{(\gamma,\delta)} \mu \text{ and so}$$
$$\mu \approx_{(\gamma,\delta)} \kappa_H^{\gamma,\delta} \odot \mu \cap \mu \odot \kappa_H^{\gamma,\delta}. \tag{26}$$

On the other hand, it is clear that $\mu \odot \kappa_H^{\gamma,\delta}$ and $\kappa_H^{\gamma,\delta} \odot \mu$ are an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right hyperideal and an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left hyperideal of H, respectively. By the assumption, we have $(\mu \odot \kappa_H^{\gamma,\delta}) \odot (\mu \odot \kappa_H^{\gamma,\delta}) \approx_{(\gamma,\delta)} \mu \odot \kappa_H^{\gamma,\delta}$, $(\kappa_H^{\gamma,\delta} \odot \mu) \odot (\kappa_H^{\gamma,\delta} \odot \mu) \approx_{(\gamma,\delta)} \kappa_H^{\gamma,\delta} \odot \mu$, and $(\mu \odot \kappa_H^{\gamma,\delta}) \odot (\kappa_H^{\gamma,\delta} \odot \mu)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-hyperideal of H. Thus, we have

$$\begin{aligned}
\mu &\approx_{(\gamma,\delta)} \mu \odot \kappa_H^{\gamma,\delta} \cap \kappa_H^{\gamma,\delta} \odot \mu \\
&\approx_{(\gamma,\delta)} \left(\mu \odot \kappa_H^{\gamma,\delta}\right) \odot \left(\mu \odot \kappa_H^{\gamma,\delta}\right) \cap \left(\kappa_H^{\gamma,\delta} \odot \mu\right) \odot \left(\kappa_H^{\gamma,\delta} \odot \mu\right) \\
&\qquad \left(\text{since } \kappa_H^{\gamma,\delta} \text{ is an } \left(\in_\gamma, \in_\gamma \vee q_\delta\right)\text{-fuzzy hyperideal of } H \text{ and so } \kappa_H^{\gamma,\delta} \odot \kappa_H^{\gamma,\delta} \approx_{(\gamma,\delta)} \kappa_H^{\gamma,\delta}\right) \\
&\approx_{(\gamma,\delta)} \mu \odot \left(\kappa_H^{\gamma,\delta} \odot \kappa_H^{\gamma,\delta}\right) \odot \mu \odot \kappa_H^{\gamma,\delta} \cap \kappa_H^{\gamma,\delta} \odot \mu \odot \left(\kappa_H^{\gamma,\delta} \odot \kappa_H^{\gamma,\delta}\right) \odot \mu \\
&\approx_{(\gamma,\delta)} \left(\left(\mu \odot \kappa_H^{\gamma,\delta}\right) \odot \left(\kappa_H^{\gamma,\delta} \odot \mu\right)\right) \odot \kappa_H^{\gamma,\delta} \cap \kappa_H^{\gamma,\delta} \odot \left(\left(\mu \odot \kappa_H^{\gamma,\delta}\right) \odot \left(\kappa_H^{\gamma,\delta} \odot \mu\right)\right) \\
&\approx_{(\gamma,\delta)} \left(\mu \odot \kappa_H^{\gamma,\delta}\right) \odot \left(\kappa_H^{\gamma,\delta} \odot \mu\right) \\
&\approx_{(\gamma,\delta)} \mu \odot \left(\kappa_H^{\gamma,\delta} \odot \kappa_H^{\gamma,\delta}\right) \odot \mu \subseteq_{(\gamma,\delta)} \mu \odot \kappa_H^{\gamma,\delta} \odot \mu \subseteq_{(\gamma,\delta)} \mu,
\end{aligned} \tag{27}$$

which implies that $\mu \approx_{(\gamma,\delta)} \mu \odot \kappa_H^{\gamma,\delta} \odot \mu$. By Theorem 27, H is hyperregular. □

Corollary 36. *A semihyperring H is hyperregular if and only if all right hyperideals and all left hyperideals of H are idempotent and for every right hyperideal R and every left hyperideal L of H, the set $R \circ L$ is a quasi-hyperideal of H.*

