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Research Article

The Validity of Dimensional Regularization Method on Fractal Spacetime

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Svozil developed a regularization method for quantum field theory on fractal spacetime (1987). Such a method can be applied to the low-order perturbative renormalization of quantum electrodynamics but will depend on a conjectural integral formula on non-integer-dimensional topological spaces. The main purpose of this paper is to construct a fractal measure so as to guarantee the validity of the conjectural integral formula.

1. Introduction

The quantum field theory is one of the oldest fundamental and most widely used tools in physics. It is spectacularly successful that the value of theoretical calculation is precisely in agreement with experimental data, for example, the anomalous magnet moment of electron. Nevertheless, such a precise calculation is on the basis of some regularization methods, for example, dimensional regularization [1]. The dimensional regularization requires that **S** matrix should be calculated in a non-integer-dimensional spacetime. Following the spirit of this heuristic calculation, Svozil [2] developed the quantum field theory on fractal spacetime (QFTFS). This approach not only can be applied to the low-order perturbative renormalization of quantum electrodynamics but also preserves the gauge invariance and covariance of physical equations. Svozil's work implied that, for a D-dimensional spacetime, there might be $D = 4$ for macroscopic and $D < 4$ for microscopic events [2].

Interestingly, recently, the investigations for a consistent theory of quantum gravity strongly indicate that a power-counting renormalizable gravity model can be achieved in a fractional dimensional spacetime, for example, the Horava-Lifshitz (HL) gravity model [3, 4]. Unfortunately, HL gravity model is not Lorentz invariant. To maintain the Lorentz invariance, Calcagni [5, 6] extended the theoretical framework of Svozil's QFTFS so as to include the description for gravity. Calcagni's work showed that if the Hausdorff dimension of spacetime $D \sim 2$, then the ultraviolet divergence could be removed.

In fact, the notion that "the Universe is fractal" at quantum scales has become popular [5–7]. Thus, the demand for a generalized calculus theory on the non-integer-dimensional topological spaces is strongly increasing. Unfortunately, there is still not a rigorous calculus theory for analytically describing fractal so that, as Svozil has mentioned [2], some of the approaches of fractal calculus are essentially conjectural. It is worth mentioning that Svozil's QFTFS is just on the basis of a conjectural integral formula on the non-integer-dimensional topological spaces. Because of the importance of Svozil's approach in studying quantum gravity, we suggest to construct a fractal measure so as to guarantee the validity of Svozil's conjectural integral formula.

2. Hausdorff Measure

The mathematical basis of QFTFS is the Hausdorff measure [2]. The introduction for Hausdorff measure can be found in Appendix A. If a D-dimensional fractal Ω is embedded in R^n, then it can be tessellated into (regular) polyhedra. In particular, it is always possible to divide R^n into parallelepipeds of the form [2]:

$$E_{i_1,\dots,i_n} = \left\{(x_1,\dots,x_n) \in \Omega : x_j = \left(i_j - 1\right)\Delta x_j + \alpha_j,\right.$$
$$\left. 0 \le \alpha_j \le \Delta x_j,\ j = 1,\dots,n\right\}. \tag{1}$$

Based on such a set of parallelepipeds, Svozil [2] conjectured that the local Hausdorff measure $d\mu_H(\Omega)$ of Ω should yield the following form:

$$d\mu_H(\Omega) = \lim_{d(E_{i_1,\dots,i_n})\to 0} \prod_{j=1}^{n} \left(\Delta x_j\right)^{D/n} = \prod_{j=1}^{n} d^{D/n} x_j, \tag{2}$$

where $d(E_{i_1,\dots,i_n})$ denotes the diameter of parallelepiped $E_{i_1,\dots,i_n}$ and $d^{D/n}$ denotes the differential operator of order D/n.

With these preparations above, Svozil [2] proved that if formula (2) holds, then the integral of a spherically symmetric function $f(r)$ on Ω can be written as

$$\int_\Omega f(r)\, d\mu_H = \frac{2\pi^{D/2}}{\Gamma(D/2)} \int_0^R f(r)\, r^{D-1} dr. \tag{3}$$

The integral formula (3) is the starting point of QFTFS; therefore, we must pay much attention to the validity of formula (2). Nevertheless, formula (2) does not always hold whenever $D \neq n$; for example, using the fractional derivative [8], it is easy to check that $d^{D/n}x_j/(dx_j)^{D/n} \neq 1$ if $D \neq n$. This means that the integral formula (3) does not always hold in the framework of Hausdorff measure.

In fact, Hausdorff measure is not an ideal mathematical framework for describing fractal. Next, we will see that Hausdorff measure indeed determines the dimension of a fractal curve but does not describe its analytic properties, for example, the self-similarity between local and global shapes of a fractal curve. To realize this fact, we attempt to check the case of the Cantor set; see Figure 1 [8, 9].

