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Research Article

Controllability Robustness of Linear Interval Systems with/without State Delay and with Unstructured Parametric Uncertainties

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The robust controllability problem for the linear interval systems with/without state delay and with unstructured parametric uncertainties is studied in this paper. The rank preservation problem is converted to the nonsingularity analysis problem of the minors of the matrix under discussion. Based on some essential properties of matrix measures, two new sufficient algebraically elegant criteria for the robust controllability of linear interval systems with/without state delay and with unstructured parametric uncertainties are established. Two numerical examples are given to illustrate the applications of the proposed sufficient algebraic criteria, where one example is also presented to show that the proposed sufficient condition for the linear interval systems having no state delay and no unstructured parametric uncertainties can obtain less conservative results than the existing ones reported recently in the literature.

1. Introduction

It is well known that time delay effect may occur naturally because of the inherent characteristics of some system components or part of the control process [1, 2]. In addition, the controllability is of particular importance in control theory and plays an important role in dynamic control systems [3, 4]. Then, the controllability problem of continuous linear time delay systems has been studied by some researchers (see, e.g., [2, 5–15]). On the other hand, the problems of controlling objects whose models contain interval uncertainties arise from the control theory, differential games, operations research, and other areas of engineering and natural sciences [16]. However, the results reported in the literature [2, 5–15] cannot be applied to solve the robust controllability problems of the linear interval systems with state delay.

For the time delay systems, there are two cases considered in the literature: (i) delay in state and (ii) delay in control input. The authors of this paper have studied the controllability problem of the *uncertain/interval* system *with delay in control input* [17–19], whereas the controllability problem of the *interval* system *with delay in state* is considered in this paper. Here it should be noticed that the controllability problem of the continuous linear systems with both parametric uncertainties and delay in state has been considered by Chen and Chou [20]. The same mathematical means as that used by Chen et al. [17, 18] and Chen and Chou [19] is used in this paper, but the rationale, formulation, and concept of analyzing controllability for the delay in state case are very different from those for the delay in control input case. On the other hand, here it should be also noticed that, in the works of Chen et al. [18] and Chen and Chou [19], all the elements in the interval system matrix and in the interval input matrices, respectively, are assumed to vary with both synchronous direction and same magnitude. So, the results of Chen et al. [18] and Chen and Chou [19] cannot be used to cover all matrices in the interval system.

Recently, the robustness issues of interval multiple-input-multiple-output (MIMO) systems without state delay have been studied by many researchers (see, e.g., [16, 17, 21–31] and references therein). But, till now, only a few researchers studied the controllability issue of the interval MIMO systems without state delay [16, 19–25, 31]. The approaches proposed by Zhirabok [24] and Ashchepkov [16, 25] need to consider the solvability of dynamic systems. Most notably, the methods proposed by Cheng and Zhang [21], Ahn et al. [22], Chen et al. [23] as well as Chen and Chou [19, 20, 31] give algebraically elegant derivations. However, the interval matrices considered by Cheng and Zhang [21] must satisfy the sign-invariant condition, and all the interval matrices considered by Chen and Chou [31] must have the same variations.

On the other hand, it is well known that an approximate system model is always used in practice, and sometimes the approximation error should be covered by introducing both structured (elemental) and unstructured (norm-bounded) uncertainties in control system analysis and design [32]. That is, it is not unusual that at times we have to deal with a system simultaneously consisting of two parts: one part has only the structured parameter perturbations and the other part has the unstructured parameter uncertainties. Here it should be noticed that the system with structured uncertainties may be viewed as a special case of the interval system [33–35]. To the authors' best knowledge, the robust controllability problem of linear interval systems with/without state delay and with unstructured parametric uncertainties has not been studied in the literature.

The purpose of this paper is to study the robust controllability problem of linear interval MIMO systems with/without state delay and with unstructured parametric uncertainties. Based on some essential properties of matrix measures, two new sufficient algebraic criteria are proposed to guarantee the controllability robustness of linear interval MIMO systems with/without state delay and with unstructured parametric uncertainties. The proposed approach gives the algebraically elegant derivations. Two numerical examples are given in this paper to illustrate the applications of the proposed sufficient algebraic criteria. And, for the linear interval systems without both state delay and unstructured parametric uncertainties, the result is also given to compare with those results obtained from the existing methods reported in the literature.

2. Linear Interval Systems with Both State Delay and Unstructured Uncertainties

Let $\underline{D} = \{\underline{d}_{ij}\}$ and $\overline{D} = \{\overline{d}_{ij}\}$ be real $\alpha \times \beta$ matrices satisfying $\underline{D} \le \overline{D}$, that is, $\underline{d}_{ij} \le \overline{d}_{ij}$, $i = 1, 2, \ldots, \alpha$ and $j = 1, 2, \ldots, \beta$. The set of matrices $[\underline{D}, \overline{D}] = \{D; \underline{D} \le D \le \overline{D}\}$ is called an interval matrix. Consider a linear interval MIMO system with both state delay and unstructured parametric uncertainties as the following form:

$$\begin{aligned}\dot{x}(t) = {} & Ax(t) + \widetilde{A}x(t) + Bx(t-\tau) \\ & + \widetilde{B}x(t-\tau) + Cu(t) + \widetilde{C}u(t),\end{aligned} \tag{1}$$

where $x(t) \in R^n$ is the system state vector, $u(t) \in R^m$ is the control input vector, $\tau > 0$ denotes the time delay, $A \in [\underline{A}, \overline{A}]$, $B \in [\underline{B}, \overline{B}]$, and $C \in [\underline{C}, \overline{C}]$ are, respectively, the $n \times n$, $n \times n$, and $n \times m$ interval matrices, and the unstructured parametric matrices $\widetilde{A}$, $\widetilde{B}$, and $\widetilde{C}$ are assumed to be bounded, that is,

$$\|\widetilde{A}\| \le \beta_1, \qquad \|\widetilde{B}\| \le \beta_2, \qquad \|\widetilde{C}\| \le \beta_3, \tag{2}$$

where β_1, β_2, and β_3 are nonnegative real constant numbers, and $\|\cdot\|$ denotes any matrix norm. Let $\widehat{\mathrm{B}}_a$ be the Banach space of real n-vector-valued continuous functions defined on the interval $[t_0 - \tau, t_0]$ with the uniform norm; that is, if $\Phi \in \widehat{\mathrm{B}}_a$, we have $\|\Phi\| = \max_{t \in [t_0-\tau, t_0]} |\Phi(t)|$. The initial function space is assumed to be $\widehat{\mathrm{B}}_a$, the space of continuous functions mapping $[t_0 - \tau, t_0]$ into R^n, and the R^m-valued control function $u(t)$ is measurable and bounded on every finite time interval [6]. The system in (1), called the linear interval MIMO system with both state delay and unstructured parametric uncertainties, is said to be controllable if each combination $(\widehat{A}, \widehat{B}, \widehat{C})$ is controllable, where $\widehat{A} = A + \widetilde{A}$, $\widehat{B} = B + \widetilde{B}$, $\widehat{C} = C + \widetilde{C}$, $A \in [\underline{A}, \overline{A}]$, $B \in [\underline{B}, \overline{B}]$, and $C \in [\underline{C}, \overline{C}]$.