Proof. Assume that H is hyperregular. Let R and L be any right hyperideal and any left hyperideal of S, respectively. Then by Lemma 19, $\kappa_R^{\gamma,\delta}$ and $\kappa_L^{\gamma,\delta}$ are an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right hyperideal and an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left hyperideal of H, respectively. By the assumption and Theorem 35, we know that $\kappa_R^{\gamma,\delta} \odot \kappa_L^{\gamma,\delta} \approx_{(\gamma,\delta)} \kappa_{R\circ L}^{\gamma,\delta}$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-hyperideal of H and $\kappa_R^{\gamma,\delta} \approx_{(\gamma,\delta)} \kappa_R^{\gamma,\delta} \odot \kappa_R^{\gamma,\delta}$, $\kappa_L^{\gamma,\delta} \approx_{(\gamma,\delta)} \kappa_L^{\gamma,\delta} \odot \kappa_L^{\gamma,\delta}$. Consequently, $R \circ L$ is a quasi-hyperideal of H and $R = R \circ R$, $L = L \circ L$.

Conversely, assume that the given conditions hold. Let Q be any quasi-hyperideal of H. Analogous to the proof of Theorem 35, we may show $Q = Q \circ H \circ Q$. Therefore, H is hyperregular by Lemma 26. □

Corollary 37. *Let H be a hyperregular semihyperring and μ any fuzzy subset of H. Then the following conditions are equivalent.*

(1) *μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-hyperideal of H.*

(2) *$\mu \approx_{(\gamma,\delta)} \nu \odot \omega$ for some $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right hyperideal ν and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left hyperideal ω of H.*

Proof. (1)⇒(2) Let μ be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-hyperideal of H. Since H is hyperregular, by Remark 22 and Theorem 27, we have $\mu \approx_{(\gamma,\delta)} \mu \odot \kappa_H^{\gamma,\delta} \odot \mu \approx_{(\gamma,\delta)} \mu \odot (\kappa_H^{\gamma,\delta} \odot \kappa_H^{\gamma,\delta}) \odot \mu = (\mu \odot \kappa_H^{\gamma,\delta}) \odot (\kappa_H^{\gamma,\delta} \odot \mu)$. Evidently, $\mu \odot \kappa_H^{\gamma,\delta}$ and $\kappa_H^{\gamma,\delta} \odot \mu$ are an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right hyperideal and an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left hyperideal of H, respectively. Hence, (2) holds.

(2)⇒(1) It is straightforward by Theorem 35. □

4. Characterizations of Hyperregular and Left Duo Semihyperrings

Definition 38. A semihyperring H is called left duo if every left hyperideal of H is a hyperideal of H. A semihyperring H is called $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left duo if every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left hyperideal of H is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy hyperideal of H.

Lemma 39. *A hyperregular semihyperring H is left duo if and only if for any $x \in H$ one has $x \circ H \subseteq H \circ x$.*

Proof. Assume that the hyperregular semihyperring H is left duo. Let x be any element of H. Clearly, $H \circ x$ is a left hyperideal of H. Since H is both hyperregular and left duo, then $H \circ x$ is also a right hyperideal of H. Then by the assumption, we have

$$x \circ H \subseteq (x \circ H \circ x) \circ H \subseteq (H \circ x) \circ H \subseteq H \circ x. \tag{28}$$

Conversely, assume that the given condition holds. Let L be any left hyperideal of H. Then $L \circ H \subseteq H \circ L \subseteq L$. That is, L is also a right hyperideal of H. Therefore, H is left duo. □

Theorem 40. *A hyperregular semihyperring H is left duo if and only if it is $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left duo.*

Proof. Assume that the hyperregular semihyperring H is left duo. Let μ be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left hyperideal and x, y any elements of H. Then since H is both hyperregular and left duo, by Lemma 39, there exists $z \in H$ such that $x \circ y \subseteq z \circ x$. Thus, we have

$$\max\left\{\inf_{a\in x\circ y} \mu(a), \gamma\right\} \ge \max\left\{\inf_{a\in z\circ x} \mu(a), \gamma\right\} \ge \min\{\mu(x), \delta\}. \tag{29}$$

This implies that μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right hyperideal of H. Therefore, H is $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left duo.

