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Research Article

Oscillatory and Asymptotic Properties on a Class of Third Nonlinear Dynamic Equations with Damping Term on Time Scales

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We establish some new oscillatory and asymptotic criteria for a class of third-order nonlinear dynamic equations with damping term on time scales. The established results on one hand extend some known results in the literature on the other hand unify continuous and discrete analysis. For illustrating the validity of the established results, we also present some applications for them.

1. Introduction

The theory of time scale, which was initiated by Hilger [1], trying to treat continuous and discrete analysis in a consistent way, has received a lot of attention in recent years. Various investigations have been done by many authors. Among these investigations, some authors have taken research in the oscillatory and asymptotic properties of dynamic equations on time scales, and there has been increasing interest in obtaining sufficient conditions for the oscillation and asymptotic behavior of solutions of various dynamic equations on time scales (e.g., we refer the reader to [2–20]). But we notice that most of the investigations are concerned with oscillatory and asymptotic properties of solutions of first- or second-order dynamic equations on time scales, while relatively less attention has been paid to oscillatory and asymptotic properties of third-order dynamic equations on time scales. For recent results about the oscillation and asymptotic behavior of solutions of third-order dynamic equations on time scales, we refer the reader to [21–33]. In [34, 35], Saker researched oscillation of the following third-order dynamic equations:

$$\left(p(t)\left[\left(r(t)x^{\Delta}(t)\right)^{\Delta}\right]^{\gamma}\right)^{\Delta}+q(t)f(x(\tau(t)))=0. \quad (1)$$

Based on the Riccati substitution and the analysis of the associated Riccati dynamic inequality, some new sufficient oscillatory conditions were presented.

Moreover, to our best knowledge, none of the existing results deal with oscillatory and asymptotic behavior of solutions of third-order nonlinear dynamic equations with damping term on time scales, in which the damping term brings new difficulty in obtaining oscillatory and asymptotic criteria. We now list some important results.

In this paper, we are concerned with oscillatory and asymptotic behavior of solutions of the third-order nonlinear dynamic equation with damping term on time scales of the following form:

$$\left(a(t)\left(\left[r(t)x^{\Delta}(t)\right]^{\Delta}\right)^{\gamma}\right)^{\Delta} + p(t)\left(\left[r(t)x^{\Delta}(t)\right]^{\Delta}\right)^{\gamma}+q(t)f(x(t))=0, \quad t\in\mathbb{T}_0, \quad (2)$$

where $\mathbb{T}$ is an arbitrary time scale, $\mathbb{T}_0=[t_0,\infty)\bigcap\mathbb{T}$, a, r, p, $q\in C_{\mathrm{rd}}(\mathbb{T}_0,\mathbb{R}_+)$, $f\in C(\mathbb{R},\mathbb{R})$ satisfying $xf(x)>0$, $f(x)/x^{\gamma}\geq L>0$ for $x\neq 0$, and $\gamma\geq 1$ is a quotient of two odd positive integers.

A solution of (2) is said to be oscillatory if it is neither eventually positive nor eventually negative otherwise it is nonoscillatory. Equation (2) is said to be oscillatory in case all its solutions are oscillatory.

We will establish some new criteria of oscillatory and asymptotic behavior for (2) by a generalized Riccati transformation technique in Section 2 and present some applications for our results in Section 3. Throughout this paper, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}_+ = (0, \infty)$, while $\mathbb{Z}$ denotes the set of integers. $\mathbb{T}$ denotes an arbitrary time scale and $t_i \in \mathbb{T}$, $i = 1, 2, \dots, 5$. On $\mathbb{T}$ we define the forward and backward jump operators $\sigma \in (\mathbb{T}, \mathbb{T})$ and $\rho \in (\mathbb{T}, \mathbb{T})$ such that $\sigma(t) = \inf\{s \in \mathbb{T}, s > t\}$, $\rho(t) = \sup\{s \in \mathbb{T}, s < t\}$. A point $t \in \mathbb{T}$ with $t > \inf \mathbb{T}$ is said to be left-dense if $\rho(t) = t$, right-dense if $\sigma(t) = t$, left-scattered if $\rho(t) < t$, and right-scattered if $\sigma(t) > t$. A function $f \in (\mathbb{T}, \mathbb{R})$ is called rd-continuous if it is continuous in right-dense points and if the left-sided limits exist in left-dense points, while f is called regressive if $1 + \mu(t) f(t) \neq 0$, where $\mu(t) = \sigma(t) - t$. C_{rd} denotes the set of rd-continuous functions, while $\mathfrak{R}$ denotes the set of all regressive and rd-continuous functions, and $\mathfrak{R}^+ = \{f \mid f \in \mathfrak{R}, 1 + \mu(t) f(t) > 0, \forall t \in \mathbb{T}\}$.

Definition 1. For $p \in \mathfrak{R}$, the exponential function is defined by

$$e_p(t,s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\,\Delta\tau\right) \quad \text{for } s, t \in \mathbb{T}. \tag{3}$$

Remark 2. If $\mathbb{T} = \mathbb{R}$, then

$$e_p(t,s) = \exp\left(\int_s^t p(\tau)\,d\tau\right), \quad \text{for } s, t \in \mathbb{R}. \tag{4}$$

If $\mathbb{T} = \mathbb{Z}$, then

$$e_p(t,s) = \prod_{\tau=s}^{t-1}[1 + p(\tau)], \quad \text{for } s, t \in \mathbb{Z},\ s < t. \tag{5}$$

The following two theorems include some known properties on the exponential function.

Theorem 3 (see [36, Theorem 5.1]). *If $p \in \mathfrak{R}$ and fix $t_0 \in \mathbb{T}$, then the exponential function $e_p(t, t_0)$ is the unique solution of the following initial value problem*

$$\begin{gathered} y^\Delta(t) = p(t)\,y(t), \\ y(t_0) = 1. \end{gathered} \tag{6}$$

Theorem 4 (see [36, Theorem 5.2]). *If $p \in \mathfrak{R}^+$, then $e_p(t,s) > 0$ for $\forall s, t \in \mathbb{T}$.*

For more details about the calculus of time scales, we refer to [37].

2. Main Results

For the sake of convenience, in the rest of this paper, we set $\delta_1(t,a) = \int_a^t ([e_{-p/a}(s,t_0)]^{1/\gamma}/a^{1/\gamma}(s))\Delta s$, $\delta_2(t,a) = \int_a^t (\delta_1(s,a)/r(s))\Delta s$.

Lemma 5. *Suppose $-p/a \in \mathfrak{R}_+$, and assume that*

$$\int_{t_0}^{\infty} \frac{[e_{-p/a}(s,t_0)]^{1/\gamma}}{a^{1/\gamma}(s)}\Delta s = \infty, \tag{7}$$

$$\int_{t_0}^{\infty} \frac{1}{r(s)}\Delta s = \infty, \tag{8}$$

and (2) has an eventually positive solution x. Then there exists a sufficiently large $T_1^* \in \mathbb{T}$ such that

$$\begin{gathered} \left(\frac{a(t)\left(\left[r(t)x^\Delta(t)\right]^\Delta\right)^\gamma}{e_{-p/a}(t,t_0)}\right)^\Delta < 0, \\ \left[r(t)x^\Delta(t)\right]^\Delta > 0 \quad \text{on } [T_1^*, \infty)_{\mathbb{T}}. \end{gathered} \tag{9}$$

Proof. By $-p/a \in \mathfrak{R}_+$, we have $e_{-p/a}(t,t_0) > 0$. Since x is eventually a positive solution of (2), there exists a sufficiently large t_1 such that $x(t) > 0$ on $[t_1, \infty)_{\mathbb{T}}$, and for $t \in [t_1, \infty)_{\mathbb{T}}$, we obtain that

$$\begin{aligned} &\left(\frac{a(t)\left(\left[r(t)x^\Delta(t)\right]^\Delta\right)^\gamma}{e_{-p/a}(t,t_0)}\right)^\Delta \\ &= \left(e_{-p/a}(t,t_0)\left(a(t)\left(\left[r(t)x^\Delta(t)\right]^\Delta\right)^\gamma\right)^\Delta \right. \\ &\quad \left. -\left(e_{-p/a}(t,t_0)\right)^\Delta a(t)\left(\left[r(t)x^\Delta(t)\right]^\Delta\right)^\gamma\right) \\ &\quad \times \left(e_{-p/a}(t,t_0)\,e_{-p/a}(\sigma(t),t_0)\right)^{-1} \\ &= \frac{\left(a(t)\left(\left[r(t)x^\Delta(t)\right]^\Delta\right)^\gamma\right)^\Delta + p(t)\left(\left[r(t)x^\Delta(t)\right]^\Delta\right)^\gamma}{e_{-p/a}(\sigma(t),t_0)} \\ &= \frac{-q(t)\,f(x(t))}{e_{-p/a}(\sigma(t),t_0)} < 0. \end{aligned} \tag{10}$$

Then $a(t)([r(t)x^\Delta(t)]^\Delta)^\gamma / e_{-p/a}(t,t_0)$ is strictly decreasing on $[t_1, \infty)_{\mathbb{T}}$, and together with $a(t) > 0$, $e_{-p/a}(t,t_0) > 0$ we deduce that $[r(t)x^\Delta(t)]^\Delta$ is eventually of one sign. We claim $[r(t)x^\Delta(t)]^\Delta > 0$ on $[t_2, \infty)_{\mathbb{T}}$. Otherwise, assume there exists a sufficiently large t_3 such that $[r(t)x^\Delta(t)]^\Delta < 0$ on $[t_3, \infty)_{\mathbb{T}}$. Then

$$\begin{aligned} &r(t)x^\Delta(t) - r(t_3)x^\Delta(t_3) \\ &= \int_{t_3}^t \frac{\left[e_{-p/a}(s,t_0)\,a(s)\right]^{1/\gamma}\left[r(s)x^\Delta(s)\right]^\Delta}{\left[e_{-p/a}(s,t_0)\,a(s)\right]^{1/\gamma}}\Delta s \\ &\le \frac{a^{1/\gamma}(t_3)\left[r(t_3)x^\Delta(t_3)\right]^\Delta}{\left[e_{-p/a}(t_3,t_0)\right]^{1/\gamma}}\int_{t_3}^t \frac{\left[e_{-p/a}(s,t_0)\right]^{1/\gamma}}{a^{1/\gamma}(s)}\Delta s. \end{aligned} \tag{11}$$

By (7), we have $\lim_{t\to\infty} r(t)x^\Delta(t) = -\infty$, and thus there exists a sufficiently large $t_4 \in [t_3, \infty)_{\mathbb{T}}$ such that $r(t)x^\Delta(t) < 0$ on $[t_4, \infty)_{\mathbb{T}}$. By the assumption $[r(t)x^\Delta(t)]^\Delta < 0$ one can see $r(t)x^\Delta(t)$ is strictly decreasing on $[t_4, \infty)_{\mathbb{T}}$, and then

$$x(t) - x(t_4) = \int_{t_4}^t \frac{r(s)x^\Delta(s)}{r(s)}\Delta s \le r(t_4)x^\Delta(t_4)\int_{t_4}^t \frac{1}{r(s)}\Delta s. \tag{12}$$

Using (8), we have $\lim_{t\to\infty} x(t) = -\infty$, which leads to a contradiction. So $[r(t)x^{\Delta}(t)]^{\Delta} > 0$ on $[t_2, \infty)_{\mathbb{T}}$, and the proof is complete with taking $T_1^* = t_2$. □

Lemma 6. *Under the conditions of Lemma 5, furthermore, assume that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[\frac{1}{r(\xi)} \times \int_{\xi}^{\infty} \left(\frac{e_{-p/a}(\tau, t_0)}{a(\tau)} \times \int_{\tau}^{\infty} \frac{q(s)}{e_{-p/a}(\sigma(s), t_0)} \Delta s \right)^{1/\gamma} \Delta\tau \right] \Delta\xi = \infty. \tag{13}$$

Then either there exists a sufficiently large $T_2^* \in \mathbb{T}$ such that $x^{\Delta}(t) > 0$ on $[T_2^*, \infty)_{\mathbb{T}}$ or $\lim_{t\to\infty} x(t) = 0$.