As shown by Figure 1, the Cantor set is a fractal. Using the Hausdorff measure (A.8) (see Appendix A) we can compute the dimension of the Cantor set as [9]

$$D = \frac{\ln 2}{\ln 3} = 0.6309\cdots. \tag{4}$$

Nevertheless, for the Cantor set, we do not realize any correlation between its local and global segments (i.e., self-similarity) via the Hausdorff measure. For instance, the Hausdorff distance between points $x_2^{(3)}$ and $x_1^{(3)}$ is denoted by

$$H^D\left(x_2^{(3)}, x_1^{(3)}\right) = \left|x_2^{(3)} - x_1^{(3)}\right|^D. \tag{5}$$

Obviously, Hausdorff distance (5) is independent of the values of points $x_i^{(3)}$, where i runs from 3 to 8. Nevertheless, because of the self-similarity between parts of the Cantor set, any displacement of point $x_i^{(3)}$ $(i = 3, 4, \dots, 8)$ should influence the distance between $x_2^{(3)}$ and $x_1^{(3)}$. This is undoubtedly a nonlocal property. Unfortunately, Hausdorff distance (5) fails to show this property. In the next section, we will construct a new fractal measure so as to exhibit such a nonlocal property.

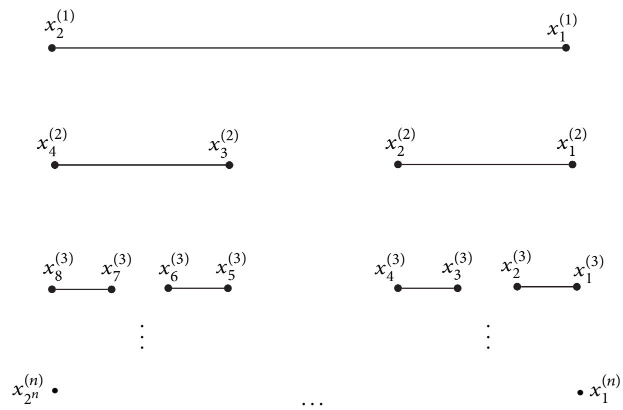

Figure 1: The Cantor ternary set is defined by repeatedly removing the middle thirds of line segments [8, 9]. (a) One starts by removing the middle third from the interval $[x_2^{(1)}, x_1^{(1)}]$, leaving $[x_4^{(2)}, x_3^{(2)}]$ and $[x_2^{(2)}, x_1^{(2)}]$. (b) Next, the "middle third" of all remaining intervals is removed. (c) This process is continued ad infinitum. Finally, the Cantor ternary set consists of all points in the interval $[x_2^{(1)}, x_1^{(1)}]$ that are not removed at any step in this infinite process.

3. Fractal Measure

In Section 2, we have noted that the key point of guaranteeing the validity of the integral formula (3) is that the Hausdorff measure is compelled to equal some differences of fractional order; that is, formula (2) holds. Such a fact reminds us that the differences of fractional order itself may be a type of measure. An interesting thought is that whether or not the differences of fractional order can describe the nonlocal property of a fractal curve. To this end, we attempt to check a m-dimensional volume:

$$x(l) = \omega(m)\, l^m, \tag{6}$$

where $\omega(m)$ is a constant factor that depends only on the dimension m and m may be a fraction.

The fractional derivatives of order m of $x(l)$ give [8]

$$d^m x(l) = \Gamma(m+1)\,(dl)^m \sim (dl)^m. \tag{7}$$

Obviously, $(dl)^m$, as a m-dimensional volume, is a m-dimensional Hausdorff measure; therefore, formula (7) implies that the differences of order m, $d^m x(l)$, can be also thought of as a measure for describing the length of a m-dimensional fractal curve. In this case, the order of differences $d^m x(l)$ represents the Hausdorff dimension m.

Using the differences of order m, we define a new distance—call it the "nonlocal distance"—in the form (see (A.19) in Appendix A):

$$\left|\Delta_m[x(l), x(l-\Delta l)]\right| = \left|\sum_{j=0}^{\infty} \frac{m(m-1)\cdots(m-j+1)(-1)^j}{j!} x(l - j\Delta l)\right|, \tag{8}$$

where $|\Delta_m[x(l), x(l-\Delta l)]|$ denotes the nonlocal distance between points $x(l)$ and $x(l-\Delta l)$.

It is carefully noted that every $x(l - j\Delta l)$ represents a point on a m-dimensional fractal curve, where, $j = 0, 1, 2, \ldots$.

Clearly, according to (8), the distance between points $x(l)$ and $x(l - \Delta l)$ would depend on all the points $x(l - j\Delta l)$, where $j = 0, 1, 2, \ldots$.

Moreover, it's easy to check that $|\Delta_{m=1}[x(l), x(l - \Delta l)]| = |x(l) - x(l - \Delta l)|$. This means that the Euclidean distance is a special case of nonlocal distance whenever the dimension of the fractal curve, m, equals 1.