For an interval matrix $[\underline{D}, \overline{D}]$ and for $\underline{d}_{ij} - d_{0ij} \le \varepsilon_{ij} \le \overline{d}_{ij} - d_{0ij}$, the $\alpha \times \beta$ matrix $D = D_0 + \sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} \varepsilon_{ij} D_{ij}$ denotes that it varies between $\underline{D}$ and $\overline{D}$, in which $\underline{D} = [\underline{d}_{ij}]$ and $\overline{D} = [\overline{d}_{ij}]$ are, respectively, the lower bound and upper bound matrices of interval matrix, D_{ij} is an $\alpha \times \beta$ constant matrix with 1 in the ijth entry and 0 elsewhere, and $D_0 = [d_{0ij}] \in [\underline{D}, \overline{D}]$ is any given constant matrix. Then, the interval matrices $[\underline{A}, \overline{A}]$, $[\underline{B}, \overline{B}]$, and $[\underline{C}, \overline{C}]$ can be written as

$$\begin{aligned}\left[\underline{A}, \overline{A}\right] &= A_0 + \sum_{i=1}^{n} \sum_{j=1}^{n} \varepsilon_{aij} A_{ij}, \\ \left[\underline{B}, \overline{B}\right] &= B_0 + \sum_{i=1}^{n} \sum_{j=1}^{n} \varepsilon_{bij} B_{ij}, \\ \left[\underline{C}, \overline{C}\right] &= C_0 + \sum_{j=1}^{n} \sum_{k=1}^{m} \varepsilon_{cjk} C_{jk},\end{aligned} \tag{3}$$

where $\underline{A} = [\underline{a}_{ij}]$, $\overline{A} = [\overline{a}_{ij}]$, $\underline{B} = [\underline{b}_{ij}]$, $\overline{B} = [\overline{b}_{ij}]$, $\underline{C} = [\underline{c}_{ij}]$, $\overline{C} = [\overline{c}_{ij}]$, A_{ij}, B_{ij}, and C_{jk} are, respectively, $n \times n$, $n \times n$, and $n \times m$ constant matrices with 1 in the ijth or jkth entry and 0 elsewhere (for $i, j = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, m$), $\underline{a}_{ij} - a_{0ij} \le \varepsilon_{aij} \le \overline{a}_{ij} - a_{0ij}$, $\underline{b}_{ij} - b_{0ij} \le \varepsilon_{bij} \le \overline{b}_{ij} - b_{0ij}$, and $\underline{c}_{jk} - c_{0jk} \le \varepsilon_{cjk} \le \overline{c}_{jk} - c_{0jk}$, and $A_0 = [a_{0ij}] \in [\underline{A}, \overline{A}]$, $B_0 = [b_{0ij}] \in [\underline{B}, \overline{B}]$, and $C_0 = [c_{0jk}] \in [\underline{C}, \overline{C}]$ are, respectively, any given $n \times n$, $n \times n$, and $n \times m$ constant matrices with that the combination (A_0, B_0, C_0) is controllable.

Before we investigate the property of robust controllability for the linear interval system with both state delay and unstructured parametric uncertainties of (1), the following definitions and lemmas need to be introduced first.

Definition 1 (see [6]). The system $\dot{x}(t) = Lx(t) + Mx(t-\tau) + Nu(t)$ with $t > t_0$ and any $\tau > 0$ is controllable to the origin from time t_0 if for each $\Phi \in \widehat{\mathrm{B}}_a$, there exists a finite time $t_1 > t_0$ and an admissible input $u(t)$ defined on $[t_0, t_1]$ such that $x(t_1, t_0, \Phi, u) = 0$, where $x(t_1, t_0, \Phi, u)$ denotes a solution to $\dot{x}(t) = Lx(t) + Mx(t-\tau) + Nu(t)$ at time t_1 corresponding to initial time t_0, initial function $\Phi \in \widehat{\mathrm{B}}_a$, and input $u(t)$, in which L, M, and N are, respectively, the $n\times n$, $n\times n$, and $n\times m$ matrices.

Definition 2 (see [36]). The measure of an $n \times n$ complex matrix $\overline{W}$ is defined as

$$\mu\left(\overline{W}\right) \equiv \lim_{\theta\to 0} \frac{\left(\left\|I + \theta\overline{W}\right\| - 1\right)}{\theta}, \tag{4}$$

where $\|\cdot\|$ is the induced matrix norm on the $n \times n$ complex matrix.

Lemma 3 (see [7, 8]). *If the system $\dot{x}(t) = (L+M)x(t) + Nu(t)$ with $t > t_0$ is controllable, then the linear time delay system $\dot{x}(t) = Lx(t) + Mx(t-\tau) + Nu(t)$ with $t > t_0$ is controllable in sense of Weiss [6] for any $\tau > 0$.*

Lemma 4. *For any $\tau > 0$, the linear time delay system $\dot{x}(t) = Lx(t) + Mx(t-\tau) + Nu(t)$ with $t > t_0$ is controllable in sense of Weiss [6] if the following $n^2 \times n(n+m-1)$ controllability matrix*

$$E = \begin{bmatrix} I_n & 0 & \cdots & 0 & 0 & \cdots & 0 & N \\ -(L+M) & I_n & \cdots & 0 & 0 & \cdots & N & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & I_n & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & -(L+M) & N & \cdots & 0 & 0 \end{bmatrix} \tag{5}$$

has rank n^2, *where $L, M \in R^{n\times n}$, $N \in R^{n\times m}$, and I_n denotes the $n \times n$ identity matrix.*

Proof. Following the same proof procedure as that given by Chen and Chou [20], in the above matrix E of (5), add $(L+M)$ times the first (block) row to the second, then add $(L+M)$ times the second row to the third, and so on. The result is a matrix

$$\left[\begin{array}{cccccc|ccccc} I_n & 0 & \cdot & \cdot & \cdot & 0 & 0 & \cdot & \cdot & 0 & N \\ 0 & I_n & \cdot & \cdot & \cdot & 0 & 0 & \cdot & \cdot & N & (L+M)N \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & I_n & 0 & \cdot & \cdot & \cdot & (L+M)^{n-2}N \\ \hline 0 & 0 & \cdot & \cdot & \cdot & 0 & N & (L+M)N & \cdot & \cdot & (L+M)^{n-1}N \end{array}\right]. \tag{6}$$

The controllability matrix $\left[N \;\; (L+M)N \;\; \cdots \;\; (L+M)^{n-1}N\right]$ is of rank n if and only if the matrix in (5) has rank n^2 (i.e., the matrix in (5) has rank n^2). So, from Lemma 3, we can conclude that if the matrix in (5) has rank n^2, then, for any $\tau > 0$, the linear time delay system $\dot{x}(t) = Lx(t) + Mx(t-\tau) + Nu(t)$ with $t > t_0$ is controllable in sense of Weiss [6]. $\square$

Remark 5. From Lemma 3, we know that the robust controllability problem of linear system with state delay can be converted to the rank preservation problem of controllability matrix. Due to the parametric uncertainties being interval matrices and unstructured uncertainties, it is difficult to calculate their matrix exponentiation and product operations for checking the rank of controllability matrix for linear time delay systems. To solve this difficulty, we can apply Lemma 4 to check the rank of controllability matrix in (5).

Lemma 6 (see [36]). *The matrix measures of the matrices $\overline{W}$ and $\overline{V}$, namely, $\mu(\overline{W})$ and $\mu(\overline{V})$, respectively, are well defined for any norm and have the following properties:*

(i) $\mu(\pm I) = \pm 1$, *for the identity matrix I;*

(ii) $-\|\overline{W}\| \le -\mu(-\overline{W}) \le \mathrm{Re}(\lambda(\overline{W})) \le \mu(\overline{W}) \le \|\overline{W}\|$, *for any norm $\|\cdot\|$ and any $n\times n$ complex matrix $\overline{W}$;*

(iii) $\mu(\overline{W} + \overline{V}) \le \mu(\overline{W}) + \mu(\overline{V})$, *for any two $n\times n$ complex matrices $\overline{W}$ and $\overline{V}$;*

(iv) $\mu(\gamma\overline{W}) = \gamma\mu(\overline{W})$, *for any matrix $\overline{W} \in C^{n\times n}$ and any nonnegative real number γ;*

where $\lambda(\overline{W})$ denotes any eigenvalue of $\overline{W}$ and $\mathrm{Re}(\lambda(\overline{W}))$ *denotes the real part of $\lambda(\overline{W})$.*

While the induced matrix norms are 1-norm, 2-norm, and ∞-norm, the corresponding matrix measures $\mu_k(\cdot)$, where $k = 1, 2, \infty$, can be easily calculated as

(i) $\mu_1(\overline{W}) = \max_j[\mathrm{Re}(\overline{w}_{jj}) + \sum_{i=1, i\ne j}^{n} |\overline{w}_{ij}|]$;

(ii) $\mu_2(\overline{W}) = \max_i[\lambda_i(\overline{W} + \overline{W}^*)/2]$;

(iii) $\mu_\infty(\overline{W}) = \max_i[\mathrm{Re}(\overline{w}_{ii}) + \sum_{j=1, j\ne i}^{n} |\overline{w}_{ij}|]$;

in which $\overline{w}_{ij}$ is the ijth element of the matrix $\overline{W}$ and $\lambda_i(\cdot)$ denotes the ith eigenvalue.