Proof. By Lemma 5, we deduce that $x^{\Delta}(t)$ is eventually of one sign. So there exists a sufficiently large $t_5 > t_4$ such that either $x^{\Delta}(t) > 0$ or $x^{\Delta}(t) < 0$ on $[t_5, \infty)_{\mathbb{T}}$, where t_4 is defined as in Lemma 5. If $x^{\Delta}(t) < 0$, together with $x(t)$ is an eventually positive solution of (2), we obtain $\lim_{t\to\infty} x(t) = \alpha \geq 0$ and $\lim_{t\to\infty} r(t)x^{\Delta}(t) = \beta \leq 0$. We claim $\alpha = 0$. Otherwise, assume $\alpha > 0$. Then $x(t) \geq \alpha$ on $[t_5, \infty)_{\mathbb{T}}$, and for $t \in [t_5, \infty) \bigcap \mathbb{T}$, an integration for (10) from t to ∞ yields

$$\begin{aligned} &-\frac{a(t)\left(\left[r(t)x^{\Delta}(t)\right]^{\Delta}\right)^{\gamma}}{e_{-p/a}(t, t_0)} \\ &= -\lim_{t\to\infty} \frac{a(t)\left(\left[r(t)x^{\Delta}(t)\right]^{\Delta}\right)^{\gamma}}{e_{-p/a}(t, t_0)} + \int_{t}^{\infty} \frac{-q(s) f(x(s))}{e_{-p/a}(\sigma(s), t_0)} \Delta s \\ &\leq -\lim_{t\to\infty} \frac{a(t)\left(\left[r(t)x^{\Delta}(t)\right]^{\Delta}\right)^{\gamma}}{e_{-p/a}(t, t_0)} + \int_{t}^{\infty} \frac{-Lq(s) x^{\gamma}(s)}{e_{-p/a}(\sigma(s), t_0)} \Delta s \\ &\leq -L \int_{t}^{\infty} \frac{q(s) x^{\gamma}(s)}{e_{-p/a}(\sigma(s), t_0)} \Delta s \\ &\leq -L\alpha^{\gamma} \int_{t}^{\infty} \frac{q(s)}{e_{-p/a}(\sigma(s), t_0)} \Delta s, \end{aligned} \tag{14}$$

which is followed by

$$-\left[r(t)x^{\Delta}(t)\right]^{\Delta} \leq -\left\{ L\alpha^{\gamma} \left[\frac{e_{-p/a}(t, t_0)}{a(t)} \int_{t}^{\infty} \frac{q(s)}{e_{-p/a}(\sigma(s), t_0)} \Delta s \right] \right\}^{1/\gamma}. \tag{15}$$

Substituting t with τ in (15), an integration for (15) with respect to τ from t to ∞ yields

$$\begin{aligned} r(t)x^{\Delta}(t) &= \lim_{t\to\infty} r(t)x^{\Delta}(t) - \alpha L^{1/\gamma} \times \int_{t}^{\infty} \left(\frac{e_{-p/a}(\tau, t_0)}{a(\tau)} \int_{\tau}^{\infty} \frac{q(s)}{e_{-p/a}(\sigma(s), t_0)} \Delta s \right)^{1/\gamma} \Delta\tau \\ &= \beta - \alpha L^{1/\gamma} \int_{t}^{\infty} \left(\frac{e_{-p/a}(\tau, t_0)}{a(\tau)} \int_{\tau}^{\infty} \frac{q(s)}{e_{-p/a}(\sigma(s), t_0)} \Delta s \right)^{1/\gamma} \Delta\tau \\ &\leq -\alpha L^{1/\gamma} \int_{t}^{\infty} \left(\frac{e_{-p/a}(\tau, t_0)}{a(\tau)} \int_{\tau}^{\infty} \frac{q(s)}{e_{-p/a}(\sigma(s), t_0)} \Delta s \right)^{1/\gamma} \Delta\tau, \end{aligned} \tag{16}$$

which implies

$$x^{\Delta}(t) \leq -\alpha L^{1/\gamma} \frac{1}{r(t)} \int_{t}^{\infty} \left(\frac{e_{-p/a}(\tau, t_0)}{a(\tau)} \times \int_{\tau}^{\infty} \frac{q(s)}{e_{-p/a}(\sigma(s), t_0)} \Delta s \right)^{1/\gamma} \Delta\tau. \tag{17}$$

Substituting t with ξ in (17), an integration for (17) with respect to ξ from t_5 to t yields

$$x(t) - x(t_5) \leq -\alpha L^{1/\gamma} \int_{t_5}^{t} \left[\frac{1}{r(\xi)} \times \int_{\xi}^{\infty} \left(\frac{e_{-p/a}(\tau, t_0)}{a(\tau)} \times \int_{\tau}^{\infty} \frac{q(s)}{e_{-p/a}(\sigma(s), t_0)} \Delta s \right)^{1/\gamma} \Delta\tau \right] \Delta\xi. \tag{18}$$

By (18) and (13) we have $\lim_{t\to\infty} x(t) = -\infty$, which leads to a contradiction. So one has $\alpha = 0$, and the proof is complete. □

Lemma 7. *Suppose* $-p/a \in \mathfrak{R}_+$, *and assume that* x *is a positive solution of* (2) *such that*

$$\left[r(t)x^{\Delta}(t)\right]^{\Delta} > 0, \quad x^{\Delta}(t) > 0 \text{ on } [T_3^*, \infty)_{\mathbb{T}}, \tag{19}$$

where $T_3^* \in \mathbb{T}$ *is sufficiently large. Then for* $t \in [T_3^*, \infty)_{\mathbb{T}}$, *we have*

$$\begin{aligned} x^{\Delta}(t) &\geq \frac{\delta_1(t, T_3^*)}{r(t)} \left\{ \frac{a^{1/\gamma}(t)\left[r(t)x^{\Delta}(t)\right]^{\Delta}}{\left[e_{-p/a}(t,t_0)\right]^{1/\gamma}} \right\}, \\ x(t) &\geq \delta_2(t, T_3^*) \left\{ \frac{a^{1/\gamma}(t)\left[r(t)x^{\Delta}(t)\right]^{\Delta}}{\left[e_{-p/a}(t,t_0)\right]^{1/\gamma}} \right\}. \end{aligned} \tag{20}$$

Proof. Take $T_3^* > \max(T_1^*, T_2^*)$, where T_1^*, T_2^* are defined as in Lemmas 5 and 6, respectively. By Lemma 5 we have $a(t)([r(t)x^{\Delta}(t)]^{\Delta})^{\gamma}/e_{-p/a}(t,t_0)$ strictly decreasing on $[T_3^*, \infty)$. So

$$\begin{aligned} r(t)x^{\Delta}(t) &\geq r(t)x^{\Delta}(t) - r(T_3^*)x^{\Delta}(T_3^*) \\ &= \int_{T_3^*}^{t} \frac{\left[e_{-p/a}(s,t_0)a(s)\right]^{1/\gamma}\left[r(s)x^{\Delta}(s)\right]^{\Delta}}{\left[e_{-p/a}(s,t_0)a(s)\right]^{1/\gamma}} \Delta s \\ &\geq \frac{a^{1/\gamma}(t)\left[r(t)x^{\Delta}(t)\right]^{\Delta}}{\left[e_{-p/a}(t,t_0)\right]^{1/\gamma}} \\ &\quad \times \int_{T_3^*}^{t} \frac{\left[e_{-p/a}(s,t_0)\right]^{1/\gamma}}{a^{1/\gamma}(s)} \Delta s \\ &= \delta_1(t, T_3^*) \frac{a^{1/\gamma}(t)\left[r(t)x^{\Delta}(t)\right]^{\Delta}}{\left[e_{-p/a}(t,t_0)\right]^{1/\gamma}}, \end{aligned} \tag{21}$$

and then

$$x^{\Delta}(t) \geq \frac{\delta_1(t, T_3^*)}{r(t)} \left\{ \frac{a^{1/\gamma}(t)\left[r(t)x^{\Delta}(t)\right]^{\Delta}}{\left[e_{-p/a}(t,t_0)\right]^{1/\gamma}} \right\}. \tag{22}$$

Furthermore,

$$\begin{aligned} x(t) &\geq x(t) - x(T_3^*) \\ &= \int_{T_3^*}^{t} x^{\Delta}(s)\Delta s \\ &\geq \int_{T_3^*}^{t} \frac{\delta_1(s, T_3^*)}{r(s)} \left\{ \frac{a^{1/\gamma}(s)\left[r(s)x^{\Delta}(s)\right]^{\Delta}}{\left[e_{-p/a}(s,t_0)\right]^{1/\gamma}} \right\} \Delta s \\ &\geq \left\{ \frac{a^{1/\gamma}(t)\left[r(t)x^{\Delta}(t)\right]^{\Delta}}{\left[e_{-p/a}(t,t_0)\right]^{1/\gamma}} \right\} \\ &\quad \times \int_{T_3^*}^{t} \frac{\delta_1(s, T_3^*)}{r(s)} \Delta s \\ &= \delta_2(t, T_3^*) \left\{ \frac{a^{1/\gamma}(t)\left[r(t)x^{\Delta}(t)\right]^{\Delta}}{\left[e_{-p/a}(t,t_0)\right]^{1/\gamma}} \right\}, \end{aligned} \tag{23}$$

which is the desired result. □

Lemma 8 (see [38, Theorem 41]). *Assume that* X *and* Y *are nonnegative real numbers. Then*

$$\lambda X Y^{\lambda-1} - X^{\lambda} \leq (\lambda - 1) Y^{\lambda} \quad \forall \lambda > 1. \tag{24}$$

Theorem 9. *Suppose* $-p/a \in \mathfrak{R}_+$, *and assume that* (7), (8), *and* (13) *hold, and for all sufficiently large* T,

$$\begin{aligned} \limsup_{t\to\infty} \Bigg\{ \int_T^t \Bigg\{ & L \frac{q(s)\rho(s)}{e_{-p/a}(\sigma(s),t_0)} - \rho(t)\left[a(s)\eta(s)\right]^{\Delta} + \frac{\rho(s)\delta_1(s,T)\left[a(\sigma(s))\eta(\sigma(s))\right]^{1+1/\gamma}}{r(s)} \\ & - \left[\frac{r(s)\rho^{\Delta}(s) + (\gamma+1)\rho(s)\delta_1(s,T)\left[a(\sigma(s))\eta(\sigma(s))\right]^{1/\gamma}}{(\gamma+1) r^{1/(\gamma+1)}(s)\rho^{\gamma/(\gamma+1)}(s)\delta_1^{\gamma/(\gamma+1)}(s,T)} \right]^{\gamma+1} \Bigg\} \Delta s \Bigg\} = \infty, \end{aligned} \tag{25}$$

where ρ, η *are two given nonnegative functions on* $\mathbb{T}$ *with* $\rho(t) > 0$. *Then every solution of* (2) *is oscillatory or tends to zero.*