Conversely, assume that the hyperregular semihyperring H is $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left duo. Let L be any left hyperideal of H. Then, by Lemma 19, $\kappa_L^{\gamma,\delta}$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left hyperideal of H. By the assumption, $\kappa_L^{\gamma,\delta}$ is also an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right hyperideal of H. Using Lemma 19 again, L is also a right hyperideal of H. Therefore, H is left duo. □

Lemma 41. *Let H be a semihyperring which is both hyperregular and left duo. Then every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-hyperideal of H is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right hyperideal of H.*

Proof. Let μ be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-hyperideal and x, y any elements of H. Since H is both hyperregular and left duo, there exists elements $a, b \in H$ such that $x \in x \circ a \circ x$ and $x \circ y \subseteq b \circ x$. Thus, we have

$$\begin{aligned} x \circ y &\subseteq (x \circ a \circ x) \circ y = (x \circ a) \circ (x \circ y) \\ &\subseteq (x \circ a) \circ (b \circ x) = x \circ (a \circ b) \circ x. \end{aligned} \tag{30}$$

Hence, there exists $c \in H$ such that $x \circ y \subseteq x \circ c \circ x$. Since μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-hyperideal of H, we have

$$\max\left\{\inf_{z \in x \circ y} \mu(z), \gamma\right\} \geq \max\left\{\inf_{z \in x \circ c \circ x} \mu(z), \gamma\right\} \geq \min\{\mu(x), \delta\}. \tag{31}$$

This implies that μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right hyperideal of H. □

Theorem 42. *Let H be a semihyperring. Then the following conditions are equivalent.*

(1) *H is both hyperregular and left duo.*
(2) *H is both hyperregular and left $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy duo.*
(3) *$\mu \cap \nu \approx_{(\gamma,\delta)} \mu \odot \nu$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-hyperideal μ and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left hyperideal of H.*
(4) *$\mu \cap \nu \approx_{(\gamma,\delta)} \mu \odot \nu$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-hyperideal μ and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left hyperideal of H.*
(5) *$\mu \cap \nu \approx_{(\gamma,\delta)} \mu \odot \nu$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-hyperideal μ and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy hyperideal of H.*
(6) *$\mu \cap \nu \approx_{(\gamma,\delta)} \mu \odot \nu$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-hyperideal μ and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy hyperideal of H.*
(7) *$\mu \cap \nu \approx_{(\gamma,\delta)} \mu \odot \nu$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left or right hyperideal μ and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy hyperideal of H.*

Proof. Firstly, by Theorem 40, (1) and (2) are equivalent. Assume that (1) holds. Let μ and ν be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-hyperideal and any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left hyperideal of H, respectively. Then since H is hyperregular, it follows from Theorem 31 that $\mu \cap \nu \subseteq_{(\gamma,\delta)} \mu \odot \nu$. On the other hand, since μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-hyperideal of H, by Lemma 41, μ is also an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right hyperideal of H. Thus we have $\mu \odot \nu \subseteq_{(\gamma,\delta)} \mu \odot \kappa_H^{\gamma,\delta} \subseteq_{(\gamma,\delta)} \mu$ and $\mu \odot \nu \subseteq_{(\gamma,\delta)} \kappa_H^{\gamma,\delta} \odot \nu \subseteq_{(\gamma,\delta)} \nu$. Hence, $\mu \odot \nu \subseteq_{(\gamma,\delta)} \mu \cap \nu$ and so $\mu \odot \nu \approx_{(\gamma,\delta)} \mu \odot \nu$. Therefore, (3) holds. It is clear that (3)⇒(4)⇒(6)⇒(7) and (3)⇒(5)⇒(7). Assume that (7) holds. Let μ and ν be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right hyperideal and any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left hyperideal of H, respectively. Then since $\kappa_H^{\gamma,\delta}$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy hyperideal of H, by the assumption, we have $\nu \approx_{(\gamma,\delta)} \nu \cap \kappa_H^{\gamma,\delta} \approx_{(\gamma,\delta)} \nu \odot \kappa_H^{\gamma,\delta}$. Thus, ν is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right hyperideal of H; that is, ν is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy hyperideal of H. Hence, H is left duo. On the other hand, by the assumption, we have $\mu \cap \nu \approx_{(\gamma,\delta)} \mu \odot \nu$. Then it follows from Theorem 23 that H is hyperregular. This completes the proof. □