Lemma 7. *For any $\gamma < 0$ and any $n \times n$ complex matrix W, $\mu(\gamma W) = -\gamma\mu(-W)$.*

Proof. From the property (iv) in Lemma 8, this lemma can be immediately obtained. $\square$

Lemma 8 (see [37]). *Let $\overline{N} \in C^{n\times n}$. If $\mu(-\overline{N}) < 1$, then* $\det(I + \overline{N}) \ne 0$.

From Lemma 4, it is known that, for any $\tau > 0$, the interval system with both state delay and unstructured parametric uncertainties in (1) is robustly controllable on $[0, T]$ in sense of Weiss [6] if the $n^2 \times n(n+m-1)$ matrix E has full row rank n^2, where

$$\begin{aligned} E = E_0 &+ \sum_{i=1}^{n}\sum_{j=1}^{n} \varepsilon_{aij} F_{ij} + \sum_{i=1}^{n}\sum_{j=1}^{n} \varepsilon_{bij} G_{ij} \\ &+ \sum_{j=1}^{n}\sum_{k=1}^{m} \varepsilon_{cjk} H_{jk} + \widetilde{F} + \widetilde{G} + \widetilde{H}, \end{aligned} \tag{7}$$

for $\underline{a}_{ij} - a_{0ij} \le \varepsilon_{aij} \le \overline{a}_{ij} - a_{0ij}$, $\underline{b}_{ij} - b_{0ij} \le \varepsilon_{bij} \le \overline{b}_{ij} - b_{0ij}$, $\underline{c}_{jk} - c_{0jk} \le \varepsilon_{cjk} \le \overline{c}_{jk} - c_{0jk}$, $i, j = 1, 2, \ldots, n$, and $k = 1, 2, \ldots, m$, in which

$$E_0 = \begin{bmatrix} I_n & 0 & \cdots & 0 & 0 & \cdots & 0 & C_0 \\ -(A_0 + B_0) & I_n & \cdots & 0 & 0 & \cdots & C_0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & I_n & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & -(A_0 + B_0) & C_0 & \cdots & 0 & 0 \end{bmatrix}, \tag{8}$$

$$F_{ij} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ -A_{ij} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & -A_{ij} & 0 & \cdots & 0 & 0 \end{bmatrix}, \tag{9}$$

$$G_{ij} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ -B_{ij} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & -B_{ij} & 0 & \cdots & 0 & 0 \end{bmatrix}, \tag{10}$$

$$H_{jk} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & C_{jk} \\ 0 & 0 & \cdots & 0 & 0 & \cdots & C_{jk} & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & C_{jk} & \cdots & 0 & 0 \end{bmatrix}, \tag{11}$$

$$\widetilde{F} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ -\widetilde{A} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & -\widetilde{A} & 0 & \cdots & 0 & 0 \end{bmatrix}, \tag{12}$$

$$\widetilde{G} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ -\widetilde{B} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & -\widetilde{B} & 0 & \cdots & 0 & 0 \end{bmatrix}, \tag{13}$$

$$\widetilde{H} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \widetilde{C} \\ 0 & 0 & \cdots & 0 & 0 & \cdots & \widetilde{C} & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & \widetilde{C} & \cdots & 0 & 0 \end{bmatrix}. \tag{14}$$

Let the singular value decomposition of E_0, which has rank n^2 due to that the given combination (A_0, B_0, C_0) is controllable, be

$$E_0 = U \left[S \ \ 0_{n^2 \times n(m-1)} \right] V^H, \tag{15}$$

where $U \in R^{n^2 \times n^2}$ and $V \in R^{n(n+m-1) \times n(n+m-1)}$ are the unitary matrices, V^H denotes the complex-conjugate transpose of matrix V, $S = \operatorname{diag}[\sigma_1, \ldots, \sigma_{n^2}]$, and $\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_{n^2} > 0$ are the singular values of E_0.

In what follows, we present a sufficient criterion for ensuring that, for any $\tau > 0$, the linear interval system with both state delay and unstructured parametric uncertainties of (1) is robustly controllable on $[0, T]$ in sense of Weiss [6].

Theorem 9. *For any $\tau > 0$, the linear interval MIMO system with both state delay and unstructured parametric uncertainties of* (1) *is robustly controllable in sense of Weiss [6] if the matrix E_0 in* (8) *has a full row rank, and if the following conditions simultaneously hold:*

$$\begin{aligned} &\sum_{i=1}^{n} \sum_{j=1}^{n} \varepsilon_{aij} \phi_{ij} + \sum_{i=1}^{n} \sum_{j=1}^{n} \varepsilon_{bij} \varphi_{ij} \\ &\quad + \sum_{j=1}^{n} \sum_{k=1}^{m} \varepsilon_{cjk} \theta_{jk} + \overline{\beta}_1 + \overline{\beta}_2 + \overline{\beta}_3 < 1, \end{aligned} \tag{16}$$

where

$$\phi_{ij} = \begin{cases} \mu\left(-S^{-1} U^H F_{ij} V \left[I_{n^2}, 0_{n^2 \times n(m-1)}\right]^T\right), & \textit{for } \varepsilon_{aij} \ge 0, \\ -\mu\left(S^{-1} U^H F_{ij} V \left[I_{n^2}, 0_{n^2 \times n(m-1)}\right]^T\right), & \textit{for } \varepsilon_{aij} < 0, \end{cases} \tag{17a}$$

$$\varphi_{ij} = \begin{cases} \mu\left(-S^{-1} U^H G_{ij} V \left[I_{n^2}, 0_{n^2 \times n(m-1)}\right]^T\right), & \textit{for } \varepsilon_{bij} \ge 0, \\ -\mu\left(S^{-1} U^H G_{ij} V \left[I_{n^2}, 0_{n^2 \times n(m-1)}\right]^T\right), & \textit{for } \varepsilon_{bij} < 0, \end{cases} \tag{17b}$$

$$\theta_{jk} = \begin{cases} \mu\left(-S^{-1} U^H H_{jk} V \left[I_{n^2}, 0_{n^2 \times n(m-1)}\right]^T\right), & \textit{for } \varepsilon_{cjk} \ge 0, \\ -\mu\left(S^{-1} U^H H_{jk} V \left[I_{n^2}, 0_{n^2 \times n(m-1)}\right]^T\right), & \textit{for } \varepsilon_{cjk} < 0, \end{cases} \tag{17c}$$

$$\overline{\beta}_1 = \beta_1 \left\| S^{-1} U^H \right\| \left\| V \left[I_{n^2}, 0_{n^2 \times n(m-1)}\right]^T \right\|, \tag{17d}$$

$$\overline{\beta}_2 = \beta_2 \left\| S^{-1} U^H \right\| \left\| V \left[I_{n^2}, 0_{n^2 \times n(m-1)}\right]^T \right\|, \tag{17e}$$

$$\overline{\beta}_3 = \beta_3 \left\| S^{-1} U^H \right\| \left\| V \left[I_{n^2}, 0_{n^2 \times n(m-1)}\right]^T \right\|, \tag{17f}$$

$i, j = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, m$, in which the matrices F_{ij}, G_{ij}, H_{jk}, S, U, and V are, respectively, defined in (9)–(11) *and* (15) *and I_{n^2} denotes the $n^2 \times n^2$ identity matrix.*

Proof. If the matrix E in (7) has full row rank, then the linear interval system with both state delay and unstructured parametric uncertainties of (1) is robustly controllable. Since the matrix E_0 in (8) has a full row rank due to that the given combination (A_0, B_0, C_0) is controllable, and since we know that

$$\operatorname{rank}(E_0) = \operatorname{rank}\left(S^{-1}U^H E_0 V\right), \tag{18}$$

thus, instead of rank $(E_0+\sum_{i=1}^n \sum_{j=1}^n \varepsilon_{aij}F_{ij}+\sum_{i=1}^n \sum_{j=1}^n \varepsilon_{bij}G_{ij}+\sum_{j=1}^n \sum_{k=1}^m \varepsilon_{cjk}H_{jk} + \widetilde{F} + \widetilde{G} + \widetilde{H})$, we can discuss the rank of

$$\begin{aligned}&\left[I_{n^2} \quad 0_{n^2\times n(m-1)}\right] + \sum_{i=1}^n \sum_{j=1}^n \varepsilon_{aij} S^{-1}U^H F_{ij} V \\ &\quad + \sum_{i=1}^n \sum_{j=1}^n \varepsilon_{bij} S^{-1}U^H G_{ij} V + \sum_{j=1}^n \sum_{k=1}^m \varepsilon_{cjk} S^{-1}U^H H_{jk} V \\ &\quad + S^{-1}U^H\widetilde{F}V + S^{-1}U^H\widetilde{G}V + S^{-1}U^H\widetilde{H}V.\end{aligned} \tag{19}$$