If we use the nonlocal distance (8) to measure the distance between points $x_2^{(3)}$ and $x_1^{(3)}$ (see Figure 1), then we will surprisingly find that the nonlocal distance

$$\begin{aligned}&\left|\Delta_{m=D}\left[x_2^{(3)}, x_1^{(3)}\right]\right|\\&\quad = \left|\sum_{j=1}^{8} \frac{D(D-1)\cdots(D-j+1)(-1)^j}{j!} x_j^{(3)}\right|,\end{aligned} \tag{9}$$

which remarkably differs from the Hausdorff distance (5), would depend on the values of points $x_i^{(3)}$ $(i = 3, 4, \ldots, 8)$. This means that any displacement of point $x_i^{(3)}$ $(i = 3, 4, \ldots, 8)$ would change the nonlocal distance between points $x_2^{(3)}$ and $x_1^{(3)}$. Consequently, nonlocal distance (8) is indeed an intrinsic way of describing self-similar fractal, since it not only determines the dimension of a fractal curve (e.g., Cantor ternary set) but also reflects the correlation between its parts. Interestingly, the nonlocal distance seems to have some connection with quantum behavior; for details see Appendix C.

Using the nonlocal distance we have given a definition for fractal measure in Appendix A (see (A.23)).

To study the analytic properties of a fractal curve, we define the fractal derivative (see (A.26) in Appendix A) in the form:

$$\frac{{}_lD_\omega f(x)}{{}_lD_\omega x} = \lim_{\Delta l \to 0} \frac{\Delta_\omega\left[\overline{f}(l), \overline{f}(l - \Delta l)\right]}{\Delta_\omega\left[x(l), x(l - \Delta l)\right]}, \tag{10}$$

where $\overline{f}(l) = f[x(l)]$ is a differentiable function with respect to coordinate l, l is a parameter (e.g., the single parameter of Peano's curve [10]; for details see Appendix A) which completely determines the generation of a ω-dimensional fractal curve, and $x(l)$ denotes the length of the corresponding fractal curve. (We introduce a simple way of understanding the fractal derivative (10). For the case of the Newton-Leibniz derivative of $y = f(x)$, x is a 1-dimensional coordinate axis and hence can be measured by a Euclidean scale (ruler). Thus, the differential element of x is a 1-dimensional Euclidean length dx, which gives rise to the Newton-Leibniz derivative $df(x)/dx$. Nevertheless, if x is a ω-dimensional fractal curve, then it can not be measured by the Euclidean scale (ruler). In this case, the differential element of x should be a ω-dimensional volume ${}_lD_\omega x$, which gives rise to the fractal derivative ${}_lD_\omega f(x)/{}_lD_\omega x$. For details see Appendix A and Figure 3.)

In particular, the fractal derivative (10) will return to the well-known Newton-Leibniz derivative whenever $\omega = 1$.

Using the formula of fractional derivative [8], the fractal derivative can be rewritten as (see (A.27)–(A.29) in Appendix A)

$$\frac{{}_lD_\omega f(x)}{{}_lD_\omega x} = \frac{d^\omega \overline{f}(l)/dl^\omega}{d^\omega x(l)/dl^\omega}. \tag{11}$$

By formula (11) we can easily compute the fractal derivative of any differentiable function using the fractional derivative; for concrete examples see Appendix B.

4. Fractal Integral

In Section 3, we have proposed a definition for fractal derivative. Correspondingly, we can now present a convenient definition for fractal integral as follow.

Definition 1. If ${}_lD_m f(x)/{}_lD_m x = g(x)$ then the fractal integral of $g(x)$ on a m-dimensional fractal curve $\beta_m(l)$ is defined in the form:

$$\begin{aligned}\int {}_lD_m f(x) &= \int_{\beta_m(l)} g(x)\, {}_lD_m x\\&= \int_W g(x) \frac{d^m x}{dl^m}(dl)^m,\end{aligned} \tag{12}$$

where W denotes the definitional domain of the characteristic parameter l; also, the parameter l completely determines the generation of the m-dimensional fractal curve $\beta_m(l)$.

Using such a definition of fractal integral we can prove the following proposition.

Proposition 2. *If $x(r) = \omega(m)r^m$ not only describes the length of a m-dimensional fractal curve $\beta_m(r)$ but also denotes the volume of a m-dimensional sphere (for example, the m-dimensional fractal curve $\beta_m(r)$ fills up the entire sphere Ω) Ω and if $f(x) = \overline{f}(r)$ is a spherically symmetric function, then the fractal integration of $f(x)$ on the m-dimensional fractal curve $\beta_m(r)$ equals*

$$\int_{\beta_m(r)} f(x)\, {}_rD_m x = \frac{2\pi^{m/2}}{\Gamma(m/2)} \int_0^R \overline{f}(r) r^{m-1} dr, \tag{13}$$

where $\omega(m) = \pi^{m/2}/\Gamma((m/2) + 1)$.