Proof. Assume (2) has a nonoscillatory solution x on $\mathbb{T}_0$. Without loss of generality, we may assume $x(t) > 0$ on $[t_1, \infty)_{\mathbb{T}}$, where t_1 is sufficiently large. By Lemmas 5 and 6, there exists sufficiently large t_2 such that $[r(t)x^{\Delta}(t)]^{\Delta} > 0$ on $[t_2, \infty)_{\mathbb{T}}$, and either $x^{\Delta}(t) > 0$ on $[t_2, \infty)_{\mathbb{T}}$ or $\lim_{t\to\infty} x(t) = 0$. Now we assume $x^{\Delta}(t) > 0$ on $[t_2, \infty)_{\mathbb{T}}$. Define the generalized Riccati function:

$$\omega(t) = \rho(t)a(t)\left[\frac{\left(\left[r(t)x^{\Delta}(t)\right]^{\Delta}\right)^{\gamma}}{x^{\gamma}(t)e_{-p/a}(t,t_0)} + \eta(t) \right]. \tag{26}$$

Then for $t \in [t_2, \infty)_{\mathbb{T}}$, we have

$$\begin{aligned}
\omega^{\Delta}(t) &= \frac{\rho(t)}{x^{\gamma}(t)} \left\{ \frac{a(t)\left(\left[r(t)x^{\Delta}(t)\right]^{\Delta}\right)^{\gamma}}{e_{-p/a}(t,t_0)} \right\}^{\Delta} + \left[\frac{\rho(t)}{x^{\gamma}(t)}\right]^{\Delta} \\
&\quad \times \frac{a(\sigma(t))\left(\left[r(\sigma(t))x^{\Delta}(\sigma(t))\right]^{\Delta}\right)^{\gamma}}{e_{-p/a}(\sigma(t),t_0)} \\
&\quad + \rho(t)\left[a(t)\eta(t)\right]^{\Delta} + \rho^{\Delta}(t)a(\sigma(t))\eta(\sigma(t)) \\
&= \frac{\rho(t)}{x^{\gamma}(t)} \\
&\quad \times \Big\{ \left(e_{-p/a}(t,t_0)\left(a(t)\left(\left[r(t)x^{\Delta}(t)\right]^{\Delta}\right)^{\gamma}\right)^{\Delta} \right. \\
&\qquad \left. -\left(e_{-p/a}(t,t_0)\right)^{\Delta} a(t)\left(\left[r(t)x^{\Delta}(t)\right]^{\Delta}\right)^{\gamma}\right) \\
&\qquad \times \left(e_{-p/a}(t,t_0)\,e_{-p/a}(\sigma(t),t_0)\right)^{-1} \Big\} \\
&\quad + \left[\frac{x^{\gamma}(t)\rho^{\Delta}(t) - \left(x^{\gamma}(t)\right)^{\Delta}\rho(t)}{x^{\gamma}(t)\,x^{\gamma}(\sigma(t))}\right] \\
&\quad \times \frac{a(\sigma(t))\left(\left[r(\sigma(t))x^{\Delta}(\sigma(t))\right]^{\Delta}\right)^{\gamma}}{e_{-p/a}(\sigma(t),t_0)} \\
&\quad + \rho(t)\left[a(t)\eta(t)\right]^{\Delta} + \rho^{\Delta}(t)a(\sigma(t))\eta(\sigma(t)) \\
&= \frac{\rho(t)}{x^{\gamma}(t)} \\
&\quad \times \Big[\left(\left(a(t)\left(\left[r(t)x^{\Delta}(t)\right]^{\Delta}\right)^{\gamma}\right)^{\Delta} \right. \\
&\qquad \left. + p(t)\left(\left[r(t)x^{\Delta}(t)\right]^{\Delta}\right)^{\gamma}\right) \\
&\qquad \times \left(e_{-p/a}(\sigma(t),t_0)\right)^{-1} \Big] \\
&\quad + \frac{\rho^{\Delta}(t)}{\rho(\sigma(t))}\omega(\sigma(t)) - \left[\frac{\rho(t)\left(x^{\gamma}(t)\right)^{\Delta}}{x^{\gamma}(t)}\right] \\
&\quad \times \frac{a(\sigma(t))\left(\left[r(\sigma(t))x^{\Delta}(\sigma(t))\right]^{\Delta}\right)^{\gamma}}{x^{\gamma}(\sigma(t))\,e_{-p/a}(\sigma(t),t_0)} \\
&\quad + \rho(t)\left[a(t)\eta(t)\right]^{\Delta} \\
&= -\frac{\rho(t)}{x^{\gamma}(t)}\left[\frac{q(t)f(x(t))}{e_{-p/a}(\sigma(t),t_0)}\right] + \frac{\rho^{\Delta}(t)}{\rho(\sigma(t))}\omega(\sigma(t)) \\
&\quad - \left[\frac{\rho(t)\left(x^{\gamma}(t)\right)^{\Delta}}{x^{\gamma}(t)}\right]\frac{a(\sigma(t))\left(\left[r(\sigma(t))x^{\Delta}(\sigma(t))\right]^{\Delta}\right)^{\gamma}}{x^{\gamma}(\sigma(t))\,e_{-p/a}(\sigma(t),t_0)} \\
&\quad + \rho(t)\left[a(t)\eta(t)\right]^{\Delta} \\
&\le -L\frac{q(t)\rho(t)}{e_{-p/a}(\sigma(t),t_0)} + \frac{\rho^{\Delta}(t)}{\rho(\sigma(t))}\omega(\sigma(t)) \\
&\quad - \left[\frac{\rho(t)\left(x^{\gamma}(t)\right)^{\Delta}}{x^{\gamma}(t)}\right] \\
&\quad \times \frac{a(\sigma(t))\left(\left[r(\sigma(t))x^{\Delta}(\sigma(t))\right]^{\Delta}\right)^{\gamma}}{x^{\gamma}(\sigma(t))\,e_{-p/a}(\sigma(t),t_0)} \\
&\quad + \rho(t)\left[a(t)\eta(t)\right]^{\Delta}.
\end{aligned} \tag{27}$$

By [37, Theorem 1.93], we have $(x^{\gamma}(t))^{\Delta} \ge \gamma x^{\gamma-1}(t)x^{\Delta}(t)$. Then

$$\begin{aligned}
\omega^{\Delta}(t) \le &-L\frac{q(t)\rho(t)}{e_{-p/a}(\sigma(t),t_0)} + \frac{\rho^{\Delta}(t)}{\rho(\sigma(t))}\omega(\sigma(t)) \\
&- \rho(t)\left[\frac{\gamma x^{\gamma-1}(t)x^{\Delta}(t)}{x^{\gamma}(t)}\right] \\
&\times \frac{a(\sigma(t))\left(\left[r(\sigma(t))x^{\Delta}(\sigma(t))\right]^{\Delta}\right)^{\gamma}}{x^{\gamma}(\sigma(t))\,e_{-p/a}(\sigma(t),t_0)} \\
&+ \rho(t)\left[a(t)\eta(t)\right]^{\Delta} \\
\le &-L\frac{q(t)\rho(t)}{e_{-p/a}(\sigma(t),t_0)} + \frac{\rho^{\Delta}(t)}{\rho(\sigma(t))}\omega(\sigma(t)) \\
&- \left[\frac{\gamma\rho(t)}{x(\sigma(t))}\right] \\
&\times \left\{\frac{\delta_1(t,t_2)}{r(t)}\left[\frac{a^{1/\gamma}(t)\left[r(t)x^{\Delta}(t)\right]^{\Delta}}{\left[e_{-p/a}(t,t_0)\right]^{1/\gamma}}\right]\right\} \\
&\times \frac{a(\sigma(t))\left(\left[r(\sigma(t))x^{\Delta}(\sigma(t))\right]^{\Delta}\right)^{\gamma}}{x^{\gamma}(\sigma(t))\,e_{-p/a}(\sigma(t),t_0)} \\
&+ \rho(t)\left[a(t)\eta(t)\right]^{\Delta} \\
= &-L\frac{q(t)\rho(t)}{e_{-p/a}(\sigma(t),t_0)} + \frac{\rho^{\Delta}(t)}{\rho(\sigma(t))}\omega(\sigma(t)) \\
&- \gamma\frac{\rho(t)\delta_1(t,t_2)}{r(t)} \\
&\times \left[\frac{\omega(\sigma(t))}{\rho(\sigma(t))} - a(\sigma(t))\eta(\sigma(t))\right]^{1+1/\gamma} \\
&+ \rho(t)\left[a(t)\eta(t)\right]^{\Delta}.
\end{aligned} \tag{28}$$

Using the following inequality (see [25, (2.17)]):

$$(u-v)^{1+1/\gamma} \ge u^{1+1/\gamma} + \frac{1}{\gamma}v^{1+1/\gamma} - \left(1+\frac{1}{\gamma}\right)v^{1/\gamma}u, \tag{29}$$

where u, v are constants and $\gamma \geq 1$ is a quotient of two odd positive integers, we obtain

$$\left[\frac{\omega(\sigma(t))}{\rho(\sigma(t))} - a(\sigma(t))\eta(\sigma(t))\right]^{1+1/\gamma} \geq \frac{\omega^{1+1/\gamma}(\sigma(t))}{\rho^{1+1/\gamma}(\sigma(t))} + \frac{1}{\gamma}\left[a(\sigma(t))\eta(\sigma(t))\right]^{1+1/\gamma} - \left(1+\frac{1}{\gamma}\right)\frac{\left[a(\sigma(t))\eta(\sigma(t))\right]^{1/\gamma}\omega(\sigma(t))}{\rho(\sigma(t))}. \tag{30}$$

A combination of (28) and (30) yields

$$\begin{aligned}\omega^{\Delta}(t) &\leq -L\frac{q(t)\rho(t)}{e_{-p/a}(\sigma(t),t_0)} + \rho(t)\left[a(t)\eta(t)\right]^{\Delta} - \frac{\rho(t)\delta_1(t,t_2)\left[a(\sigma(t))\eta(\sigma(t))\right]^{1+1/\gamma}}{r(t)} \\ &+ \frac{r(t)\rho^{\Delta}(t) + (\gamma+1)\rho(t)\delta_1(t,t_2)\left[a(\sigma(t))\eta(\sigma(t))\right]^{1/\gamma}}{r(t)\rho(\sigma(t))} \\ &\times\omega(\sigma(t)) - \gamma\frac{\rho(t)\delta_1(t,t_2)}{r(t)}\frac{\omega^{1+1/\gamma}(\sigma(t))}{\rho^{1+1/\gamma}(\sigma(t))}.\end{aligned} \tag{31}$$