Theorem 43. *Let H be a semihyperring. Then the following conditions are equivalent.*

(1) *H is both hyperregular and left duo.*
(2) *$\mu \cap \nu \approx_{(\gamma,\delta)} \mu \odot \nu \odot \mu$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-hyperideal μ and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left hyperideal ν of H.*
(3) *$\mu \cap \nu \approx_{(\gamma,\delta)} \mu \odot \nu \odot \mu$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-hyperideal μ and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left hyperideal ν of H.*

Proof. (1)⇒(2) Assume that (1) holds. Let μ and ν be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-hyperideal and any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left hyperideal of H, respectively. Then

$$\mu \odot \nu \odot \mu \subseteq_{(\gamma,\delta)} \mu \odot \kappa_H^{\gamma,\delta} \odot \mu \subseteq_{(\gamma,\delta)} \mu. \tag{32}$$

On the other hand, since H is left duo, ν is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right hyperideal of H. Hence, we have

$$\mu \odot \nu \odot \mu \subseteq_{(\gamma,\delta)} \kappa_H^{\gamma,\delta} \odot \nu \odot \kappa_H^{\gamma,\delta} \subseteq_{(\gamma,\delta)} \nu \odot \kappa_H^{\gamma,\delta} \subseteq_{(\gamma,\delta)} \nu. \tag{33}$$

Therefore, $\mu \odot \nu \odot \mu \subseteq_{(\gamma,\delta)} \mu \cap \nu$. Now let x be any element of H. Since H is both hyperregular and left duo, there exist elements a and b of S such that $x \in x \circ a \circ x$ and $x \circ a \subseteq b \circ x$. Then we have

$$\begin{aligned} x \in x \circ a \circ x &\subseteq (x \circ a \circ x) \circ a \circ (x \circ a \circ x) \\ &= x \circ a \circ x \circ a \circ (x \circ a) \circ x \\ &\subseteq x \circ a \circ x \circ a \circ (b \circ x) \circ x \\ &= (x \circ a \circ x) \circ (a \circ b \circ x) \circ x. \end{aligned} \tag{34}$$

Thus, there exist elements c, d, e, and f such that $c \in x \circ a \circ x$, $d \in a \circ b$, $e \in d \circ x$, $f \in c \circ e$, and $x \in f \circ x$. Since μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-hyperideal and ν an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left hyperideal of S, by Remark 15, we have $\max\{\mu(c), \gamma\} \geq \min\{\mu(x), \delta\}$ and $\max\{\nu(e), \gamma\} \geq \min\{\nu(x), \delta\}$. Thus, we have

$$\begin{aligned} &\max\{(\mu \odot \nu \odot \mu)(x), \gamma\} \\ &\quad = \max\left\{\sup_{x \in \sum_{i=1}^{n} a_i \circ b_i} \min\{(\mu \odot \nu)(a_i), \omega(b_i)\}, \gamma\right\} \\ &\quad \geq \max\{\min\{(\mu \odot \nu)(f), \mu(x)\}, \gamma\} \end{aligned}$$

$$= \max\left\{\sup_{f\in\sum_{j=1}^{m}c_j\circ d_j}\min\left\{\min\left\{\mu\left(c_j\right),\nu\left(d_j\right)\right\},\mu\left(x\right)\right\},\gamma\right\}$$
$$\geq \max\left\{\min\left\{\mu\left(c\right),\nu\left(e\right),\mu\left(x\right)\right\},\gamma\right\}$$
$$= \min\left\{\max\left\{\mu\left(c\right),\gamma\right\},\max\left\{\nu\left(e\right),\gamma\right\},\max\left\{\mu\left(x\right),\gamma\right\}\right\}$$
$$\geq \min\left\{\min\left\{\mu\left(x\right),\delta\right\},\min\left\{\nu\left(x\right),\delta\right\},\max\left\{\mu\left(x\right),\gamma\right\}\right\}$$
$$= \min\left\{\left(\mu\cap\nu\right)\left(x\right),\delta\right\}. \tag{35}$$

This implies that $\mu\cap\nu\subseteq_{(\gamma,\delta)}\mu\odot\nu\odot\mu$. Therefore, $\mu\cap\nu\approx_{(\gamma,\delta)}\mu\odot\nu\odot\mu$.