Since a matrix has at least rank n^2 if it has at least one nonsingular $n^2 \times n^2$ submatrix, a sufficient condition for the matrix in (19) to have rank n^2 is the nonsingularity of

$$\begin{aligned}\overline{L} = I_{n^2} &+ \sum_{i=1}^n \sum_{j=1}^n \varepsilon_{aij}\overline{F}_{ij} + \sum_{i=1}^n \sum_{j=1}^n \varepsilon_{bij}\overline{G}_{ij} \\ &+ \sum_{j=1}^n \sum_{k=1}^m \varepsilon_{cjk}\overline{H}_{jk} + \overline{F} + \overline{G} + \overline{H},\end{aligned} \tag{20}$$

where $\overline{F}_{ij} = S^{-1}U^H F_{ij} V[I_{n^2},\ 0_{n^2\times n(m-1)}]^T$, $\overline{G}_{ij} = S^{-1}U^H G_{ij} V[I_{n^2},\ 0_{n^2\times n(m-1)}]^T$, $\overline{H}_{jk} = S^{-1}U^H H_{jk} V[I_{n^2},\ 0_{n^2\times n(m-1)}]^T$, $\overline{F} = S^{-1}U^H \widetilde{F} V[I_{n^2},\ 0_{n^2\times n(m-1)}]^T$, $\overline{G} = S^{-1}U^H \widetilde{G} V[I_{n^2},\ 0_{n^2\times n(m-1)}]^T$, and $\overline{H} = S^{-1}U^H \widetilde{H} V[I_{n^2},\ 0_{n^2\times n(m-1)}]^T$, for $i, j = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, m$.

Using the properties in Lemmas 6 and 7, and from (2) and (16), we have

$$\begin{aligned}&\mu\left(-\sum_{i=1}^n \sum_{j=1}^n \varepsilon_{aij}\overline{F}_{ij} - \sum_{i=1}^n \sum_{j=1}^n \varepsilon_{bij}\overline{G}_{ij} - \sum_{j=1}^n \sum_{k=1}^m \varepsilon_{cjk}\overline{H}_{jk} - \overline{F} - \overline{G} - \overline{H}\right) \\ &\quad \le \mu\left(-\sum_{i=1}^n \sum_{j=1}^n \varepsilon_{aij}\overline{F}_{ij}\right) + \mu\left(-\sum_{i=1}^n \sum_{j=1}^n \varepsilon_{bij}\overline{G}_{ij}\right) \\ &\qquad + \mu\left(-\sum_{j=1}^n \sum_{k=1}^m \varepsilon_{cjk}\overline{H}_{jk}\right) + \mu\left(-\overline{F}\right) + \mu\left(-\overline{G}\right) + \mu\left(-\overline{H}\right) \\ &\quad \le \mu\left(-\sum_{i=1}^n \sum_{j=1}^n \varepsilon_{aij}\overline{F}_{ij}\right) + \mu\left(-\sum_{i=1}^n \sum_{j=1}^n \varepsilon_{bij}\overline{G}_{ij}\right) \\ &\qquad + \mu\left(-\sum_{j=1}^n \sum_{k=1}^m \varepsilon_{cjk}\overline{H}_{jk}\right) + \left\|\overline{F}\right\| + \left\|\overline{G}\right\| + \left\|\overline{H}\right\| \\ &\quad \le \sum_{i=1}^n \sum_{j=1}^n \mu\left(-\varepsilon_{aij}\overline{F}_{ij}\right) + \sum_{i=1}^n \sum_{j=1}^n \mu\left(-\varepsilon_{bij}\overline{G}_{ij}\right) \\ &\qquad + \sum_{j=1}^n \sum_{k=1}^m \mu\left(-\varepsilon_{cjk}\overline{H}_{jk}\right) + \left\|\overline{F}\right\| + \left\|\overline{G}\right\| + \left\|\overline{H}\right\| \\ &\quad \le \sum_{i=1}^n \sum_{j=1}^n \varepsilon_{aij}\phi_{ij} + \sum_{i=1}^n \sum_{j=1}^n \varepsilon_{bij}\varphi_{ij} \\ &\qquad + \sum_{j=1}^n \sum_{k=1}^m \varepsilon_{cjk}\theta_{jk} + \overline{\beta}_1 + \overline{\beta}_2 + \overline{\beta}_3 < 1.\end{aligned} \tag{21}$$

Thus, from Lemma 8, we have

$$\begin{aligned}\det\left(\overline{L}\right) = \det\Bigg(I_{n^2} &+ \sum_{i=1}^n \sum_{j=1}^n \varepsilon_{aij}\overline{F}_{ij} + \sum_{i=1}^n \sum_{j=1}^n \varepsilon_{bij}\overline{G}_{ij} \\ &+ \sum_{j=1}^n \sum_{k=1}^m \varepsilon_{cjk}\overline{H}_{jk} + \overline{F} + \overline{G} + \overline{H}\Bigg) \neq 0.\end{aligned} \tag{22}$$

Hence, the matrix $\overline{L}$ in (20) is nonsingular. That is, the matrix $E = E_0 + \sum_{i=1}^n \sum_{j=1}^n \varepsilon_{aij}F_{ij} + \sum_{i=1}^n \sum_{j=1}^n \varepsilon_{bij}G_{ij} + \sum_{j=1}^n \sum_{k=1}^m \varepsilon_{cjk}H_{ik} + \widetilde{F} + \widetilde{G} + \widetilde{H}$, for $i, j = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, m$, has full row rank n^2. So, from the results mentioned previously and Lemma 4, it is ensured that, for any $\tau > 0$, the linear interval MIMO system with both state delay and structured parametric uncertainties of (1) is robustly controllable in sense of Weiss [6]. □

3. Linear Interval Systems without State Delay

In this section, we consider a linear interval MIMO system with unstructured parametric uncertainties having the form of

$$\dot{x}(t) = Ax(t) + \widetilde{A}x(t) + Cu(t) + \widetilde{C}u(t). \tag{23}$$

The system in (23), called the linear interval MIMO system with unstructured parametric uncertainties, is said to be controllable if each pair $(\widehat{A}, \widehat{C})$ is controllable, where $\widehat{A} = A + \widetilde{A}$, $\widehat{C} = C + \widetilde{C}$, $A \in [\underline{A}, \overline{A}]$, and $C \in [\underline{C}, \overline{C}]$.

Following the same proof procedure given in the aforementioned theorem, we can get the following corollary to ensure that the linear interval system with unstructured parametric uncertainties is robustly controllable if the matrix $\overline{M}$ has full row rank n^2, where $\overline{M} = M_0 + \sum_{i=1}^n \sum_{j=1}^n \varepsilon_{aij}F_{ij} + \sum_{j=1}^n \sum_{k=1}^m \varepsilon_{cjk}H_{jk} + \widetilde{F} + \widetilde{H}$, and

$$M_0 = \begin{bmatrix} I_n & 0 & \cdots & 0 & 0 & \cdots & 0 & C_0 \\ -A_0 & I_n & \cdots & 0 & 0 & \cdots & C_0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & I_n & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & -A_0 & C_0 & \cdots & 0 & 0 \end{bmatrix}, \tag{24}$$

in which $\underline{a}_{ij}-a_{0ij} \le \varepsilon_{aij} \le \overline{a}_{ij}-a_{0ij}$, $\underline{c}_{jk}-c_{0jk} \le \varepsilon_{cjk} \le \overline{c}_{jk}-c_{0jk}$, for $i, j = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, m$. $A_0 \in [\underline{A}, \overline{A}]$ and $C_0 \in [\underline{C}, \overline{C}]$ are given constant matrices, and the pair (A_0, C_0) is controllable.