Setting

$$\lambda = 1+\frac{1}{\gamma},$$
$$X^{\lambda} = \gamma\frac{\rho(t)\delta_1(t,t_2)}{r(t)}\frac{\omega^{1+1/\gamma}(\sigma(t))}{\rho^{1+1/\gamma}(\sigma(t))},$$
$$Y^{\lambda-1} = \gamma^{1/(\gamma+1)} \times\left[\frac{r(t)\rho^{\Delta}(t) + (\gamma+1)\rho(t)\delta_1(t,t_2)\left[a(\sigma(t))\eta(\sigma(t))\right]^{1/\gamma}}{(\gamma+1)r^{1/(\gamma+1)}(t)\rho^{\gamma/(\gamma+1)}(t)\delta_1^{\gamma/(\gamma+1)}(t,t_2)}\right]. \tag{32}$$

Using Lemma 8 in (31) we get that

$$\begin{aligned}\omega^{\Delta}(t) &\leq -L\frac{q(t)\rho(t)}{e_{-p/a}(\sigma(t),t_0)} + \rho(t)\left[a(t)\eta(t)\right]^{\Delta} - \frac{\rho(t)\delta_1(t,t_2)\left[a(\sigma(t))\eta(\sigma(t))\right]^{1+1/\gamma}}{r(t)} \\ &+ \left[\frac{r(t)\rho^{\Delta}(t) + (\gamma+1)\rho(t)\delta_1(t,t_2)\left[a(\sigma(t))\eta(\sigma(t))\right]^{1/\gamma}}{(\gamma+1)r^{1/(\gamma+1)}(t)\rho^{\gamma/(\gamma+1)}(t)\delta_1^{\gamma/(\gamma+1)}(t,t_2)}\right]^{\gamma+1}.\end{aligned} \tag{33}$$

Substituting t with s in (33), an integration for (33) with respect to s from t_2 to t yields

$$\begin{aligned}\int_{t_2}^{t}&\left\{L\frac{q(s)\rho(s)}{e_{-p/a}(\sigma(s),t_0)} - \rho(t)\left[a(s)\eta(s)\right]^{\Delta} + \frac{\rho(s)\delta_1(s,t_2)\left[a(\sigma(s))\eta(\sigma(s))\right]^{1+1/\gamma}}{r(s)}\right. \\ &\left. - \left[\frac{r(s)\rho^{\Delta}(s) + (\gamma+1)\rho(s)\delta_1(s,t_2)\left[a(\sigma(s))\eta(\sigma(s))\right]^{1/\gamma}}{(\gamma+1)r^{1/(\gamma+1)}(s)\rho^{\gamma/(\gamma+1)}(s)\delta_1^{\gamma/(\gamma+1)}(s,t_2)}\right]^{\gamma+1}\right\} \\ &\times\Delta s \leq \omega(t_2) - \omega(t) \leq \omega(t_2) < \infty,\end{aligned} \tag{34}$$

which contradicts (25). So the proof is complete. □

In Theorem 9, if we take $\mathbb{T}$ for some special cases, then we can obtain the following corollaries.

Corollary 10. *Let $\mathbb{T} = \mathbb{R}$. Assume that*

$$\int_{t_0}^{\infty}\frac{\left[e_{-p/a}(s,t_0)\right]^{1/\gamma}}{a^{1/\gamma}(s)}ds = \infty,$$
$$\int_{t_0}^{\infty}\frac{1}{r(s)}ds = \infty,$$
$$\int_{t_0}^{\infty}\left[\frac{1}{r(\xi)}\int_{\xi}^{\infty}\left(\frac{e_{-p/a}(\tau,t_0)}{a(\tau)} \times\int_{\tau}^{\infty}\frac{q(s)}{e_{-p/a}(s,t_0)}ds\right)^{1/\gamma}d\tau\right]d\xi = \infty, \tag{35}$$

and for all sufficiently large T,

$$\limsup_{t\to\infty}\left\{\int_T^t\left\{L\frac{q(s)\rho(s)}{e_{-p/a}(s,t_0)}-\rho(t)\left[a(s)\eta(s)\right]'+\frac{\rho(s)\delta_1(s,T)\left[a(s)\eta(s)\right]^{1+1/\gamma}}{r(s)}\right.\right.$$
$$\left.\left.-\left[\frac{r(s)\rho'(s)+(\gamma+1)\rho(s)\delta_1(s,T)\left[a(s)\eta(\sigma(s))\right]^{1/\gamma}}{(\gamma+1)r^{1/(\gamma+1)}(s)\rho^{\gamma/(\gamma+1)}(s)\delta_1^{\gamma/(\gamma+1)}(s,T)}\right]^{\gamma+1}\right\}ds\right\}=\infty. \tag{36}$$

Then every solution of (2) *is oscillatory or tends to zero.*

Corollary 11. *Let* $\mathbb{T}=\mathbb{Z}$ *and* $-p/a\in\mathfrak{R}_+$. *Assume that*

$$\sum_{s=t_0}^{\infty}\frac{\left[e_{-p/a}(s,t_0)\right]^{1/\gamma}}{a^{1/\gamma}(s)}=\infty,$$

$$\sum_{s=t_0}^{\infty}\frac{1}{r(s)}=\infty,$$

$$\sum_{\xi=t_0}^{\infty}\left[\frac{1}{r(\xi)}\sum_{\tau=\xi}^{\infty}\left(\frac{e_{-p/a}(\tau,t_0)}{a(\tau)}\sum_{s=\tau}^{\infty}\frac{q(s)}{e_{-p/a}(s+1,t_0)}\right)^{1/\gamma}\right]=\infty, \tag{37}$$

and for all sufficiently large T,

$$\limsup_{t\to\infty}\left\{\sum_{s=T}^{t-1}\left\{L\frac{q(s)\rho(s)}{e_{-p/a}(s+1,t_0)}-\rho(t)\left[a(s+1)\eta(s+1)-a(s)\eta(s)\right]+\frac{\rho(s)\delta_1(s,T)\left[a(s+1)\eta(s+1)\right]^{1+1/\gamma}}{r(s)}\right.\right.$$
$$\left.\left.-\left[\frac{r(s)(\rho(s+1)-\rho(s))+(\gamma+1)\rho(s)\delta_1(s,T)\left[a(s+1)\eta(s+1)\right]^{1/\gamma}}{(\gamma+1)r^{1/(\gamma+1)}(s)\rho^{\gamma/(\gamma+1)}(s)\delta_1^{\gamma/(\gamma+1)}(s,T)}\right]^{\gamma+1}\right\}\right\}=\infty. \tag{38}$$

Then every solution of (2) *is oscillatory or tends to zero.*

Theorem 12. *Suppose* $-p/a\in\mathfrak{R}_+$, *and assume that* (7), (8), *and* (13) *hold, and for all sufficiently large* T,

$$\limsup_{t\to\infty}\left\{\int_T^t\left\{L\frac{q(s)\rho(s)}{e_{-p/a}(\sigma(s),t_0)}-\rho(s)\left[a(s)\eta(s)\right]^{\Delta}+\frac{\gamma\rho(s)\delta_1(s,T)\delta_2^{\gamma-1}(\sigma(s),T)a^2(\sigma(s))\eta^2(\sigma(s))}{r(s)}\right.\right.$$
$$\left.\left.-\frac{\left[r(s)\rho^{\Delta}(s)+2\gamma\rho(s)\delta_1(s,T)\delta_2^{\gamma-1}(\sigma(s),T)a(\sigma(s))\eta(\sigma(s))\right]^2}{4\gamma r(s)\rho(s)\delta_1(s,T)\delta_2^{\gamma-1}(\sigma(s),T)}\right\}\Delta s\right\}=\infty, \tag{39}$$

where ρ, η *are defined as in Theorem 9, then every solution of* (2) *is oscillatory or tends to zero.*

Proof. Assume (2) has a nonoscillatory solution x on $\mathbb{T}_0$. Similar to Theorem 9, we may assume $x(t)>0$ on $[t_1,\infty)_{\mathbb{T}}$, where t_1 is sufficiently large. By Lemmas 5 and 6, there exists sufficiently large t_2 such that $[r(t)x^{\Delta}(t)]^{\Delta}>0$ on $[t_2,\infty)_{\mathbb{T}}$, and either $x^{\Delta}(t)>0$ on $[t_2,\infty)_{\mathbb{T}}$ or $\lim_{t\to\infty}x(t)=0$. Now we assume $x^{\Delta}(t)>0$ on $[t_2,\infty)_{\mathbb{T}}$. Let $\omega(t)$ be defined as in Theorem 9. By Lemma 7, we have the following observation:

$$\frac{x^{\Delta}(t)}{x(t)}$$
$$\geq\frac{\delta_1(t,t_2)}{r(t)}\left\{\frac{a^{1/\gamma}(t)\left[r(t)x^{\Delta}(t)\right]^{\Delta}}{\left[e_{-p/a}(t,t_0)\right]^{1/\gamma}}\right\}\frac{1}{x(\sigma(t))}$$
$$=\frac{\delta_1(t,t_2)}{r(t)}\left\{\frac{a^{1/\gamma}(t)\left[r(t)x^{\Delta}(t)\right]^{\Delta}}{\left[e_{-p/a}(t,t_0)\right]^{1/\gamma}x^{\gamma}(\sigma(t))}\right\}x^{\gamma-1}(\sigma(t))$$
$$\geq\frac{\delta_1(t,t_2)}{r(t)}\left\{\frac{a^{1/\gamma}(\sigma(t))\left[r(\sigma(t))x^{\Delta}(\sigma(t))\right]^{\Delta}}{\left[e_{-p/a}(\sigma(t),t_0)\right]^{1/\gamma}x^{\gamma}(\sigma(t))}\right\}x^{\gamma-1}(\sigma(t))$$
$$\geq\frac{\delta_1(t,t_2)}{r(t)}\left\{\frac{a^{1/\gamma}(\sigma(t))\left[r(\sigma(t))x^{\Delta}(\sigma(t))\right]^{\Delta}}{\left[e_{-p/a}(\sigma(t),t_0)\right]^{1/\gamma}x^{\gamma}(\sigma(t))}\right\}\delta_2^{\gamma-1}$$
$$\times(\sigma(t),t_2)\left\{\frac{a^{1/\gamma}(\sigma(t))\left[r(\sigma(t))x^{\Delta}(\sigma(t))\right]^{\Delta}}{\left[e_{-p/a}(\sigma(t),t_0)\right]^{1/\gamma}}\right\}^{\gamma-1}$$

$$= \frac{\delta_1(t,t_2)\delta_2^{\gamma-1}(\sigma(t),t_2)}{r(t)} \times \left\{ \frac{a(\sigma(t))\left(\left[r(\sigma(t))x^{\Delta}(\sigma(t))\right]^{\Delta}\right)^{\gamma}}{e_{-p/a}(\sigma(t),t_0)x^{\gamma}(\sigma(t))} \right\}. \tag{40}$$