Proof. The Riemann-Liouville fractional integrals of order m of $f(l)$ are defined in the form [8]:

$$\int f(l)(dl)^m = \frac{1}{\Gamma(m)} \int_y^R (l - y)^{m-1} f(l)\, dl. \tag{14}$$

Using (A.29) and (B.3) we have

$${}_rD_m x = \frac{d^m x}{dr^m}(dr)^m = \omega(m)\Gamma(1 + m)(dr)^m. \tag{15}$$

Using (12) and (15) we arrive at

$$\int_{\beta_m(r)} f(x)\, {}_rD_m x = \int_\Omega \overline{f}(r)\omega(m)\Gamma(1 + m)(dr)^m. \tag{16}$$

Inserting (14) into (16) leads to

$$\int_{\beta_m(r)} f(x)\ {}_rD_m x = \omega(m)\frac{\Gamma(1+m)}{\Gamma(m)} \times \int_y^R (r-y)^{m-1}\overline{f}(r)\,dr. \tag{17}$$

Using $\omega(m) = \pi^{m/2}/\Gamma((m/2)+1)$ and the formula (B.7), (17) can be rewritten as

$$\int_{\beta_m(r)} f(x)\ {}_rD_m x = \frac{\pi^{m/2} m}{\Gamma((m/2)+1)}\int_y^R (r-y)^{m-1}\overline{f}(r)\,dr. \tag{18}$$

For $y = 0$, (18) yields

$$\int_{\beta_m(r)} f(x)\ {}_rD_m x = \frac{\pi^{m/2} m}{\Gamma((m/2)+1)}\int_0^R r^{m-1}\overline{f}(r)\,dr. \tag{19}$$

Using the formula (B.7) we have

$$\frac{m}{2}\Gamma\left(\frac{m}{2}\right) = \Gamma\left(\frac{m}{2}+1\right). \tag{20}$$

Substituting (20) into (19) leads to

$$\int_{\beta_m(r)} f(x)\ {}_rD_m x = \frac{2\pi^{m/2}}{\Gamma(m/2)}\int_0^R \overline{f}(r)\,r^{m-1}dr. \tag{21}$$

The proof is complete. □

Clearly, for $f(x) = 1$, formula (13) gives the volume of a m-dimensional sphere, $\pi^{m/2}/\Gamma((m/2)+1)R^m$.

Before proceeding to arrive at the main result of this paper, let us consider three measurable sets W_i with the dimension m_i, where $i = 1, 2, 3$. According to Fubini's theorem, the Cartesian product of the sets W_i can produce a set W with the dimension $m = m_1 + m_2 + m_3$, where $W = W_1 \otimes W_2 \otimes W_3$. The integration over a function $f(r_1, r_2, r_3)$ on W can be written in the form:

$$\int_W f(r_1,r_2,r_3)\,d\mu(W) = \int_{W_1}\int_{W_2}\int_{W_3} f(r_1,r_2,r_3)\,d\mu(W_1)\cdot d\mu(W_2)\cdot d\mu(W_3). \tag{22}$$

Then we have the following lemma.

Lemma 3. *If $d\mu(W_i)$ is denoted by*

$$d\mu(W_i) = \frac{2\pi^{m_i/2}}{\Gamma(m_i/2)} r_i^{m_i-1}dr_i \quad (i = 1,2,3) \tag{23}$$

and if $f(r_1, r_2, r_3)$ is specified by a spherically symmetric function $f(r)$ with $r^2 = r_1^2 + r_2^2 + r_3^2$, then we have

$$\int_{W_1}\int_{W_2}\int_{W_3} f(r_1,r_2,r_3)\,d\mu(W_1)\cdot d\mu(W_2)\cdot d\mu(W_3) = \frac{2\pi^{m/2}}{\Gamma(m/2)}\int_0^R f(r)\,r^{m-1}dr. \tag{24}$$

The proof of Lemma 3 can be found in page 28 in the literature [8].

With the preparations above, we can introduce the main result of this paper as follow.

Proposition 4. *If $x_i(r_i) = \omega(D_i)r_i^{D_i}$ $(i = 1, 2, 3)$ not only describes the length of a D_i-dimensional fractal curve $\beta_{D_i}(r_i)$ but also denotes the volume of a D_i-dimensional sphere Ω_i and if $f(r) = f(r_1, r_2, r_3)$ is a spherically symmetric function, then the fractal integration of $f(r)$ on the fractal graph $\beta_{D_1}(r_1) \otimes \beta_{D_2}(r_2) \otimes \beta_{D_3}(r_3)$ equals*

$$\int_{\beta_{D_1}(r_1)}\int_{\beta_{D_2}(r_2)}\int_{\beta_{D_3}(r_3)} f(r_1,r_2,r_3)\prod_{i=1}^{3} {}_{r_i}D_{D_i}x_i = \frac{2\pi^{D/2}}{\Gamma(D/2)}\int_0^R f(r)\,r^{D-1}dr, \tag{25}$$

where $\omega(D_i) = \pi^{D_i/2}/\Gamma((D_i/2)+1)$, $r^2 = r_1^2 + r_2^2 + r_3^2$, and $D = D_1 + D_2 + D_3$.