Let the singular value decomposition of M_0, which has rank n^2 due to that the given pair (A_0, C_0) is controllable, be

$$M_0 = \widetilde{U}\left[\widetilde{S} \;\; 0_{n^2\times n(m-1)}\right]\widetilde{V}^H, \tag{25}$$

where $\widetilde{U} \in R^{n^2\times n^2}$ and $\widetilde{V} \in R^{n(n+m-1)\times n(n+m-1)}$ are the unitary matrices, $\widetilde{V}^H$ denotes the complex-conjugate transpose of matrix $\widetilde{V}$, $\widetilde{S} = \operatorname{diag}[\widetilde{\sigma}_1, \ldots, \widetilde{\sigma}_{n^2}]$, and $\widetilde{\sigma}_1 \ge \widetilde{\sigma}_2 \ge \cdots \ge \widetilde{\sigma}_{n^2} > 0$ are the singular values of M_0.

Corollary 10. *The linear interval MIMO system with unstructured parametric uncertainties in* (23) *is robustly controllable, if the given pair* (A_0, C_0) *is controllable, and if the following conditions simultaneously hold:*

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\varepsilon_{aij}\widetilde{\phi}_{ij} + \sum_{j=1}^{n}\sum_{k=1}^{m}\varepsilon_{cjk}\widetilde{\theta}_{jk} + \widetilde{\beta}_1 + \widetilde{\beta}_3 < 1, \tag{26}$$

where

$$\widetilde{\phi}_{ij} = \begin{cases}\mu\left(-\widetilde{S}^{-1}\widetilde{U}^H F_{ij}\widetilde{V}[I_{n^2}, 0_{n^2\times n(m-1)}]^T\right), & \textit{for } \varepsilon_{aij} \ge 0,\\ -\mu\left(\widetilde{S}^{-1}\widetilde{U}^H F_{ij}\widetilde{V}[I_{n^2}, 0_{n^2\times n(m-1)}]^T\right), & \textit{for } \varepsilon_{aij} < 0,\end{cases} \tag{27a}$$

$$\widetilde{\theta}_{jk} = \begin{cases}\mu\left(-\widetilde{S}^{-1}\widetilde{U}^H H_{jk}\widetilde{V}[I_{n^2}, 0_{n^2\times n(m-1)}]^T\right), & \textit{for } \varepsilon_{cjk} \ge 0,\\ -\mu\left(\widetilde{S}^{-1}\widetilde{U}^H H_{jk}\widetilde{V}[I_{n^2}, 0_{n^2\times n(m-1)}]^T\right), & \textit{for } \varepsilon_{cjk} < 0,\end{cases} \tag{27b}$$

$$\widetilde{\beta}_1 = \beta_1 \left\|\widetilde{S}^{-1}\widetilde{U}^H\right\| \left\|\widetilde{V}[I_{n^2}, 0_{n^2\times n(m-1)}]^T\right\|, \tag{27c}$$

$$\widetilde{\beta}_3 = \beta_3 \left\|\widetilde{S}^{-1}\widetilde{U}^H\right\| \left\|\widetilde{V}[I_{n^2}, 0_{n^2\times n(m-1)}]^T\right\|, \tag{27d}$$

$i, j = 1, 2, \ldots, n$ *and* $k = 1, 2, \ldots, m$, *in which the matrices* F_{ij}, H_{jk}, $\widetilde{S}$, $\widetilde{U}$, *and* $\widetilde{V}$ *are, respectively, defined in* (9), (11), *and* (25) *and* I_{n^2} *denotes the* $n^2 \times n^2$ *identity matrix.*

Proof. The proof procedure is the same as that of the aforementioned Theorem, hence omitted here. □

Remark 11. If we only consider the robust controllability problem of linear interval systems with/without state delay (i.e., $\widetilde{A} = 0$, $\widetilde{B} = 0$, and $\widetilde{C} = 0$), the sufficient condition in (16) or (26) can be, respectively, simplified as

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\varepsilon_{aij}\phi_{ij} + \sum_{i=1}^{n}\sum_{j=1}^{n}\varepsilon_{bij}\varphi_{ij} + \sum_{j=1}^{n}\sum_{k=1}^{m}\varepsilon_{cjk}\theta_{jk} < 1, \tag{28}$$

or

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\varepsilon_{aij}\widetilde{\phi}_{ij} + \sum_{j=1}^{n}\sum_{k=1}^{m}\varepsilon_{cjk}\widetilde{\theta}_{jk} < 1. \tag{29}$$

Furthermore, it can be found that the result given by Chen and Chou [20] can be viewed as a special case of (16) and (28). On the other hand, by applying the evolutionary optimization methods [38] to choose the best matrices A_0, B_0, and C_0, the conservatism of (16), (26), (28), and (29) can be reduced.

4. Illustrative Examples

In this section, two numerical examples are given to illustrate the applications of the proposed sufficient algebraic criteria. We will also compare the conservatism of the proposed sufficient condition for linear interval system having no state delay and no unstructured parametric uncertainties with those results reported recently in the literature.

Example 1. Consider a linear interval system having no state delay and no unstructured parametric uncertainties as

$$\dot{x}(t) = Ax(t) + Cu(t), \tag{30}$$

with

$$A = \begin{bmatrix} 1^{+0.4}_{-0.2} & 0 & 0\\ 0 & 1^{+0.1}_{-0.1} & 1\\ 0 & -2 & 4^{+0.6}_{-0.4}\end{bmatrix}, \qquad C = \begin{bmatrix} 1^{+0.6}_{-0.3} & 0\\ 0 & 0\\ 0 & 1\end{bmatrix}, \tag{31}$$

which is slightly modified from the example given by Chen et al. [23].

Letting

$$A_0 = \begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 1\\ 0 & -2 & 4\end{bmatrix}, \qquad C_0 = \begin{bmatrix} 1 & 0\\ 0 & 0\\ 0 & 1\end{bmatrix}, \tag{32}$$

then the interval matrices A and C can be represented as $A = A_0 + \varepsilon_{a11}A_{11} + \varepsilon_{a22}A_{22} + \varepsilon_{a33}A_{33}$ and $C = C_0 + \varepsilon_{c11}C_{11}$ with $A_{11} = \left[\begin{smallmatrix}1&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right]$, $A_{22} = \left[\begin{smallmatrix}0&0&0\\0&1&0\\0&0&0\end{smallmatrix}\right]$, $A_{33} = \left[\begin{smallmatrix}0&0&0\\0&0&0\\0&0&1\end{smallmatrix}\right]$, $C_{11} = \left[\begin{smallmatrix}1&0\\0&0\\0&0\end{smallmatrix}\right]$, $\varepsilon_{a11} \in [-0.2,\ 0.4]$, $\varepsilon_{a22} \in [-0.1,\ 0.1]$, $\varepsilon_{a33} \in [-0.4,\ 0.6]$, and $\varepsilon_{c11} \in [-0.3,\ 0.6]$.

First, following the approach of Cheng and Zhang [21], we have

$$Q_0 = [C_0 \;\; A_0C_0] = \begin{bmatrix} 1 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 1 & 0 & 4\end{bmatrix},$$
$$Q_\delta = [|C_0| \;\; |A_0||C_0|] = \begin{bmatrix} 1 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 1 & 0 & 4\end{bmatrix}, \tag{33}$$

$\lambda_{\min}(Q_0{Q_0}^T) = 0.055728$, $\lambda_{\max}(Q_\delta{Q_\delta}^T) = 17.944$, $\delta = 0.027486$, $|\delta^0_{\mathrm{ij}}| \nless \delta$, and $|\sigma^0_{ij}| \nless \delta$ for all i and j, where δ, δ^0_{ij}, and σ^0_{ij} are detailedly defined in the work of Cheng and Zhang [21]. Hence, the condition of Cheng and Zhang [21] is not satisfied. Then, no conclusion can be made. That is, the sufficient condition of Cheng and Zhang [21] cannot be applied in this example.