(2)⇒(3) It is straightforward.

(3)⇒(1) Assume that (3) holds. Let μ be any $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy quasi-hyperideal of H. Since $\kappa_H^{\gamma,\delta}$ is an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy left hyperideal of H, by the assumption, we have $\mu\approx_{(\gamma,\delta)}\mu\cap\kappa_H^{\gamma,\delta}\approx_{(\gamma,\delta)}\mu\odot\kappa_H^{\gamma,\delta}\odot\mu$. Then it follows from Theorem 27 that H is hyperregular. Next, let ν be any $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy left hyperideal of H. Then, since $\kappa_H^{\gamma,\delta}$ is an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy bi-hyperideal of H, by the assumption, we have

$$\nu\approx_{(\gamma,\delta)}\kappa_H^{\gamma,\delta}\cap\nu\subseteq_{(\gamma,\delta)}\kappa_H^{\gamma,\delta}\odot\nu\odot\kappa_H^{\gamma,\delta}. \tag{36}$$

Hence,

$$\nu\odot\kappa_H^{\gamma,\delta}\subseteq_{(\gamma,\delta)}\left(\kappa_H^{\gamma,\delta}\odot\nu\odot\kappa_H^{\gamma,\delta}\right)\odot\kappa_H^{\gamma,\delta}$$
$$\approx_{(\gamma,\delta)}\kappa_H^{\gamma,\delta}\odot\nu\odot\kappa_H^{\gamma,\delta}\subseteq_{(\gamma,\delta)}\nu. \tag{37}$$

This implies that μ is an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy right hyperideal of H. Therefore, μ is an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy hyperideal of H. It follows from Theorem 40 that H is left duo. □

Theorem 44. *Let H be a semihyperring. Then the following conditions are equivalent.*

(1) *H is both hyperregular and left duo.*

(2) *$\mu\cap\nu\approx_{(\gamma,\delta)}\mu\odot\kappa_H^{\gamma,\delta}\odot\nu$ for every $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy bi-hyperideal μ and every $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy left hyperideal ν of H.*

(3) *$\mu\cap\nu\approx_{(\gamma,\delta)}\mu\odot\kappa_H^{\gamma,\delta}\odot\nu$ for every $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy quasi-hyperideal μ and every $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy left hyperideal ν of H.*

Proof. (1)⇒(2) Assume that (1) holds. Let μ and ν be any $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy bi-hyperideal and any $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy left hyperideal of H, respectively. Then by Lemma 41, μ is an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy right hyperideal of H. Thus, we have

$$\mu\odot\kappa_H^{\gamma,\delta}\odot\nu\subseteq_{(\gamma,\delta)}\mu\odot\nu\subseteq_{(\gamma,\delta)}\mu\odot\kappa_H^{\gamma,\delta}\subseteq_{(\gamma,\delta)}\mu,$$
$$\mu\odot\kappa_H^{\gamma,\delta}\odot\nu\subseteq_{(\gamma,\delta)}\mu\odot\nu\subseteq_{(\gamma,\delta)}\kappa_H^{\gamma,\delta}\odot\nu\subseteq_{(\gamma,\delta)}\nu. \tag{38}$$

Therefore, $\mu\odot\kappa_H^{\gamma,\delta}\odot\nu\subseteq_{(\gamma,\delta)}\mu\cap\nu$. Now let x be any element of H. Since H is hyperregular, there exists an element a of S such that $x\in x\circ a\circ x\subseteq(x\circ a\circ x)\circ a\circ x$. Thus, there exist elements b and c of H such that $b\in a\circ x\circ a$, $c\in x\circ b$, and $x\in c\circ x$. Then we have