Proof. Using (13) and (24) we easily arrive at (25).

The proof is complete. □

Proposition 4 shows that we have arrived at the conjectural integral formula (3) through the fractal measure (A.23). In particular, if $R \to \infty$, then the integral formula (25) will exhibit linearity, translational invariance, and scaling property. These properties are natural and necessary in applying dimensional regularization to quantum field theory [2]. It is worth mentioning that Proposition 4 can be generalized so that i might take any positive integer, for example, $i = 1, 2, \ldots, \infty$. If $i = 1, 2, 3, 4$, then $\beta_{D_i}(r_i)$ may denote coordinate axis with the dimension D_i and so $\beta_{D_1}(r_1) \otimes \beta_{D_2}(r_2) \otimes \beta_{D_3}(r_3) \otimes \beta_{D_4}(r_4)$ may denote spacetime.

5. Conclusion

Hausdorff measure is not an ideal mathematical framework for describing fractal since it fails to describe the nonlocal property of fractal (e.g., self-similarity). However, the fractal measure constructed by this paper not only shows the dimension of a fractal but also describes its analytic properties (e.g., nonlocal property). Not only so, using this fractal measure we can derive Svozil's conjectural integral formula (3) which is the starting point of quantum field theory on fractal spacetime. Therefore, our fractal measure may be regarded as a possible mathematical basis of establishing quantum field theory on fractal spacetime.

Appendices

A. Mathematical Preparations

In Euclidean geometry, the dimension of a geometric graph is determined by the number of independent variables (i.e., the number of degrees of freedom). For example, every point on a plane can be represented by 2-tuples real number (x_1, x_2); then the dimension of the plane is denoted by

2. Nevertheless, the existence of Peano's curve powerfully refutes this viewpoint. Peano's curve, which is determined by an independent characteristic parameter (i.e., fill parameter), would fill up the entire plane [10]. Therefore, mathematicians have to reconsider the definition of dimension. The most famous one of all definitions of dimension is the Hausdorff dimension, which is defined through the Hausdorff measure [8].

A.1. Hausdorff Measure and Hausdorff Dimension. In order to bring the definition of Hausdorff dimension, we firstly introduce the Hausdorff measure [8].

Let W be a nonempty subset of n-dimensional Euclidean space R^n; the diameter of W is defined as

$$\operatorname{diam}(W) = \sup\{d(x,y), x, y \in W\}, \tag{A.1}$$

where $d(x,y)$, which is the distance between points x and y, is a real-valued function on $W \otimes W$, such that the following four conditions are satisfied:

$$d(x,y) \geq 0 \quad \forall x, y \in W; \tag{A.2}$$

$$d(x,y) = 0 \quad \text{iff } x = y; \tag{A.3}$$

$$d(x,y) = d(y,x) \quad \forall x, y \in W; \tag{A.4}$$

$$d(x,z) \leq d(x,y) + d(y,z) \quad \forall x, y, z \in W. \tag{A.5}$$

For example, the distance of n-dimensional Euclidean space R^n can be defined as

$$d_E(x,y) = |x-y| = \left(\sum_{i=1}^{n} |x_i - y_i|^2\right)^{1/2}. \tag{A.6}$$

Then, it is easy to check that (A.6) satisfies conditions (A.2)–(A.5).

Now, let us consider a countable set $\{E_i\}$ of subsets of diameter at most ε that covers W; that is,

$$W \subset \bigcup_{i=1}^{\infty} E_i, \quad \operatorname{diam}(E_i) \leq \varepsilon \quad \forall i. \tag{A.7}$$

For a positive D and each $\varepsilon > 0$, we consider covers of W by countable families $\{E_i\}$ of (arbitrary) sets E_i with diameter less than ε and take the infimum of the sum of $[\operatorname{diam}(E_i)]^D$. Then we have

$$H_\varepsilon^D(W) = \inf\left\{\sum_{i=1}^{\infty}[\operatorname{diam}(E_i)]^D : W \subset \bigcup_{i=1}^{\infty} E_i,\ \operatorname{diam}(E_i) \leq \varepsilon\right\}. \tag{A.8}$$

If the following limit exists

$$H^D(W) = \lim_{\varepsilon \to 0} H_\varepsilon^D(W) = \text{finite}, \tag{A.9}$$

then the value $H^D(W)$ is called the D-dimensional Hausdorff measure.

A.2. Shortcoming of Hausdorff Measure. In general, D may be a fraction. In 1967, Mandelbrot realized that [11] the length of coastline can be measured using Hausdorff measure (A.8) rather than Euclidean measure (A.6), and then the dimension of coastline is a fraction. Mandelbrot called such geometric graphs the "fractal".