Using (40) in (28) we get that

$$\begin{aligned}
\omega^{\Delta}(t) \leq & -L\frac{q(t)\rho(t)}{e_{-p/a}(\sigma(t),t_0)} + \frac{\rho^{\Delta}(t)}{\rho(\sigma(t))}\omega(\sigma(t)) - \rho(t)\left[\frac{\gamma x^{\Delta}(t)}{x(t)}\right] \\
& \times \frac{a(\sigma(t))\left[\left(r(\sigma(t))x^{\Delta}(\sigma(t))\right)^{\Delta}\right]^{\gamma}}{x^{\gamma}(\sigma(t))e_{-p/a}(\sigma(t),t_0)} + \rho(t)\left[a(t)\eta(t)\right]^{\Delta} \\
\leq & -L\frac{q(t)\rho(t)}{e_{-p/a}(\sigma(t),t_0)} + \frac{\rho^{\Delta}(t)}{\rho(\sigma(t))}\omega(\sigma(t)) - \gamma\rho(t)\frac{\delta_1(t,t_2)\delta_2^{\gamma-1}(\sigma(t),t_2)}{r(t)} \\
& \times \left\{\frac{a(\sigma(t))\left(\left[r(\sigma(t))x^{\Delta}(\sigma(t))\right]^{\Delta}\right)^{\gamma}}{e_{-p/a}(\sigma(t),t_0)x^{\gamma}(\sigma(t))}\right\}^2 + \rho(t)\left[a(t)\eta(t)\right]^{\Delta} \\
= & -L\frac{q(t)\rho(t)}{e_{-p/a}(\sigma(t),t_0)} + \frac{\rho^{\Delta}(t)}{\rho(\sigma(t))}\omega(\sigma(t)) - \frac{\gamma\rho(t)\delta_1(t,t_2)\delta_2^{\gamma-1}(\sigma(t),t_2)}{r(t)} \\
& \times \left[\frac{\omega(\sigma(t))}{\rho(\sigma(t))} - a(\sigma(t))\eta(\sigma(t))\right]^2 + \rho(t)\left[a(t)\eta(t)\right]^{\Delta} \\
= & -L\frac{q(t)\rho(t)}{e_{-p/a}(\sigma(t),t_0)} + \rho(t)\left[a(t)\eta(t)\right]^{\Delta} - \frac{\gamma\rho(t)\delta_1(t,t_2)\delta_2^{\gamma-1}(\sigma(t),t_2)a^2(\sigma(t))\eta^2(\sigma(t))}{r(t)} \\
& + \left[\frac{r(t)\rho^{\Delta}(t) + 2\gamma\rho(t)\delta_1(t,t_2)\delta_2^{\gamma-1}(\sigma(t),t_2)a(\sigma(t))\eta(\sigma(t))}{r(t)\rho(\sigma(t))}\right]\omega(\sigma(t)) \\
& - \frac{\gamma\rho(t)\delta_1(t,t_2)\delta_2^{\gamma-1}(\sigma(t),t_2)}{r(t)\rho^2(\sigma(t))}\omega^2(\sigma(t)) \leq -L\frac{q(t)\rho(t)}{e_{-p/a}(\sigma(t),t_0)} + \rho(t)\left[a(t)\eta(t)\right]^{\Delta} \\
& - \frac{\gamma\rho(t)\delta_1(t,t_2)\delta_2^{\gamma-1}(\sigma(t),t_2)a^2(\sigma(t))\eta^2(\sigma(t))}{r(t)} \\
& + \frac{\left[r(t)\rho^{\Delta}(t) + 2\gamma\rho(t)\delta_1(t,t_2)\delta_2^{\gamma-1}(\sigma(t),t_2)a(\sigma(t))\eta(\sigma(t))\right]^2}{4\gamma r(t)\rho(t)\delta_1(t,t_2)\delta_2^{\gamma-1}(\sigma(t),t_2)}.
\end{aligned} \tag{41}$$

Substituting t with s in (41), an integration for (41) with respect to s from t_2 to t yields

$$\begin{aligned}
\int_{t_2}^{t} & \left\{ L\frac{q(s)\rho(s)}{e_{-p/a}(\sigma(s),t_0)} - \rho(s)\left[a(s)\eta(s)\right]^{\Delta} + \frac{\gamma\rho(s)\delta_1(s,t_2)\delta_2^{\gamma-1}(\sigma(s),t_2)a^2(\sigma(s))\eta^2(\sigma(s))}{r(s)} \right. \\
& \left. - \frac{\left[r(s)\rho^{\Delta}(s) + 2\gamma\rho(s)\delta_1(s,t_2)\delta_2^{\gamma-1}(\sigma(s),t_2)a(\sigma(s))\eta(\sigma(s))\right]^2}{4\gamma r(s)\rho(s)\delta_1(s,t_2)\delta_2^{\gamma-1}(\sigma(s),t_2)} \right\}\Delta s \leq \omega(t_2) - \omega(t) \leq \omega(t_2) < \infty,
\end{aligned} \tag{42}$$

which contradicts (39). So the proof is complete. □

Based on Theorems 9 and 12, we will establish some Philos-type oscillation criteria for (2).

Theorem 13. *Suppose $-p/a \in \Re_+$, and assume that (7), (8), and (13) hold, and define $\mathbb{D} = \{(t,s) \mid t \geq s \geq t_0\}$. If there exists a function $H \in C_{\mathrm{rd}}(\mathbb{D}, \mathbb{R})$ such that*

$$H(t,t) = 0, \quad \text{for } t \geq t_0, \qquad H(t,s) > 0, \quad \text{for } t > s \geq t_0, \tag{43}$$

and H has a nonpositive continuous $\Delta-$ partial derivative $H^{\Delta_s}(t,s)$ with respect to the second variable, and for all sufficiently large T,

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \left\{ \int_{t_0}^{t} H(t,s) \left\{ L\frac{q(s)\rho(s)}{e_{-p/a}(\sigma(s),t_0)} - \rho(t)[a(s)\eta(s)]^{\Delta} + \frac{\rho(s)\delta_1(s,T)[a(\sigma(s))\eta(\sigma(s))]^{1+1/\gamma}}{r(s)} - \left[\frac{r(s)\rho^{\Delta}(s) + (\gamma+1)\rho(s)\delta_1(s,T)[a(\sigma(s))\eta(\sigma(s))]^{1/\gamma}}{(\gamma+1)r^{1/(\gamma+1)}(s)\rho^{\gamma/(\gamma+1)}(s)\delta_1^{\gamma/(\gamma+1)}(s,T)}\right]^{\gamma+1} \right\} \Delta s \right\} = \infty, \tag{44}$$

where ρ, η are defined as in Theorem 9. Then every solution of (2) is oscillatory or tends to zero.

Proof. Assume (2) has a nonoscillatory solution x on $\mathbb{T}_0$. Without loss of generality, we may assume $x(t) > 0$ on $[t_1,\infty)_{\mathbb{T}}$, where t_1 is sufficiently large. By Lemmas 5 and 6, there exists sufficiently large t_2 such that $[r(t)x^{\Delta}(t)]^{\Delta} > 0$ on $[t_2,\infty)_{\mathbb{T}}$, and either $x^{\Delta}(t) > 0$ on $[t_2,\infty)_{\mathbb{T}}$ or $\lim_{t\to\infty}x(t) = 0$. Now we assume $x^{\Delta}(t) > 0$ on $[t_2,\infty)_{\mathbb{T}}$. Let $\omega(t)$ be defined as in Theorem 9. By (33) we have

$$L\frac{q(t)\rho(t)}{e_{-p/a}(\sigma(t),t_0)} - \rho(t)[a(t)\eta(t)]^{\Delta} + \frac{\rho(t)\delta_1(t,t_2)[a(\sigma(t))\eta(\sigma(t))]^{1+1/\gamma}}{r(t)} - \left[\frac{r(t)\rho^{\Delta}(t) + (\gamma+1)\rho(t)\delta_1(t,t_2)[a(\sigma(t))\eta(\sigma(t))]^{1/\gamma}}{(\gamma+1)r^{1/(\gamma+1)}(t)\rho^{\gamma/(\gamma+1)}(t)\delta_1^{\gamma/(\gamma+1)}(t,t_2)}\right]^{\gamma+1} \leq -\omega^{\Delta}(t). \tag{45}$$

Substituting t with s in (45) and multiplying both sides by $H(t,s)$ and then integrating with respect to s from t_2 to t yield

$$\begin{aligned}\int_{t_2}^{t} H(t,s) \left\{ L\frac{q(s)\rho(s)}{e_{-p/a}(\sigma(s),t_0)} - \rho(t)[a(s)\eta(s)]^{\Delta} + \frac{\rho(s)\delta_1(s,t_2)[a(\sigma(s))\eta(\sigma(s))]^{1+1/\gamma}}{r(s)} - \left[\frac{r(s)\rho^{\Delta}(s) + (\gamma+1)\rho(s)\delta_1(s,t_2)[a(\sigma(s))\eta(\sigma(s))]^{1/\gamma}}{(\gamma+1)r^{1/(\gamma+1)}(s)\rho^{\gamma/(\gamma+1)}(s)\delta_1^{\gamma/(\gamma+1)}(s,t_2)}\right]^{\gamma+1} \right\} \Delta s \\ \leq -\int_{t_2}^{t} H(t,s)\omega^{\Delta}(s)\Delta s = H(t,t_2)\omega(t_2) + \int_{t_2}^{t} H^{\Delta_s}(t,s)\omega(\sigma(s))\Delta s \leq H(t,t_2)\omega(t_2) \leq H(t,t_0)\omega(t_2).\end{aligned} \tag{46}$$

Then

$$\begin{aligned}&\int_{t_0}^{t} H(t,s) \left\{ L\frac{q(s)\rho(s)}{e_{-p/a}(\sigma(s),t_0)} - \rho(t)[a(s)\eta(s)]^{\Delta} + \frac{\rho(s)\delta_1(s,t_2)[a(\sigma(s))\eta(\sigma(s))]^{1+1/\gamma}}{r(s)} - \left[\frac{r(s)\rho^{\Delta}(s) + (\gamma+1)\rho(s)\delta_1(s,t_2)[a(\sigma(s))\eta(\sigma(s))]^{1/\gamma}}{(\gamma+1)r^{1/(\gamma+1)}(s)\rho^{\gamma/(\gamma+1)}(s)\delta_1^{\gamma/(\gamma+1)}(s,t_2)}\right]^{\gamma+1} \right\} \Delta s \\ &= \int_{t_0}^{t_2} H(t,s) \left\{ L\frac{q(s)\rho(s)}{e_{-p/a}(\sigma(s),t_0)} - \rho(t)[a(s)\eta(s)]^{\Delta} + \frac{\rho(s)\delta_1(s,t_2)[a(\sigma(s))\eta(\sigma(s))]^{1+1/\gamma}}{r(s)} - \left[\frac{r(s)\rho^{\Delta}(s) + (\gamma+1)\rho(s)\delta_1(s,t_2)[a(\sigma(s))\eta(\sigma(s))]^{1/\gamma}}{(\gamma+1)r^{1/(\gamma+1)}(s)\rho^{\gamma/(\gamma+1)}(s)\delta_1^{\gamma/(\gamma+1)}(s,t_2)}\right]^{\gamma+1} \right\} \Delta s\end{aligned}$$

$$
\begin{aligned}
&+\int_{t_2}^{t} H(t,s)\left\{L\frac{q(s)\rho(s)}{e_{-p/a}(\sigma(s),t_0)}-\rho(t)\left[a(s)\eta(s)\right]^{\Delta}+\frac{\rho(s)\delta_1(s,t_2)\left[a(\sigma(s))\eta(\sigma(s))\right]^{1+1/\gamma}}{r(s)}\right.\\
&\qquad\left.-\left[\frac{r(s)\rho^{\Delta}(s)+(\gamma+1)\rho(s)\delta_1(s,t_2)\left[a(\sigma(s))\eta(\sigma(s))\right]^{1/\gamma}}{(\gamma+1)r^{1/(\gamma+1)}(s)\rho^{\gamma/(\gamma+1)}(s)\delta_1^{\gamma/(\gamma+1)}(s,t_2)}\right]^{\gamma+1}\right\}\Delta s\\
&\le H(t,t_0)\omega(t_2)+H(t,t_0)\\
&\quad\times\int_{t_0}^{t_2}\left|L\frac{q(s)\rho(s)}{e_{-p/a}(\sigma(s),t_0)}-\rho(t)\left[a(s)\eta(s)\right]^{\Delta}+\frac{\rho(s)\delta_1(s,t_2)\left[a(\sigma(s))\eta(\sigma(s))\right]^{1+1/\gamma}}{r(s)}\right.\\
&\qquad\left.-\left[\frac{r(s)\rho^{\Delta}(s)+(\gamma+1)\rho(s)\delta_1(s,t_2)\left[a(\sigma(s))\eta(\sigma(s))\right]^{1/\gamma}}{(\gamma+1)r^{1/(\gamma+1)}(s)\rho^{\gamma/(\gamma+1)}(s)\delta_1^{\gamma/(\gamma+1)}(s,t_2)}\right]^{\gamma+1}\right|\Delta s.
\end{aligned}
\tag{47}
$$