Next, we use the method proposed by Ahn et al. [22] and Chen et al. [23] to test the controllability. The controllability matrix D is calculated as

$$D = \begin{bmatrix} 1_{-0.3}^{+0.6} & 0 & 1_{-0.44}^{+1.24} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 4_{-0.3}^{+0.6} \end{bmatrix}. \quad (34)$$

So, we have four subsquare matrices:

$$S^1 = \begin{bmatrix} 1_{-0.3}^{+0.6} & 0 & 1_{-0.44}^{+1.24} \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$
$$S^2 = \begin{bmatrix} 1_{-0.3}^{+0.6} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 4_{-0.3}^{+0.6} \end{bmatrix}, \quad (35)$$
$$S^3 = \begin{bmatrix} 1_{-0.3}^{+0.6} & 1_{-0.44}^{+1.24} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 4_{-0.3}^{+0.6} \end{bmatrix},$$

and $S^4 = \begin{bmatrix} 0 & 1_{-0.44}^{+1.24} & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 4_{-0.3}^{+0.6} \end{bmatrix}$. From S^2 and S^4, it can be obtained that S_0^2 and S_0^4 are nonsingular and the spectral radiuses $\rho(\|(S_0^2)^{-1}\|\Delta S^2) = 2.5416 \nless 1$ and $\rho(\|(S_0^4)^{-1}\|\Delta S^4) = 2.5416 \nless 1$; thus, the condition of Ahn et al. [22] and Chen et al. [23] is not satisfied either. Then, no conclusion can be made. That is, the sufficient condition of Ahn et al. [22] and Chen et al. [23] cannot be applied in this example. On the other hand, all the elements in the interval matrices of (30) can independently vary with different direction and magnitude. That is, the interval matrices in (30) do not satisfy the assumption that all the elements in the interval matrices must have the same variations. So, the result of Chen and Chou [19, 20, 31] cannot be used in this example.

Now, applying the sufficient criterion in the proposed condition of (29) with 2-norm-based matrix measure for the robust controllability, we have $\operatorname{rank}(M_0) = 9$, and we get

(i) $\varepsilon_{a11}\tilde{\phi}_{11} + \varepsilon_{a22}\tilde{\phi}_{22} + \varepsilon_{a33}\tilde{\phi}_{33} + \varepsilon_{c11}\tilde{\theta}_{11} \le 0.568385 < 1$, for

$$\varepsilon_{a11} \in [0,\ 0.4], \quad \varepsilon_{a22} \in [0,\ 0.1], \quad \varepsilon_{a33} \in [0,\ 0.6], \quad \varepsilon_{c11} \in [0,\ 0.6]; \quad (36a)$$

(ii) $\varepsilon_{a11}\tilde{\phi}_{11} + \varepsilon_{a22}\tilde{\phi}_{22} + \varepsilon_{a33}\tilde{\phi}_{33} + \varepsilon_{c11}\tilde{\theta}_{11} \le 0.86838 < 1$, for

$$\varepsilon_{a11} \in [0,\ 0.4], \quad \varepsilon_{a22} \in [0,\ 0.1], \quad \varepsilon_{a33} \in [0,\ 0.6], \quad \varepsilon_{c11} \in [-0.3,\ 0]; \quad (36b)$$

(iii) $\varepsilon_{a11}\tilde{\phi}_{11} + \varepsilon_{a22}\tilde{\phi}_{22} + \varepsilon_{a33}\tilde{\phi}_{33} + \varepsilon_{c11}\tilde{\theta}_{11} \le 0.76838 < 1$, for

$$\varepsilon_{a11} \in [0,\ 0.4], \quad \varepsilon_{a22} \in [0,\ 0.1], \quad \varepsilon_{a33} \in [-0.4,\ 0], \quad \varepsilon_{c11} \in [-0.3,\ 0]; \quad (36c)$$

(iv) $\varepsilon_{a11}\tilde{\phi}_{11} + \varepsilon_{a22}\tilde{\phi}_{22} + \varepsilon_{a33}\tilde{\phi}_{33} + \varepsilon_{c11}\tilde{\theta}_{11} \le 0.46838 < 1$, for

$$\varepsilon_{a11} \in [0,\ 0.4], \quad \varepsilon_{a22} \in [0,\ 0.1], \quad \varepsilon_{a33} \in [-0.4,\ 0], \quad \varepsilon_{c11} \in [0,\ 0.6]; \quad (36d)$$

(v) $\varepsilon_{a11}\tilde{\phi}_{11} + \varepsilon_{a22}\tilde{\phi}_{22} + \varepsilon_{a33}\tilde{\phi}_{33} + \varepsilon_{c11}\tilde{\theta}_{11} \le 0.56588 < 1$, for

$$\varepsilon_{a11} \in [0,\ 0.4], \quad \varepsilon_{a22} \in [-0.1,\ 0], \quad \varepsilon_{a33} \in [0,\ 0.6], \quad \varepsilon_{c11} \in [0,\ 0.6]; \quad (36e)$$

(vi) $\varepsilon_{a11}\tilde{\phi}_{11} + \varepsilon_{a22}\tilde{\phi}_{22} + \varepsilon_{a33}\tilde{\phi}_{33} + \varepsilon_{c11}\tilde{\theta}_{11} \le 0.86588 < 1$, for

$$\varepsilon_{a11} \in [0,\ 0.4], \quad \varepsilon_{a22} \in [-0.1,\ 0], \quad \varepsilon_{a33} \in [0,\ 0.6], \quad \varepsilon_{c11} \in [-0.3,\ 0]; \quad (36f)$$

(vii) $\varepsilon_{a11}\tilde{\phi}_{11} + \varepsilon_{a22}\tilde{\phi}_{22} + \varepsilon_{a33}\tilde{\phi}_{33} + \varepsilon_{c11}\tilde{\theta}_{11} \le 0.76588 < 1$, for

$$\varepsilon_{a11} \in [0,\ 0.4], \quad \varepsilon_{a22} \in [-0.1,\ 0], \quad \varepsilon_{a33} \in [-0.4,\ 0], \quad \varepsilon_{c11} \in [-0.3,\ 0]; \quad (36g)$$

(viii) $\varepsilon_{a11}\tilde{\phi}_{11} + \varepsilon_{a22}\tilde{\phi}_{22} + \varepsilon_{a33}\tilde{\phi}_{33} + \varepsilon_{c11}\tilde{\theta}_{11} \le 0.46588 < 1$, for

$$\varepsilon_{a11} \in [0,\ 0.4], \quad \varepsilon_{a22} \in [-0.1,\ 0], \quad \varepsilon_{a33} \in [-0.4,\ 0], \quad \varepsilon_{c11} \in [0,\ 0.6]; \quad (36h)$$

(ix) $\varepsilon_{a11}\tilde{\phi}_{11} + \varepsilon_{a22}\tilde{\phi}_{22} + \varepsilon_{a33}\tilde{\phi}_{33} + \varepsilon_{c11}\tilde{\theta}_{11} \le 0.59031 < 1$, for

$$\varepsilon_{a11} \in [-0.2,\ 0], \quad \varepsilon_{a22} \in [0,\ 0.1], \quad \varepsilon_{a33} \in [0,\ 0.6], \quad \varepsilon_{c11} \in [0,\ 0.6]; \quad (36i)$$

(x) $\varepsilon_{a11}\tilde{\phi}_{11} + \varepsilon_{a22}\tilde{\phi}_{22} + \varepsilon_{a33}\tilde{\phi}_{33} + \varepsilon_{c11}\tilde{\theta}_{11} \le 0.89031 < 1$, for

$$\varepsilon_{a11} \in [0,\ 0.4], \quad \varepsilon_{a22} \in [0,\ 0.1], \quad \varepsilon_{a33} \in [0,\ 0.6], \quad \varepsilon_{c11} \in [-0.3,\ 0]; \quad (36j)$$

(xi) $\varepsilon_{a11}\tilde{\phi}_{11} + \varepsilon_{a22}\tilde{\phi}_{22} + \varepsilon_{a33}\tilde{\phi}_{33} + \varepsilon_{c11}\tilde{\theta}_{11} \le 0.790315 < 1$, for

$$\varepsilon_{a11} \in [0,\ 0.4], \quad \varepsilon_{a22} \in [0,\ 0.1], \quad \varepsilon_{a33} \in [-0.4,\ 0], \quad \varepsilon_{c11} \in [-0.3,\ 0]; \quad (36k)$$

(xii) $\varepsilon_{a11}\tilde{\phi}_{11} + \varepsilon_{a22}\tilde{\phi}_{22} + \varepsilon_{a33}\tilde{\phi}_{33} + \varepsilon_{c11}\tilde{\theta}_{11} \le 0.49031 < 1$, for

$$\varepsilon_{a11} \in [0,\ 0.4], \quad \varepsilon_{a22} \in [0,\ 0.1], \quad \varepsilon_{a33} \in [-0.4,\ 0], \quad \varepsilon_{c11} \in [0,\ 0.6]; \quad (36l)$$