The fractal is self-similar between its local and global shapes. Unfortunately, Hausdorff measure can determine the dimension of fractal but not reflect the connection (e.g., self-similarity) among the parts of the corresponding fractal. To see this, we consider Koch's curve in Figure 2.

Clearly, the congruent triangle $\Delta x_1 x_2 x_3$ is similar to $\Delta x_4 x_5 x_6$. If we use the Hausdorff measure (A.8) to measure the local distance of Koch's curve (e.g., the distance between points x_1 and x_3), then we have

$$H^D(x_1, x_3) = |x_1 - x_2|^D + |x_2 - x_3|^D, \tag{A.10}$$

where D is the dimension of theKoch curve.

Equation (A.10) shows that the Hausdorff distance between points x_1 and x_3 depends only on the positions of points x_i $(i = 1,2,3)$ and is thereby independent of the positions of points x_j $(j = 4,5,6)$. Nevertheless, because of the self-similarity of Koch's curve, any displacements of points x_j $(j = 4,5,6)$ would influence the positions of x_i $(i = 1,2,3)$ and hence change the distance between points x_1 and x_3. That is to say, the local shape (e.g., $\Delta x_1 x_2 x_3$) is closely related to the global shape (e.g., $\Delta x_4 x_5 x_6$). Unfortunately, the Hausdorff distance (A.10) undoubtedly fails to reflect this fact. Therefore, we need to find a new measure of describing the analytic properties of fractal.

A.3. Definition of Fractal Measure. Hausdorff measure (A.8) does not reflect the self-similarity of fractal, so we cannot establish the calculus theory of fractal using the Hausdorff measure. In general, people often use the fractional calculus to approximately describe the analytic properties of fractal [8, 12].

The fractional calculus is a theory of integrals and derivatives of any arbitrary real order. For example, the fractional derivatives of order m of the function $y(l) = cl^n$ equal [8]

$$\frac{d^m y(l)}{dl^m} = c\frac{\Gamma(n+1)}{\Gamma(n-m+1)} l^{n-m}, \tag{A.11}$$

where $\Gamma(x)$ denotes the Gamma function and m is an arbitrary real number.

Now, let us consider a m-dimensional volume

$$x(l) = \omega(m) l^m, \tag{A.12}$$

where $\omega(m)$ is a constant which depends only on the dimension m.

Using formula (A.11), the fractional derivatives of order m of (A.12) equal

$$\frac{d^m x(l)}{dl^m} = \Gamma(m+1)\,\omega(m). \tag{A.13}$$

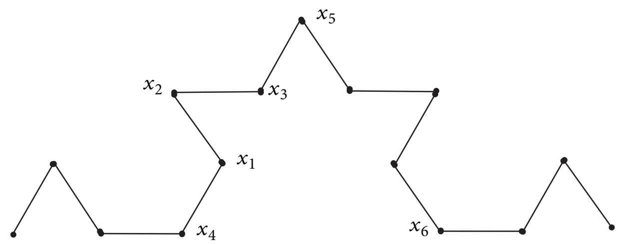

Figure 2: Koch's curve, which is similar to the generation of Cantor ternary set (see Figure 1), is defined by repeatedly adding the middle thirds of line segments [11].

Equation (A.13) can be written as

$$\Delta^m x(l) = \Gamma(m+1)\,\omega(m)\,(\Delta l)^m + o\left[(\Delta l)^m\right], \quad \text{(A.14)}$$

where $\Delta^m x(l)$ denotes the differences of order m and $o[(\Delta l)^m]$ denotes the infinitesimal terms of higher order compared to $(\Delta l)^m$.

Equation (A.14) implies that

$$\Delta^m x(l) \sim (\Delta l)^m, \quad \text{(A.15)}$$

and thereby

$$\sum_{i=1}^{N} \Delta^m x(l_i) \sim \sum_{i=1}^{N} (\Delta l)^m. \quad \text{(A.16)}$$

Obviously, $(\Delta l)^m$ is a m-dimensional Hausdorff measure, which can describe the length of a m-dimensional fractal curve. Consequently, (A.15) and (A.16) together imply that $\Delta^m x(l)$ can be also thought of as a m-dimensional measure. In this case, the order of differences $\Delta^m x(l)$ represents the Hausdorff dimension m. Because of this fact, we next attempt to use the differences of order m to define a new measure.