So

$$
\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\left\{\int_{t_0}^{t} H(t,s)\left\{L\frac{q(s)\rho(s)}{e_{-p/a}(\sigma(s),t_0)}-\rho(t)\left[a(s)\eta(s)\right]^{\Delta}+\frac{\rho(s)\delta_1(s,t_2)\left[a(\sigma(s))\eta(\sigma(s))\right]^{1+1/\gamma}}{r(s)}\right.\right.\\
&\qquad\left.\left.-\left[\frac{r(s)\rho^{\Delta}(s)+(\gamma+1)\rho(s)\delta_1(s,t_2)\left[a(\sigma(s))\eta(\sigma(s))\right]^{1/\gamma}}{(\gamma+1)r^{1/(\gamma+1)}(s)\rho^{\gamma/(\gamma+1)}(s)\delta_1^{\gamma/(\gamma+1)}(s,t_2)}\right]^{\gamma+1}\right\}\Delta s\right\}\\
&\le\omega(t_2)+\int_{t_0}^{t_2}\left|L\frac{q(s)\rho(s)}{e_{-p/a}(\sigma(s),t_0)}-\rho(t)\left[a(s)\eta(s)\right]^{\Delta}+\frac{\rho(s)\delta_1(s,t_2)\left[a(\sigma(s))\eta(\sigma(s))\right]^{1+1/\gamma}}{r(s)}\right.\\
&\qquad\left.-\left[\frac{r(s)\rho^{\Delta}(s)+(\gamma+1)\rho(s)\delta_1(s,t_2)\left[a(\sigma(s))\eta(\sigma(s))\right]^{1/\gamma}}{(\gamma+1)r^{1/(\gamma+1)}(s)\rho^{\gamma/(\gamma+1)}(s)\delta_1^{\gamma/(\gamma+1)}(s,t_2)}\right]^{\gamma+1}\right|\Delta s<\infty,
\end{aligned}
\tag{48}
$$

which contradicts (44). So the proof is complete. □

Theorem 14. *Suppose $-p/a\in\mathfrak{R}_+$, and assume that* (7), (8), *and* (13) *hold. Let H be defined as in Theorem 13, and for all sufficiently large T,*

$$
\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\left\{\int_{t_0}^{t} H(t,s)\left\{L\frac{q(s)\rho(s)}{e_{-p/a}(\sigma(s),t_0)}-\rho(s)\left[a(s)\eta(s)\right]^{\Delta}\right.\right.\\
&\qquad+\frac{\gamma\rho(s)\delta_1(s,T)\delta_2^{\gamma-1}(\sigma(s),T)a^2(\sigma(s))\eta^2(\sigma(s))}{r(s)}\\
&\qquad\left.\left.-\frac{\left[r(s)\rho^{\Delta}(s)+2\gamma\rho(s)\delta_1(s,T)\delta_2^{\gamma-1}(\sigma(s),T)a(\sigma(s))\eta(\sigma(s))\right]^2}{4\gamma r(s)\rho(s)\delta_1(s,T)\delta_2^{\gamma-1}(\sigma(s),T)}\right\}\Delta s\right\}=\infty,
\end{aligned}
\tag{49}
$$

where ρ, η are defined as in Theorem 9. Then every solution of (2) *is oscillatory or tends to zero.*

Proof. Assume (2) has a nonoscillatory solution x on $\mathbb{T}_0$. Without loss of generality, we may assume $x(t)>0$ on $[t_1,\infty)_{\mathbb{T}}$, where t_1 is sufficiently large. By Lemmas 5 and 6, there exists sufficiently large t_2 such that $[r(t)x^{\Delta}(t)]^{\Delta}>0$ on $[t_2,\infty)_{\mathbb{T}}$, and either $x^{\Delta}(t)>0$ on $[t_2,\infty)_{\mathbb{T}}$ or $\lim_{t\to\infty}x(t)=0$. Now we assume $x^{\Delta}(t)>0$ on $[t_2,\infty)_{\mathbb{T}}$. Let $\omega(t)$ be defined as in Theorem 9. By (41) we have

$$L\frac{q(t)\rho(t)}{e_{-p/a}(\sigma(t),t_0)}-\rho(t)\left[a(t)\eta(t)\right]^{\Delta}+\frac{\gamma\rho(t)\delta_1(t,t_2)\delta_2^{\gamma-1}(\sigma(t),t_2)a^2(\sigma(t))\eta^2(\sigma(t))}{r(t)}$$
$$-\frac{\left[r(t)\rho^{\Delta}(t)+2\gamma\rho(t)\delta_1(t,t_2)\delta_2^{\gamma-1}(\sigma(t),t_2)a(\sigma(t))\eta(\sigma(t))\right]^2}{4\gamma r(t)\rho(t)\delta_1(t,t_2)\delta_2^{\gamma-1}(\sigma(t),t_2)}\le-\omega^{\Delta}(t). \tag{50}$$

Substituting t with s in (45), multiplying both sides by $H(t,s)$, and then integrating with respect to s from t_2 to t yield

$$\int_{t_2}^{t}H(t,s)\left\{L\frac{q(s)\rho(s)}{e_{-p/a}(\sigma(s),t_0)}-\rho(s)\left[a(s)\eta(s)\right]^{\Delta}\right.$$
$$+\frac{\gamma\rho(s)\delta_1(s,t_2)\delta_2^{\gamma-1}(\sigma(s),t_2)a^2(\sigma(s))\eta^2(\sigma(s))}{r(s)}$$
$$\left.-\frac{\left[r(s)\rho^{\Delta}(s)+2\gamma\rho(s)\delta_1(s,t_2)\delta_2^{\gamma-1}(\sigma(s),t_2)a(\sigma(s))\eta(\sigma(s))\right]^2}{4\gamma r(s)\rho(s)\delta_1(s,t_2)\delta_2^{\gamma-1}(\sigma(s),t_2)}\right\}\Delta s \tag{51}$$
$$\le-\int_{t_2}^{t}H(t,s)\omega^{\Delta}(s)\Delta s=H(t,t_2)\omega(t_2)+\int_{t_2}^{t}H^{\Delta_s}(t,s)\omega(\sigma(s))\Delta s$$
$$\le H(t,t_2)\omega(t_2)\le H(t,t_0)\omega(t_2).$$

Then similar to Theorem 13, we obtain

$$\lim_{t\to\infty}\sup\frac{1}{H(t,t_0)}$$
$$\times\left\{\int_{t_0}^{t}H(t,s)\left\{L\frac{q(s)\rho(s)}{e_{-p/a}(\sigma(s),t_0)}-\rho(s)\left[a(s)\eta(s)\right]^{\Delta}+\frac{\gamma\rho(s)\delta_1(s,t_2)\delta_2^{\gamma-1}(\sigma(s),t_2)a^2(\sigma(s))\eta^2(\sigma(s))}{r(s)}\right.\right.$$
$$\left.\left.-\frac{\left[r(s)\rho^{\Delta}(s)+2\gamma\rho(s)\delta_1(s,t_2)\delta_2^{\gamma-1}(\sigma(s),t_2)a(\sigma(s))\eta(\sigma(s))\right]^2}{4\gamma r(s)\rho(s)\delta_1(s,t_2)\delta_2^{\gamma-1}(\sigma(s),t_2)}\right\}\Delta s\right\}$$
$$\le\omega(t_2)+\int_{t_0}^{t_2}\left|L\frac{q(s)\rho(s)}{e_{-p/a}(\sigma(s),t_0)}-\rho(s)\left[a(s)\eta(s)\right]^{\Delta}\right. \tag{52}$$
$$+\frac{\gamma\rho(s)\delta_1(s,t_2)\delta_2^{\gamma-1}(\sigma(s),t_2)a^2(\sigma(s))\eta^2(\sigma(s))}{r(s)}$$
$$\left.-\frac{\left[r(s)\rho^{\Delta}(s)+2\gamma\rho(s)\delta_1(s,t_2)\delta_2^{\gamma-1}(\sigma(s),t_2)a(\sigma(s))\eta(\sigma(s))\right]^2}{4\gamma r(s)\rho(s)\delta_1(s,t_2)\delta_2^{\gamma-1}(\sigma(s),t_2)}\right|\Delta s<\infty,$$

which contradicts (44). So the proof is complete. □

In Theorems 13 and 14, if we take $H(t,s)$ for some special functions such as $(t-s)^m$ or $\ln(t/s)$, then we can obtain some corollaries. For example, if we take $H(t,s)=(t-s)^m$, $m \geq 1$, then we have the following corollaries.

Corollary 15. *Suppose $-p/a \in \mathfrak{R}_+$, and assume that (7), (8), and (13) hold, and for all sufficiently large T,*

$$\limsup_{t\to\infty} \frac{1}{(t-t_0)^m} \times \left\{ \int_{t_0}^{t} (t-s)^m \left\{ L\frac{q(s)\rho(s)}{e_{-p/a}(\sigma(s),t_0)} - \rho(t)\left[a(s)\eta(s)\right]^{\Delta} + \frac{\rho(s)\delta_1(s,T)\left[a(\sigma(s))\eta(\sigma(s))\right]^{1+1/\gamma}}{r(s)} - \left[\frac{r(s)\rho^{\Delta}(s)+(\gamma+1)\rho(s)\delta_1(s,T)\left[a(\sigma(s))\eta(\sigma(s))\right]^{1/\gamma}}{(\gamma+1)r^{1/(\gamma+1)}(s)\rho^{\gamma/(\gamma+1)}(s)\delta_1^{\gamma/(\gamma+1)}(s,T)}\right]^{\gamma+1} \right\} \Delta s \right\} = \infty. \tag{53}$$

Then every solution of (2) is oscillatory or tends to zero.