$$\max\left\{\left(\mu\odot\kappa_H^{\gamma,\delta}\odot\nu\right)\left(x\right),\gamma\right\}$$
$$= \max\left\{\sup_{x\in\sum_{i=1}^{n}a_i\circ b_i}\min\left\{\left(\mu\odot\kappa_H^{\gamma,\delta}\right)\left(a_i\right),\nu\left(b_i\right)\right\},\gamma\right\}$$
$$\geq \max\left\{\min\left\{\left(\mu\odot\kappa_H^{\gamma,\delta}\right)\left(c\right),\nu\left(x\right)\right\},\gamma\right\}$$
$$= \max\left\{\sup_{c\in\sum_{j=1}^{m}c_j\circ d_j}\min\left\{\mu\left(c_j\right),\kappa_H^{\gamma,\delta}\left(d_j\right),\nu\left(x\right)\right\},\gamma\right\}$$
$$\geq \max\left\{\min\left\{\mu\left(x\right),\delta,\nu\left(x\right)\right\},\gamma\right\}$$
$$\geq \min\left\{\mu\left(x\right),\nu\left(x\right),\delta\right\} = \min\left\{\left(\mu\cap\nu\right)\left(x\right),\delta\right\}. \tag{39}$$

This implies that $\mu\cap\nu\subseteq_{(\gamma,\delta)}\mu\odot\kappa_H^{\gamma,\delta}\odot\nu$. Therefore, $\mu\odot\kappa_H^{\gamma,\delta}\odot\nu\approx_{(\gamma,\delta)}\mu\cap\nu$.

(2)⇒(3) It is straightforward.

(3)⇒(1) Assume that (3) holds. Let μ and ν be any $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy quasi-hyperideal and any $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy left hyperideal of H. Then by the assumption, we have

$$\mu\cap\nu\approx_{(\gamma,\delta)}\mu\odot\kappa_H^{\gamma,\delta}\odot\nu=\mu\odot\left(\kappa_H^{\gamma,\delta}\odot\nu\right)\subseteq_{(\gamma,\delta)}\mu\odot\nu. \tag{40}$$

Hence, it follows from Theorem 31 that H is hyperregular. Next, since ν is an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy left hyperideal, then it is an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy quasi-hyperideal. It is clear that $\kappa_H^{\gamma,\delta}$ is an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy left hyperideal of H; by the assumption, we have $\nu\approx_{(\gamma,\delta)}\nu\cap\kappa_H^{\gamma,\delta}\approx_{(\gamma,\delta)}\nu\odot\kappa_H^{\gamma,\delta}\odot\kappa_H^{\gamma,\delta}$. Hence,

$$\nu\odot\kappa_H^{\gamma,\delta}\approx_{(\gamma,\delta)}\nu\odot\kappa_H^{\gamma,\delta}\odot\kappa_H^{\gamma,\delta}\odot\kappa_H^{\gamma,\delta}$$
$$\subseteq_{(\gamma,\delta)}\nu\odot\kappa_H^{\gamma,\delta}\odot\kappa_H^{\gamma,\delta}\approx_{(\gamma,\delta)}\nu. \tag{41}$$

This implies that μ is an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy right hyperideal of H. Therefore, μ is an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy hyperideal of H. It follows from Theorem 40 that H is left duo. □

Theorem 45. *Let H be a semihyperring. Then the following conditions are equivalent.*

(1) *H is both hyperregular and left duo.*

(2) *$\mu\cap\nu\cap\omega\approx_{(\gamma,\delta)}\mu\odot\nu\odot\omega$ for every $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy bi-hyperideal μ, every $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy hyperideal ν, and every $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy left hyperideal ω of H.*

(3) *$\mu\cap\nu\cap\omega\approx_{(\gamma,\delta)}\mu\odot\nu\odot\omega$ for every $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy quasi-hyperideal μ, every $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy hyperideal ν, and every $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy left hyperideal ω of H.*