Let us consider the left-shift operator with step Δl and the identity operator as follows:

$$\begin{aligned} L_{\Delta l} x(l) &= x(l-\Delta l), \\ L_0 x(l) &= x(l). \end{aligned} \quad \text{(A.17)}$$

Using the left-shift operator $L_{\Delta l}$ and the identity operator L_0, we can define the difference operator of order m in the form:

$$(L_0 - L_{\Delta l})^m = \sum_{j=0}^{\infty} \frac{m(m-1)\cdots(m-j+1)(-1)^j}{j!} L_{j\Delta l}. \quad \text{(A.18)}$$

Using (A.18), we define a new distance between points $x(l)$ and $x(l-\Delta l)$ in the form:

$$\left|\Delta_m [x(l), x(l-\Delta l)]\right| = \left|\sum_{j=0}^{\infty} \frac{m(m-1)\cdots(m-j+1)(-1)^j}{j!} x(l-j\Delta l)\right|. \quad \text{(A.19)}$$

We call (A.19) the "nonlocal distance," which describes the length of a m-dimensional fractal curve.

When $m = 1$, the nonlocal distance (A.19) returns to the Euclidean distance; that is,

$$\left|\Delta_{m=1} [x(l), x(l-\Delta l)]\right| = |x(l) - x(l-\Delta l)|. \quad \text{(A.20)}$$

In general, the nonlocal distance (A.19) does not satisfy the general properties of distance, (A.4)-(A.5) but reflects the connection between local and global segments of fractal. (For example, the nonlocal distance between points $x(l)$ and $x(l+\Delta l)$, that is, $|\Delta_m[x(l), x(l+\Delta l)]|$, needs to be defined using the right-shift operator $R_{\Delta l}$ which leads to $R_{\Delta l}x(l) = x(l+\Delta l)$. The corresponding difference operator reads $(R_{\Delta l} - R_0)^m$. Then, the nonlocal distance would not satisfy condition (A.4).) To understand the latter, we need to realize that the output value of (A.19) would depend on the values of all points $x(l - j\Delta l)$ $(j = 0, 1, 2 \ldots)$ rather than only on points $x(l)$ and $x(l - \Delta l)$.

For instance, in Figure 2, the nonlocal distance $|\Delta_D[x_5, x_3]|$ between points x_3 and x_5 would depend on the positions of points x_i $(i = 1, 2, 3, 4, 5)$ rather than only on points x_3 and x_5. Therefore, the nonlocal distance (A.19) is indeed an intrinsic way of describing fractal, since it not only shows the dimension but also reflects the connection between local and global segments of fractal.

Using the nonlocal distance (A.19), we can propose a definition for fractal measure.

Let W be a nonempty subset of n-dimensional Euclidean space R^n. We consider a countable set $\{F_i\}$ of subsets of diameter at most ε that covers W; that is,

$$W \subset \bigcup_{i=1}^{\infty} F_i, \quad \operatorname{diam}^D(F_i) \le \varepsilon \quad \forall i, \quad \text{(A.21)}$$

where $\operatorname{diam}^D(F_i)$ defined by using the nonlocal distance (A.19) denotes the diameter of F_i; that is,

$$\operatorname{diam}^D(F_i) = \sup\left\{\left|\Delta_D[x, y]\right|,\ x, y \in F_i\right\}. \quad \text{(A.22)}$$

Fractal Measure. For a positive D and each $\varepsilon > 0$, we consider covers of W by countable families $\{F_i\}$ of (arbitrary) sets F_i with diameter less than ε and take the infimum of the sum of $\operatorname{diam}^D(F_i)$. Then we have

$$\Pi_\varepsilon^D(W) = \inf\left\{\sum_{i=1}^{\infty} \operatorname{diam}^D(F_i) : W \subset \bigcup_{i=1}^{\infty} F_i,\ \operatorname{diam}^D(F_i) \le \varepsilon\right\}. \quad \text{(A.23)}$$

If the following limit exists

$$\Pi^D(W) = \lim_{\varepsilon \to 0} \Pi_\varepsilon^D(W) = \text{finite}, \quad \text{(A.24)}$$

then the value $\Pi^D(W)$ is called the D-dimensional fractal measure; meanwhile, the dimension of W equals D.

A.4. Definition of Fractal Derivative. Obviously, to describe the analytic properties of fractal, we need the corresponding calculus theory.

Before proceeding to introduce the definition of fractal derivative, let us consider a ω-dimensional fractal curve $\beta_\omega(l)$ (see Figure 3), which is determined by an independent characteristic parameter l (e.g., the fill parameter of Peano's curve), filling up a ω-dimensional region. Assume that the length of the fractal curve is specified by a ω-dimensional volume $x(l)$, then the (nonlocal) length between points $a = \beta_\omega(l_0)$ and $b = \beta_\omega(l_n)$ in Figure 3 should be denoted by

$$\Pi^D([a,b]) = \lim_{\Delta l \to 0} \inf \left\{ \sum_{i=1}^{n} \left| \Delta_\omega \left[x(l_{i-1}), x(l_i) \right] \right| \right\}, \quad \text{(A.25)}$$

where we have used the fractal measure (A.23).

It is carefully noted that the length between points $a = \beta_\omega(l_0)$ and $b = \beta_\omega(l_n)$ can not be measured by Euclidean scale; see Figure 3.