Corollary 16. *Suppose $-p/a \in \mathfrak{R}_+$, and assume that (7), (8), and (13) hold, and for all sufficiently large T,*

$$\limsup_{t\to\infty} \frac{1}{(t-t_0)^m} \times \left\{ \int_{t_0}^{t} (t-s)^m \left\{ L\frac{q(s)\rho(s)}{e_{-p/a}(\sigma(s),t_0)} - \rho(s)\left[a(s)\eta(s)\right]^{\Delta} + \frac{\gamma\rho(s)\delta_1(s,T)\delta_2^{\gamma-1}(\sigma(s),T)a^2(\sigma(s))\eta^2(\sigma(s))}{r(s)} - \frac{\left[r(s)\rho^{\Delta}(s)+2\gamma\rho(s)\delta_1(s,T)\delta_2^{\gamma-1}(\sigma(s),T)a(\sigma(s))\eta(\sigma(s))\right]^2}{4\gamma r(s)\rho(s)\delta_1(s,T)\delta_2^{\gamma-1}(\sigma(s),T)} \right\} \Delta s \right\} = \infty. \tag{54}$$

Then every solution of (2) is oscillatory or tends to zero.

Remark 17. The established results above extend the main results in [25, Theorems 2.1–2.4] except that the latter is related to time delay.

Remark 18. In Theorems 12–14, if we take $\mathbb{T}$ for some special time scales, we can obtain similar results as in Corollaries 10 and 11, which are omitted here.

In Theorems 9, 12, 13, and 14, if we let $p = 0$, then $e_{-p/a}(t,t_0) \equiv 1$, and subsequently we obtain the following four corollaries concerning oscillatory criteria of the following equation:

$$\left(a(t)\left(\left[r(t)x^{\Delta}(t)\right]^{\Delta}\right)^{\gamma}\right)^{\Delta} + q(t)f(x(t)) = 0, \quad t \in \mathbb{T}_0. \tag{55}$$

Corollary 19. *Assume (8) holds. If*

$$\int_{t_0}^{\infty} \frac{1}{a^{1/\gamma}(s)}\Delta s = \infty, \tag{56}$$

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[\frac{1}{r(\xi)} \int_{\xi}^{\infty} \left(\frac{1}{a(\tau)} \int_{\tau}^{\infty} q(s)\Delta s\right)^{1/\gamma} \Delta\tau\right] \Delta\xi = \infty. \tag{57}$$

and for all sufficiently large T,

$$\limsup_{t\to\infty} \left\{ \int_{T}^{t} \left\{ Lq(s)\rho(s) - \rho(t)\left[a(s)\eta(s)\right]^{\Delta} + \frac{\rho(s)\delta_1(s,T)\left[a(\sigma(s))\eta(\sigma(s))\right]^{1+1/\gamma}}{r(s)} - \left[\frac{r(s)\rho^{\Delta}(s)+(\gamma+1)\rho(s)\delta_1(s,T)\left[a(\sigma(s))\eta(\sigma(s))\right]^{1/\gamma}}{(\gamma+1)r^{1/(\gamma+1)}(s)\rho^{\gamma/(\gamma+1)}(s)\delta_1^{\gamma/(\gamma+1)}(s,T)}\right]^{\gamma+1} \right\} \Delta s \right\} = \infty, \tag{58}$$

where $\rho,\ \eta$ are defined as in Theorem 9. Then every solution of (2) is oscillatory or tends to zero.

Corollary 20. *Assume that (8), (56), and (57) hold, and for all sufficiently large T,*

$$\lim_{t\to\infty}\sup\left\{\int_T^t\left\{Lq(s)\rho(s)-\rho(s)[a(s)\eta(s)]^{\Delta}+\frac{\gamma\rho(s)\delta_1(s,T)\delta_2^{\gamma-1}(\sigma(s),T)a^2(\sigma(s))\eta^2(\sigma(s))}{r(s)}-\frac{\left[r(s)\rho^{\Delta}(s)+2\gamma\rho(s)\delta_1(s,T)\delta_2^{\gamma-1}(\sigma(s),T)a(\sigma(s))\eta(\sigma(s))\right]^2}{4\gamma r(s)\rho(s)\delta_1(s,T)\delta_2^{\gamma-1}(\sigma(s),T)}\right\}\Delta s\right\}=\infty, \tag{59}$$

where $\rho,\ \eta$ are defined as in Theorem 9. Then every solution of (2) is oscillatory or tends to zero.

Corollary 21. *Assume that (8), (56), and (57) hold. If for all sufficiently large T,*

$$\lim_{t\to\infty}\sup\frac{1}{H(t,t_0)}\left\{\int_{t_0}^t H(t,s)\left\{Lq(s)\rho(s)-\rho(t)[a(s)\eta(s)]^{\Delta}+\frac{\rho(s)\delta_1(s,T)[a(\sigma(s))\eta(\sigma(s))]^{1+1/\gamma}}{r(s)}-\left[\frac{r(s)\rho^{\Delta}(s)+(\gamma+1)\rho(s)\delta_1(s,T)[a(\sigma(s))\eta(\sigma(s))]^{1/\gamma}}{(\gamma+1)r^{1/(\gamma+1)}(s)\rho^{\gamma/(\gamma+1)}(s)\delta_1^{\gamma/(\gamma+1)}(s,T)}\right]^{\gamma+1}\right\}\Delta s\right\}=\infty, \tag{60}$$

where $\rho,\ \eta,\ H$ are defined as in Theorems 9 and 13, respectively, then every solution of (2) is oscillatory or tends to zero.

Corollary 22. *Assume that (8), (56), and (57) hold. If for all sufficiently large T,*

$$\lim_{t\to\infty}\sup\frac{1}{H(t,t_0)}\left\{\int_{t_0}^t H(t,s)\left\{L\frac{q(s)\rho(s)}{e_{-p/a}(\sigma(s),t_0)}-\rho(s)[a(s)\eta(s)]^{\Delta}+\frac{\gamma\rho(s)\delta_1(s,T)\delta_2^{\gamma-1}(\sigma(s),T)a^2(\sigma(s))\eta^2(\sigma(s))}{r(s)}-\frac{\left[r(s)\rho^{\Delta}(s)+2\gamma\rho(s)\delta_1(s,T)\delta_2^{\gamma-1}(\sigma(s),T)a(\sigma(s))\eta(\sigma(s))\right]^2}{4\gamma r(s)\rho(s)\delta_1(s,T)\delta_2^{\gamma-1}(\sigma(s),T)}\right\}\Delta s\right\}=\infty, \tag{61}$$

where $\rho,\ \eta,\ H$ are defined as in Theorems 9 and 13, respectively, then every solution of (2) is oscillatory or tends to zero.

Remark 23. In [34, Theorems 3.3-3.4] and [35, Theorems 2.7–2.9], Saker established some new oscillatory criteria for the equation

$$\left(p(t)\left[\left(r(t)x^{\Delta}(t)\right)^{\Delta}\right]^{\gamma}\right)^{\Delta}+q(t)f(x(\tau(t)))=0 \tag{62}$$

under the condition $\tau(t)\le t$. We note that the conditions (8) and (56) in Corollaries 19–22 are consistent with those in [34, (24)] and [35, (1.12)], which were used in [34, Theorems 3.3-3.4] and [35, Theorems 2.7–2.9], respectively, while $r(t)\equiv 1$ is assumed in [34, 35]. Moreover, in the results established above, the Riccati substitution function is defined by $\omega(t)=\rho(t)a(t)[([r(t)x^{\Delta}(t)]^{\Delta})^{\gamma}/x^{\gamma}(t)e_{-p/a}(t,t_0)+\eta(t)]$ (see Theorem 9), which is different form that in [34, Theorem 3.3] and [35, Theorem 2.7], where the Riccati substitution function is defined by $u(t)=y^{[2]}/(y^{[1]})^{\gamma}=x^{\Delta\Delta}/(x^{\Delta})^{\gamma}$. Since the Riccati substitution function is the most important fact in establishing sufficient oscillatory conditions, so our results in Corollaries 19–22 are essentially different from Saker's results in [34, 35].

3. Applications

In this section, we will present some applications for the established results above. First we consider the following third-order nonlinear differential equation with damping term.

Example 24. Consider

$$\left[\left(tx''(t)\right)^{\gamma}\right]'+\frac{1}{t^{\gamma+1}}\left(x''(t)\right)^{\gamma}$$

$$+\frac{1}{t^{\gamma+1}}x^{\gamma}(t)\left[e^{x(t)}+1\right]=0,\quad t\in[2,\infty),\tag{63}$$

where $\gamma\geq 1$ is a quotient of two odd positive integers.

We have in (2) $\mathbb{T}=\mathbb{R}$, $a(t)=t^{\gamma}$, $p(t)=q(t)=1/t^{\gamma+1}$, $f(x)=x^{\gamma}[e^{x}+1]$, $r(t)=1$, $t_0=2$. Then $f(x)/x^{\gamma}\geq 1=L$, $\mu(t)=\sigma(t)-t=0$, and $-p/a\in\mathfrak{R}_{+}$. So $e_{-p/a}(t,t_0)=e_{-p/a}(t,2)=\exp(-\int_2^t(p(s)/a(s))ds)$. Moreover, we have

$$\begin{aligned}1&>\exp\left(-\int_2^t\frac{p(s)}{a(s)}ds\right)\geq 1-\int_2^t\frac{p(s)}{a(s)}ds\\&=1-\int_2^t\frac{1}{s^{2\gamma+1}}ds=1+\frac{1}{2\gamma}\left[t^{-2\gamma}-2^{-2\gamma}\right]>\frac{1}{2}.\end{aligned}\tag{64}$$

Then we have

$$\begin{gathered}\int_{t_0}^{\infty}\frac{\left[e_{-p/a}(s,t_0)\right]^{1/\gamma}}{a^{1/\gamma}(s)}ds>\frac{1}{2^{1/\gamma}}\int_2^{\infty}\frac{1}{s}ds=\infty,\\\int_{t_0}^{\infty}\frac{1}{r(s)}ds=\infty.\end{gathered}\tag{65}$$

Furthermore,

$$\begin{aligned}&\int_{t_0}^{\infty}\left[\frac{1}{r(\xi)}\int_{\xi}^{\infty}\left(\frac{e_{-p/a}(\tau,t_0)}{a(\tau)}\int_{\tau}^{\infty}\frac{q(s)}{e_{-p/a}(s,t_0)}ds\right)^{1/\gamma}d\tau\right]d\xi\\&=\int_2^{\infty}\left[\int_{\xi}^{\infty}\left(\frac{e_{-p/a}(\tau,2)}{\tau^{\gamma}}\int_{\tau}^{\infty}\frac{1}{s^{\gamma+1}e_{-p/a}(s,2)}ds\right)^{1/\gamma}d\tau\right]d\xi\\&>\frac{1}{2^{1/\gamma}}\int_2^{\infty}\left[\int_{\xi}^{\infty}\left(\frac{1}{\tau^{\gamma}}\int_{\tau}^{\infty}\frac{1}{s^{\gamma+1}}ds\right)^{1/\gamma}d\tau\right]d\xi\\&=\frac{1}{(2\gamma)^{1/\gamma}}\int_2^{\infty}\left[\int_{\xi}^{\infty}\frac{1}{\tau^2}d\tau\right]d\xi=\frac{1}{(2\gamma)^{1/\gamma}}\int_2^{\infty}\frac{1}{\xi}d\xi=\infty.\end{aligned}\tag{66}$$