(xiii) $\varepsilon_{a11}\tilde{\phi}_{11} + \varepsilon_{a22}\tilde{\phi}_{22} + \varepsilon_{a33}\tilde{\phi}_{33} + \varepsilon_{c11}\tilde{\theta}_{11} \le 0.587806 < 1$, for

$$\varepsilon_{a11} \in [0,\ 0.4], \quad \varepsilon_{a22} \in [-0.1,\ 0], \quad \varepsilon_{a33} \in [0,\ 0.6], \quad \varepsilon_{c11} \in [0,\ 0.6]; \quad (36m)$$

(xiv) $\varepsilon_{a11}\widetilde{\phi}_{11}+\varepsilon_{a22}\widetilde{\phi}_{22}+\varepsilon_{a33}\widetilde{\phi}_{33}+\varepsilon_{c11}\widetilde{\theta}_{11}\leq 0.88780<1$, for

$$\varepsilon_{a11}\in[0,\ 0.4],\qquad \varepsilon_{a22}\in[-0.1,\ 0],$$
$$\varepsilon_{a33}\in[0,\ 0.6],\qquad \varepsilon_{c11}\in[-0.3,\ 0]; \tag{36n}$$

(xv) $\varepsilon_{a11}\widetilde{\phi}_{11}+\varepsilon_{a22}\widetilde{\phi}_{22}+\varepsilon_{a33}\widetilde{\phi}_{33}+\varepsilon_{c11}\widetilde{\theta}_{11}\leq 0.78780<1$, for

$$\varepsilon_{a11}\in[0,\ 0.4],\qquad \varepsilon_{a22}\in[-0.1,\ 0],$$
$$\varepsilon_{a33}\in[-0.4,\ 0],\qquad \varepsilon_{c11}\in[-0.3,\ 0]; \tag{36o}$$

(xvi) $\varepsilon_{a11}\widetilde{\phi}_{11}+\varepsilon_{a22}\widetilde{\phi}_{22}+\varepsilon_{a33}\widetilde{\phi}_{33}+\varepsilon_{c11}\widetilde{\theta}_{11}\leq 0.48780<1$, for

$$\varepsilon_{a11}\in[0,\ 0.4],\qquad \varepsilon_{a22}\in[-0.1,\ 0],$$
$$\varepsilon_{a33}\in[-0.4,\ 0],\qquad \varepsilon_{c11}\in[0,\ 0.6]. \tag{36p}$$

Then, from the results obtained above in (36a), (36b), (36c), (36d), (36e), (36f), (36g), (36h), (36i), (36j), (36k), (36l), (36m), (36n), (36o), and (36p), we can conclude that the linear interval system in (30) is robustly controllable. From this example, it is clear that our proposed approach gives less conservative results than the existing methods of Cheng and Zhang [21], Ahn et al. [22], Chen et al. [23], and Chen and Chou [19, 20, 31].

Example 2. Consider a linear interval system with both state delay and unstructured parametric uncertainties of (1) having

$$A=\begin{bmatrix}1 & -1 & -1\\ 2^{+0.8}_{-0.6} & 0 & -3\\ -4 & -3^{+0.4}_{-0.4} & 0\end{bmatrix},$$
$$B=\begin{bmatrix}-3 & -1 & 0\\ -2 & -2 & 3\\ 5^{+1.0}_{-0.5} & -1 & 0\end{bmatrix}, \tag{37}$$
$$C=\begin{bmatrix}0 & 0\\ 0 & 1^{+1.6}_{-0.01}\\ 1 & 0\end{bmatrix},$$

with $\|\widetilde{A}\|\leq\beta$, $\|\widetilde{B}\|\leq\beta$, $\|\widetilde{C}\|\leq\beta$, and $\beta=0.01$.

Letting

$$A_0=\begin{bmatrix}1 & -1 & -1\\ 2 & 0 & -3\\ -4 & -3 & 0\end{bmatrix},$$
$$B_0=\begin{bmatrix}-3 & -1 & 0\\ -2 & -2 & 3\\ 5 & -1 & 0\end{bmatrix},\qquad C_0=\begin{bmatrix}0 & 0\\ 0 & 1\\ 1 & 0\end{bmatrix}, \tag{38}$$

then the interval matrices A, B, and C can be represented as $A=A_0+\varepsilon_{a21}A_{21}+\varepsilon_{a32}A_{32}$, $B=B_0+\varepsilon_{b31}B_{31}$, and $C=C_0+\varepsilon_{c22}C_{22}$ with $A_{21}=\left[\begin{smallmatrix}0&0&0\\1&0&0\\0&0&0\end{smallmatrix}\right]$, $A_{32}=\left[\begin{smallmatrix}0&0&0\\0&0&0\\0&1&0\end{smallmatrix}\right]$, $B_{31}=\left[\begin{smallmatrix}0&0&0\\0&0&0\\1&0&0\end{smallmatrix}\right]$, $C_{22}=\left[\begin{smallmatrix}0&0\\0&1\\0&0\end{smallmatrix}\right]$, $\varepsilon_{a21}\in[-0.6,\ 0.8]$, $\varepsilon_{a32}\in[-0.4,\ 0.4]$, $\varepsilon_{b31}\in[-0.5,\ 1]$, and $\varepsilon_{c22}\in[-0.01,\ 1.6]$. Now, applying the sufficient criterion in the proposed theorem with 2-norm-based matrix measure for the robust controllability, we get rank $(E_0)=9$, and we have

(i) $\varepsilon_{a21}\phi_{21}+\varepsilon_{a32}\phi_{32}+\varepsilon_{b31}\varphi_{31}+\varepsilon_{c22}\theta_{22}+\overline{\beta}_1+\overline{\beta}_2+\overline{\beta}_3\leq 0.92798<1$, for

$$\varepsilon_{a21}\in[0,\ 0.8],\qquad \varepsilon_{a32}\in[0,\ 0.4],$$
$$\varepsilon_{b31}\in[0,\ 1],\qquad \varepsilon_{c22}\in[0,\ 1.6]; \tag{39a}$$

(ii) $\varepsilon_{a21}\phi_{21}+\varepsilon_{a32}\phi_{32}+\varepsilon_{b31}\varphi_{31}+\varepsilon_{c22}\theta_{22}+\overline{\beta}_1+\overline{\beta}_2+\overline{\beta}_3\leq 0.94798<1$, for

$$\varepsilon_{a21}\in[0,\ 0.8],\qquad \varepsilon_{a32}\in[0,\ 0.4],$$
$$\varepsilon_{b31}\in[0,\ 1],\qquad \varepsilon_{c22}\in[-0.01,\ 0]; \tag{39b}$$

(iii) $\varepsilon_{a21}\phi_{21}+\varepsilon_{a32}\phi_{32}+\varepsilon_{b31}\varphi_{31}+\varepsilon_{c22}\theta_{22}+\overline{\beta}_1+\overline{\beta}_2+\overline{\beta}_3\leq 0.84164<1$, for

$$\varepsilon_{a21}\in[0,\ 0.8],\qquad \varepsilon_{a32}\in[0,\ 0.4],$$
$$\varepsilon_{b31}\in[-0.5,\ 0],\qquad \varepsilon_{c22}\in[-0.01,\ 0]; \tag{39c}$$

(iv) $\varepsilon_{a21}\phi_{21}+\varepsilon_{a32}\phi_{32}+\varepsilon_{b31}\varphi_{31}+\varepsilon_{c22}\theta_{22}+\overline{\beta}_1+\overline{\beta}_2+\overline{\beta}_3\leq 0.82164<1$, for

$$\varepsilon_{a21}\in[0,\ 0.8],\qquad \varepsilon_{a32}\in[0,\ 0.4],$$
$$\varepsilon_{b31}\in[-0.5,\ 0],\qquad \varepsilon_{c22}\in[0,\ 1.6]; \tag{39d}$$

(v) $\varepsilon_{a21}\phi_{21}+\varepsilon_{a32}\phi_{32}+\varepsilon_{b31}\varphi_{31}+\varepsilon_{c22}\theta_{22}+\overline{\beta}_1+\overline{\beta}_2+\overline{\beta}_3\leq 0.85587<1$, for

$$\varepsilon_{a21}\in[0,\ 0.8],\qquad \varepsilon_{a32}\in[-0.4,\ 0],$$
$$\varepsilon_{b31}\in[0,\ 1],\qquad \varepsilon_{c22}\in[0,\ 1.6]; \tag{39e}$$