Proof. (1)⇒(2) Assume that (1) holds. Let μ, ν, and ω be any $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy bi-hyperideal, any $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy hyperideal, and any $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy left hyperideal of H, respectively. Then by Lemma 41, μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right hyperideal of H. Thus, we have

$$\mu \odot \nu \odot \omega \subseteq_{(\gamma,\delta)} \kappa_H^{\gamma,\delta} \odot \kappa_H^{\gamma,\delta} \odot \omega \subseteq_{(\gamma,\delta)} \kappa_H^{\gamma,\delta} \odot \omega \subseteq_{(\gamma,\delta)} \omega,$$
$$\mu \odot \nu \odot \omega \subseteq_{(\gamma,\delta)} \kappa_H^{\gamma,\delta} \odot \nu \odot \kappa_H^{\gamma,\delta} \subseteq_{(\gamma,\delta)} \nu, \quad (42)$$
$$\mu \odot \nu \odot \omega \subseteq_{(\gamma,\delta)} \left(\mu \odot \kappa_H^{\gamma,\delta}\right) \odot \kappa_H^{\gamma,\delta} \subseteq_{(\gamma,\delta)} \mu \odot \kappa_H^{\gamma,\delta} \subseteq_{(\gamma,\delta)} \mu.$$

Therefore, $\mu \odot \nu \odot \omega \subseteq_{(\gamma,\delta)} \mu \cap \nu \cap \omega$. Now let x be any element of H. Since H is both hyperregular and left duo, analogous to the proof of Theorem 43, there exist elements a, b, c, d, e, and f such that $c \in x \circ a \circ x$, $d \in a \circ b$, $e \in d \circ x$, $f \in c \circ e$, and $x \in f \circ x$. It follows from Remark 15 that $\max\{\mu(c), \gamma\} \geq \min\{\mu(x), \delta\}$ and $\max\{\nu(e), \gamma\} \geq \min\{\nu(x), \delta\}$. Thus, we have

$$\begin{aligned}
&\max\{(\mu \odot \nu \odot \omega)(x), \gamma\} \\
&\quad = \max\left\{\sup_{x \in \sum_{i=1}^{n} a_i \circ b_i} \min\{(\mu \odot \nu)(a_i), \omega(b_i)\}, \gamma\right\} \\
&\quad \geq \max\{\min\{(\mu \odot \nu)(f), \omega(x)\}, \gamma\} \\
&\quad = \max\left\{\sup_{f \in \sum_{j=1}^{m} c_j d_j} \min\{\mu(c_j), \nu(d_j), \omega(x)\}, \gamma\right\} \\
&\quad \geq \max\{\min\{\mu(c), \nu(e), \omega(x)\}, \gamma\} \\
&\quad = \min\{\max\{\mu(c), \gamma\}, \max\{\nu(e), \gamma\}, \max\{\omega(x), \gamma\}\} \\
&\quad \geq \min\{\min\{\mu(x), \delta\}, \min\{\nu(x), \delta\}, \max\{\omega(x), \gamma\}\} \\
&\quad = \min\{(\mu \cap \nu \cap \omega)(x), \delta\}.
\end{aligned} \quad (43)$$

This implies that $\mu \cap \nu \cap \omega \subseteq_{(\gamma,\delta)} \mu \odot \nu \odot \omega$. Therefore, $\mu \cap \nu \cap \omega \approx_{(\gamma,\delta)} \mu \odot \nu \odot \omega$.

(2)⇒(3) It is straightforward.

(3)⇒(1) Assume that (3) holds. Let μ and ν be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-hyperideal and any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left hyperideal of H, respectively. Since $\kappa_H^{\gamma,\delta}$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy hyperideal of H, then by the assumption, we have

$$\mu \cap \nu \approx_{(\gamma,\delta)} \mu \cap \kappa_H^{\gamma,\delta} \cap \nu \approx_{(\gamma,\delta)} \mu \odot \kappa_H^{\gamma,\delta} \odot \nu. \quad (44)$$

Then it follows from Theorem 44 that H is both hyperregular and left duo. □

5. Conclusions

In this paper, we investigate some new characterizations of some kinds of semihyperrings. Semihyperrings owe their importance to the fact that so many models arising in the solutions of specific problems turn out to be semihyperrings. For this reason, the basic concepts introduced here have exhibited some universality and are applicable in so many diverse contexts. These concepts are important and effective tools in (hyper)algebraic systems, automata, and artificial intelligence. Our future work on this topic will focus on studying intuitionistic (soft) or interval-valued fuzzy sets in semihyperrings and other algebraic constructions of semihyperrings.

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