On the other hand, for a sufficiently large T, we have

$$\delta_1(t,T)=\int_T^t\frac{\left[e_{-p/a}(s,t_0)\right]^{1/\gamma}}{a^{1/\gamma}(s)}ds>\frac{1}{2^{1/\gamma}}\int_T^t\frac{1}{s}ds\longrightarrow\infty.\tag{67}$$

So there exists a sufficiently large $T^*>T$ such that $\delta_1(t,T)>1$ for $t\in[T^*,\infty)$. Taking $\rho(t)=t^{\gamma}$, $\eta(t)=0$ in (36), we get that

$$\begin{aligned}&\limsup_{t\to\infty}\left\{\int_T^t\left\{L\frac{q(s)\rho(s)}{e_{-p/a}(s,t_0)}-\left[\frac{r(s)\rho'(s)}{(\gamma+1)r^{1/(\gamma+1)}(s)\rho^{\gamma/(\gamma+1)}(s)\delta_1^{\gamma/(\gamma+1)}(s,T)}\right]^{\gamma+1}\right\}ds\right\}\\&=\limsup_{t\to\infty}\left\{\int_T^{T^*}\left\{L\frac{q(s)\rho(s)}{e_{-p/a}(s,t_0)}-\left[\frac{r(s)\rho'(s)}{(\gamma+1)r^{1/(\gamma+1)}(s)\rho^{\gamma/(\gamma+1)}(s)\delta_1^{\gamma/(\gamma+1)}(s,T)}\right]^{\gamma+1}\right\}ds\right.\\&\qquad\left.+\int_{T^*}^t\left\{L\frac{q(s)\rho(s)}{e_{-p/a}(s,t_0)}-\left[\frac{r(s)\rho'(s)}{(\gamma+1)r^{1/(\gamma+1)}(s)\rho^{\gamma/(\gamma+1)}(s)\delta_1^{\gamma/(\gamma+1)}(s,T)}\right]^{\gamma+1}\right\}ds\right\}\\&>\limsup_{t\to\infty}\left\{\int_T^{T^*}\left\{L\frac{q(s)\rho(s)}{e_{-p/a}(s,t_0)}-\left[\frac{r(s)\rho'(s)}{(\gamma+1)r^{1/(\gamma+1)}(s)\rho^{\gamma/(\gamma+1)}(s)\delta_1^{\gamma/(\gamma+1)}(s,T)}\right]^{\gamma+1}\right\}ds\right.\\&\qquad\left.+\int_{T^*}^t\left[1-\left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1}\right]\frac{1}{s}ds\right\}\longrightarrow\infty.\end{aligned}\tag{68}$$

So (35)–(36) all hold, and by Corollary 10 we deduce that every solution of (63) is oscillatory or tends to zero.

Next we consider the following third-order difference equation:

Example 25. Consider

$$\begin{aligned}\Delta\left[\left(t\Delta^2x(t)\right)^{\gamma}\right]+\frac{1}{t^{\gamma+1}}\left(\Delta^2x(t)\right)^{\gamma}+\frac{M}{t^{\gamma+1}}x^{\gamma}(t)=0,&\\t\in[2,\infty)_{\mathbb{Z}},&\end{aligned}\tag{69}$$

where Δ denotes the difference operator, $M>0$ is a constant, and $\gamma\geq 1$ is a quotient of two odd positive integers.

We have in (2) $\mathbb{T}=\mathbb{Z}$, $a(t)=t^{\gamma}$, $p(t)=q(t)=1/t^{\gamma+1}$, $f(x)=Mx^{\gamma}$, $r(t)=1$, $t_0=2$. Then $f(x)/x^{\gamma}\geq M=L$, $\mu(t)=\sigma(t)-t=1$, and

$$1-\mu(t)\frac{p(t)}{a(t)}=1-\frac{1}{t^{2\gamma+1}}\geq 1-\frac{1}{2}>0,\tag{70}$$

which implies $-p/a\in\mathfrak{R}_{+}$. So by [2, Lemma 2] we obtain

$$e_{-p/a}(t,t_0)=e_{-p/a}(t,2)\geq 1-\int_2^t\frac{p(s)}{a(s)}\Delta s$$

$$
\begin{aligned}
&= 1 - \int_2^t \frac{1}{s^{2\gamma+1}} \Delta s = 1 - \sum_{s=2}^{t-1} \frac{1}{s^{2\gamma+1}} \\
&\geq 1 - \int_1^{t-1} \frac{1}{s^{2\gamma+1}} ds \\
&= 1 + \frac{1}{2\gamma} \left[(t-1)^{-2\gamma} - 1 \right] > \frac{1}{2}, \\
&e_{-p/a}(t, t_0) \leq \exp\left(- \int_2^t \frac{p(s)}{a(s)} \Delta s \right) < 1.
\end{aligned} \tag{71}
$$

Then we have

$$
\begin{aligned}
&\sum_{s=t_0}^{\infty} \frac{\left[e_{-p/a}(s, t_0) \right]^{1/\gamma}}{a^{1/\gamma}(s)} \\
&= \sum_{s=2}^{\infty} \frac{\left[e_{-p/a}(s, 2) \right]^{1/\gamma}}{a^{1/\gamma}(s)} \\
&= \sum_{s=2}^{\infty} \frac{\left[e_{-p/a}(s, 2) \right]^{1/\gamma}}{s} > \frac{1}{2^{1/\gamma}} \sum_{s=2}^{\infty} \frac{1}{s} = \infty, \\
&\sum_{s=t_0}^{\infty} \frac{1}{r(s)} = \infty.
\end{aligned} \tag{72}
$$

Furthermore,

$$
\begin{aligned}
&\sum_{\xi=t_0}^{\infty} \left[\frac{1}{r(\xi)} \sum_{\tau=\xi}^{\infty} \left(\frac{e_{-p/a}(\tau, t_0)}{a(\tau)} \sum_{s=\tau}^{\infty} \frac{q(s)}{e_{-p/a}(s+1, t_0)} \right)^{1/\gamma} \right] \\
&= \sum_{\xi=2}^{\infty} \left[\frac{1}{r(\xi)} \sum_{\tau=\xi}^{\infty} \left(\frac{e_{-p/a}(\tau, 2)}{a(\tau)} \times \sum_{s=\tau}^{\infty} \frac{q(s)}{e_{-p/a}(s+1, 2)} \right)^{1/\gamma} \right] \\
&> \frac{1}{2^{1/\gamma}} \sum_{\xi=2}^{\infty} \left[\sum_{\tau=\xi}^{\infty} \left(\frac{1}{\tau^\gamma} \sum_{s=\tau}^{\infty} \frac{1}{s^{\gamma+1}} \right)^{1/\gamma} \right] \\
&> \frac{1}{2^{1/\gamma}} \sum_{\xi=2}^{\infty} \left[\sum_{\tau=\xi}^{\infty} \left(\frac{1}{\tau^\gamma} \int_{s=\tau}^{\infty} \frac{1}{s^{\gamma+1}} ds \right)^{1/\gamma} \right] \\
&= \frac{1}{(2\gamma)^{1/\gamma}} \sum_{\xi=2}^{\infty} \left[\sum_{\tau=\xi}^{\infty} \frac{1}{\tau^2} \right] \\
&> \frac{1}{(2\gamma)^{1/\gamma}} \sum_{\xi=2}^{\infty} \sum_{\tau=\xi}^{\infty} \frac{1}{\tau(\tau+1)} = \frac{1}{(2\gamma)^{1/\gamma}} \sum_{\xi=2}^{\infty} \frac{1}{\xi} = \infty.
\end{aligned} \tag{73}
$$

On the other hand, for a sufficiently large $T > 1$, we have

$$
\delta_1(t, T) = \sum_{s=T}^{t-1} \frac{\left[e_{-p/a}(s, t_0) \right]^{1/\gamma}}{a^{1/\gamma}(s)} > \frac{1}{2^{1/\gamma}} \sum_{s=T}^{t-1} \frac{1}{s} \longrightarrow \infty. \tag{74}
$$

So there exists $T^* > T$ such that $\delta_1(s, T) > 1$ for $t \in [T^*, \infty)_{\mathbb{Z}}$. Let $\rho(t) = t^\gamma$, $\eta(t) = 0$ in (38). Then by the inequality $(t+1)^\gamma - t^\gamma \leq \gamma(t+1)^{\gamma-1} < \gamma 2^{\gamma-1} t^{\gamma-1}$, $t \geq T^*$, we obtain

$$
\begin{aligned}
&\lim_{t\to\infty} \sup \left\{ \sum_{s=T}^{t-1} \left\{ L \frac{q(s)\rho(s)}{e_{-p/a}(s+1, t_0)} - \left[\frac{r(s)(\rho(s+1) - \rho(s))}{(\gamma+1) r^{1/(\gamma+1)}(s) \rho^{\gamma/(\gamma+1)}(s) \delta_1^{\gamma/(\gamma+1)}(s, T)} \right]^{\gamma+1} \right\} \right\} \\
&= \lim_{t\to\infty} \sup \left\{ \sum_{s=T}^{T^*} \left\{ L \frac{q(s)\rho(s)}{e_{-p/a}(s+1, t_0)} - \left[\frac{r(s)(\rho(s+1) - \rho(s))}{(\gamma+1) r^{1/(\gamma+1)}(s) \rho^{\gamma/(\gamma+1)}(s) \delta_1^{\gamma/(\gamma+1)}(s, T)} \right]^{\gamma+1} \right\} \right. \\
&\qquad \left. + \sum_{s=T^*}^{t-1} \left\{ L \frac{q(s)\rho(s)}{e_{-p/a}(s+1, t_0)} - \left[\frac{r(s)(\rho(s+1) - \rho(s))}{(\gamma+1) r^{1/(\gamma+1)}(s) \rho^{\gamma/(\gamma+1)}(s) \delta_1^{\gamma/(\gamma+1)}(s, T)} \right]^{\gamma+1} \right\} \right\} \\
&> \lim_{t\to\infty} \sup \left\{ \sum_{s=T}^{T^*} \left\{ L \frac{q(s)\rho(s)}{e_{-p/a}(s+1, t_0)} - \left[\frac{r(s)(\rho(s+1) - \rho(s))}{(\gamma+1) r^{1/(\gamma+1)}(s) \rho^{\gamma/(\gamma+1)}(s) \delta_1^{\gamma/(\gamma+1)}(s, T)} \right]^{\gamma+1} \right\} \right. \\
&\qquad \left. + \sum_{s=T^*}^{t-1} \left[M - \left(\frac{\gamma}{\gamma+1} \right)^{\gamma+1} 2^{\gamma^2-1} \right] \frac{1}{s} \right\} \longrightarrow \infty,
\end{aligned} \tag{75}
$$

provided that $M > (\gamma/(\gamma+1))^{\gamma+1} 2^{\gamma^2-1}$. So (37) and (38) all hold, and by Corollary 11 we obtain that every solution of (69) is oscillatory or tends to zero under the condition $M > (\gamma/(\gamma+1))^{\gamma+1} 2^{\gamma^2-1}$.

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