(vi) $\varepsilon_{a21}\phi_{21}+\varepsilon_{a32}\phi_{32}+\varepsilon_{b31}\varphi_{31}+\varepsilon_{c22}\theta_{22}+\overline{\beta}_1+\overline{\beta}_2+\overline{\beta}_3\leq 0.87587<1$, for

$$\varepsilon_{a21}\in[0,\ 0.8],\qquad \varepsilon_{a32}\in[-0.4,\ 0],$$
$$\varepsilon_{b31}\in[0,\ 1],\qquad \varepsilon_{c22}\in[-0.01,\ 0]; \tag{39f}$$

(vii) $\varepsilon_{a21}\phi_{21}+\varepsilon_{a32}\phi_{32}+\varepsilon_{b31}\varphi_{31}+\varepsilon_{c22}\theta_{22}+\overline{\beta}_1+\overline{\beta}_2+\overline{\beta}_3\leq 0.76952<1$, for

$$\varepsilon_{a21}\in[0,\ 0.8],\qquad \varepsilon_{a32}\in[-0.4,\ 0],$$
$$\varepsilon_{b31}\in[-0.5,\ 0],\qquad \varepsilon_{c22}\in[-0.01,\ 0]; \tag{39g}$$

(viii) $\varepsilon_{a21}\phi_{21}+\varepsilon_{a32}\phi_{32}+\varepsilon_{b31}\varphi_{31}+\varepsilon_{c22}\theta_{22}+\overline{\beta}_1+\overline{\beta}_2+\overline{\beta}_3\leq 0.74952<1$, for

$$\varepsilon_{a21}\in[0,\ 0.8],\qquad \varepsilon_{a32}\in[-0.4,\ 0],$$
$$\varepsilon_{b31}\in[-0.5,\ 0],\qquad \varepsilon_{c22}\in[0,\ 1.6]; \tag{39h}$$

(ix) $\varepsilon_{a21}\phi_{21}+\varepsilon_{a32}\phi_{32}+\varepsilon_{b31}\varphi_{31}+\varepsilon_{c22}\theta_{22}+\overline{\beta}_1+\overline{\beta}_2+\overline{\beta}_3\leq 0.84400<1$, for

$$\varepsilon_{a21}\in[-0.6,\ 0],\qquad \varepsilon_{a32}\in[0,\ 0.4],$$
$$\varepsilon_{b31}\in[0,\ 1],\qquad \varepsilon_{c22}\in[0,\ 1.6]; \tag{39i}$$

(x) $\varepsilon_{a21}\phi_{21} + \varepsilon_{a32}\phi_{32} + \varepsilon_{b31}\varphi_{31} + \varepsilon_{c22}\theta_{22} + \overline{\beta}_1 + \overline{\beta}_2 + \overline{\beta}_3 \leq 0.86400 < 1$, for

$$\begin{aligned} &\varepsilon_{a21} \in [-0.6,\ 0], \qquad \varepsilon_{a32} \in [0,\ 0.4], \\ &\varepsilon_{b31} \in [0,\ 1], \qquad \varepsilon_{c22} \in [-0.01,\ 0]; \end{aligned} \tag{39j}$$

(xi) $\varepsilon_{a21}\phi_{21} + \varepsilon_{a32}\phi_{32} + \varepsilon_{b31}\varphi_{31} + \varepsilon_{c22}\theta_{22} + \overline{\beta}_1 + \overline{\beta}_2 + \overline{\beta}_3 \leq 0.75766 < 1$, for

$$\begin{aligned} &\varepsilon_{a21} \in [-0.6,\ 0], \qquad \varepsilon_{a32} \in [0,\ 0.4], \\ &\varepsilon_{b31} \in [-0.5,\ 0], \qquad \varepsilon_{c22} \in [-0.01,\ 0]; \end{aligned} \tag{39k}$$

(xii) $\varepsilon_{a21}\phi_{21} + \varepsilon_{a32}\phi_{32} + \varepsilon_{b31}\varphi_{31} + \varepsilon_{c22}\theta_{22} + \overline{\beta}_1 + \overline{\beta}_2 + \overline{\beta}_3 \leq 0.73766 < 1$, for

$$\begin{aligned} &\varepsilon_{a21} \in [-0.6,\ 0], \qquad \varepsilon_{a32} \in [0,\ 0.4], \\ &\varepsilon_{b31} \in [-0.5,\ 0], \qquad \varepsilon_{c22} \in [0,\ 1.6]; \end{aligned} \tag{39l}$$

(xiii) $\varepsilon_{a21}\phi_{21} + \varepsilon_{a32}\phi_{32} + \varepsilon_{b31}\varphi_{31} + \varepsilon_{c22}\theta_{22} + \overline{\beta}_1 + \overline{\beta}_2 + \overline{\beta}_3 \leq 0.77189 < 1$, for

$$\begin{aligned} &\varepsilon_{a21} \in [-0.6,\ 0], \qquad \varepsilon_{a32} \in [-0.4,\ 0], \\ &\varepsilon_{b31} \in [0,\ 1], \qquad \varepsilon_{c22} \in [0,\ 1.6]; \end{aligned} \tag{39m}$$

(xiv) $\varepsilon_{a21}\phi_{21} + \varepsilon_{a32}\phi_{32} + \varepsilon_{b31}\varphi_{31} + \varepsilon_{c22}\theta_{22} + \overline{\beta}_1 + \overline{\beta}_2 + \overline{\beta}_3 \leq 0.79189 < 1$, for

$$\begin{aligned} &\varepsilon_{a21} \in [-0.6,\ 0], \qquad \varepsilon_{a32} \in [-0.4,\ 0], \\ &\varepsilon_{b31} \in [0,\ 1], \qquad \varepsilon_{c22} \in [-0.01,\ 0]; \end{aligned} \tag{39n}$$

(xv) $\varepsilon_{a21}\phi_{21} + \varepsilon_{a32}\phi_{32} + \varepsilon_{b31}\varphi_{31} + \varepsilon_{c22}\theta_{22} + \overline{\beta}_1 + \overline{\beta}_2 + \overline{\beta}_3 \leq 0.68554 < 1$, for

$$\begin{aligned} &\varepsilon_{a21} \in [-0.6,\ 0], \qquad \varepsilon_{a32} \in [-0.4,\ 0], \\ &\varepsilon_{b31} \in [-0.5,\ 0], \qquad \varepsilon_{c22} \in [-0.01,\ 0]; \end{aligned} \tag{39o}$$

(xvi) $\varepsilon_{a21}\phi_{21} + \varepsilon_{a32}\phi_{32} + \varepsilon_{b31}\varphi_{31} + \varepsilon_{c22}\theta_{22} + \overline{\beta}_1 + \overline{\beta}_2 + \overline{\beta}_3 \leq 0.66554 < 1$, for

$$\begin{aligned} &\varepsilon_{a21} \in [-0.6,\ 0], \qquad \varepsilon_{a32} \in [-0.4,\ 0], \\ &\varepsilon_{b31} \in [-0.5,\ 0], \qquad \varepsilon_{c22} \in [0,\ 1.6]. \end{aligned} \tag{39p}$$

Hence, from the results obtained above, we can conclude that, for any $\tau > 0$, the linear interval system with both state delay and unstructured parametric uncertainties is robustly controllable in sense of Weiss [6].

5. Conclusions

The robust controllability problem for the linear interval MIMO system with/without state delay and with unstructured parametric uncertainties has been investigated. The rank preservation problem for robust controllability of the linear interval system with/without state delay and with unstructured parametric uncertainties is converted to the nonsingularity analysis problem. Based on some essential properties of matrix measures, two new sufficient algebraically elegant criteria for the robust controllability of linear interval MIMO systems with/without state delay and with unstructured parametric uncertainties have been established. Two numerical examples have been given to illustrate the applications of the proposed sufficient algebraic criteria. It has also been shown that the proposed sufficient criterion for linear interval systems having no state delay and no unstructured parametric uncertainties can obtain less conservative results than the existing sufficient criteria given by Cheng and Zhang [21], Ahn et al. [22], Chen et al. [23], and Chen and Chou [19, 20, 31].

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