As such, we can present a definition for fractal derivative as follows.

Fractal Derivative. For any differentiable function $y = f(x)$, if $x = x(l)$ is not only a ω-dimensional volume but also describes the length of a ω-dimensional fractal curve $\gamma_\omega(l)$, then the fractal derivative of $y = f(x)$ with respect to the fractal curve $\gamma_\omega(l)$ is defined as

$$\frac{{}_lD_\omega f(x)}{{}_lD_\omega x} = \lim_{\Delta l \to 0} \frac{\Delta_\omega \left[\overline{f}(l), \overline{f}(l-\Delta l) \right]}{\Delta_\omega \left[x(l), x(l-\Delta l) \right]}, \quad \text{(A.26)}$$

where $\overline{f}(l) = f[x(l)]$.

Clearly, if $\omega = 1$, then the formula (A.26) will return to the Newton-Leibniz derivative, and meanwhile $\gamma_{\omega=1}(l)$ is restored to a 1-dimensional coordinate axis.

In general, the fractional derivative of order ω of any differentiable function $f(l)$ is defined in the form [13]:

$$\frac{d^\omega f(l)}{dl^\omega} = \lim_{\Delta l \to 0} \left(\sum_{j=0}^{\infty} \frac{\omega(\omega-1)\cdots(\omega-j+1)(-1)^j}{j!} f(l - j\Delta l) \times (\Delta l)^{-\omega} \right). \quad \text{(A.27)}$$

Comparing (A.19) and (A.27), we have

$$\frac{d^\omega f(l)}{dl^\omega} = \lim_{\Delta l \to 0} \frac{\Delta_\omega \left[f(l), f(l-\Delta l) \right]}{(\Delta l)^\omega}. \quad \text{(A.28)}$$

Using formula (A.28), the formula (A.26) can be rewritten as

$$\begin{aligned} \frac{{}_lD_\omega f(x)}{{}_lD_\omega x} &= \lim_{\Delta l \to 0} \frac{\Delta_\omega \left[\overline{f}(l), \overline{f}(l-\Delta l) \right]}{\Delta_\omega \left[x(l), x(l-\Delta l) \right]} \cdot \lim_{\Delta l \to 0} \frac{1/(\Delta l)^\omega}{1/(\Delta l)^\omega} \\ &= \frac{\lim_{\Delta l \to 0} \left(\Delta_\omega \left[\overline{f}(l), \overline{f}(l-\Delta l) \right] / (\Delta l)^\omega \right)}{\lim_{\Delta l \to 0} \left(\Delta_\omega \left[x(l), x(l-\Delta l) \right] / (\Delta l)^\omega \right)} \\ &= \frac{d^\omega \overline{f}(l) / dl^\omega}{d^\omega x(l) / dl^\omega}. \end{aligned} \quad \text{(A.29)}$$

Formula (A.29) indicates that we can compute the fractal derivative using the fractional derivative.

B. Computation Examples

In Appendix A, we have noted that the fractal derivative can be computed using the formula (A.29). In this appendix, we present two computing examples.

Example 1. If $x(l) = \omega(m) l^m$ describes the length of a m-dimensional fractal curve, then the fractal derivative of the constant function $f(x) = C$ with respect to x equals

$$\frac{{}_lD_m C}{{}_lD_m x} = \frac{C}{\Gamma(1-m)\Gamma(1+m)} \cdot \frac{1}{x}, \quad \text{(B.1)}$$

where $\omega(m)$ is a constant that depends only on m.

Proof. The fractional derivatives of order m of the constant C and the power function $y(l) = al^n$, are respectively, as follows [8]:

$$\frac{d^m C}{dl^m} = \frac{C}{\Gamma(1-m)} l^{-m}, \quad \text{(B.2)}$$

$$\frac{d^m y(l)}{dl^m} = a \frac{\Gamma(n+1)}{\Gamma(n-m+1)} l^{n-m}. \quad \text{(B.3)}$$

Using formulas (A.29), (B.2), and (B.3), the fractal derivative ${}_lD_m f(x) / {}_lD_m x$ can be computed as follows:

$$\begin{aligned} \frac{{}_lD_m C}{{}_lD_m x} &= \frac{d^m C / dl^m}{d^m [\omega(m) l^m] / dl^m} = \frac{(C/\Gamma(1-m)) l^{-m}}{\omega(m)\Gamma(1+m)} \\ &= \frac{C}{\Gamma(1-m)\Gamma(1+m)} \cdot \frac{1}{x}. \end{aligned} \quad \text{(B.4)}$$

The proof is complete. □

Example 2. If x is the characteristic parameter of a m-dimensional fractal curve and meanwhile it also describes the length of this fractal curve, then the fractal derivative of the constant function $f(x) = C$ with respect to x equals

$$\frac{{}_xD_m C}{{}_xD_m x} = C(1-m) \cdot \frac{1}{x}. \quad \text{(B.5)}$$

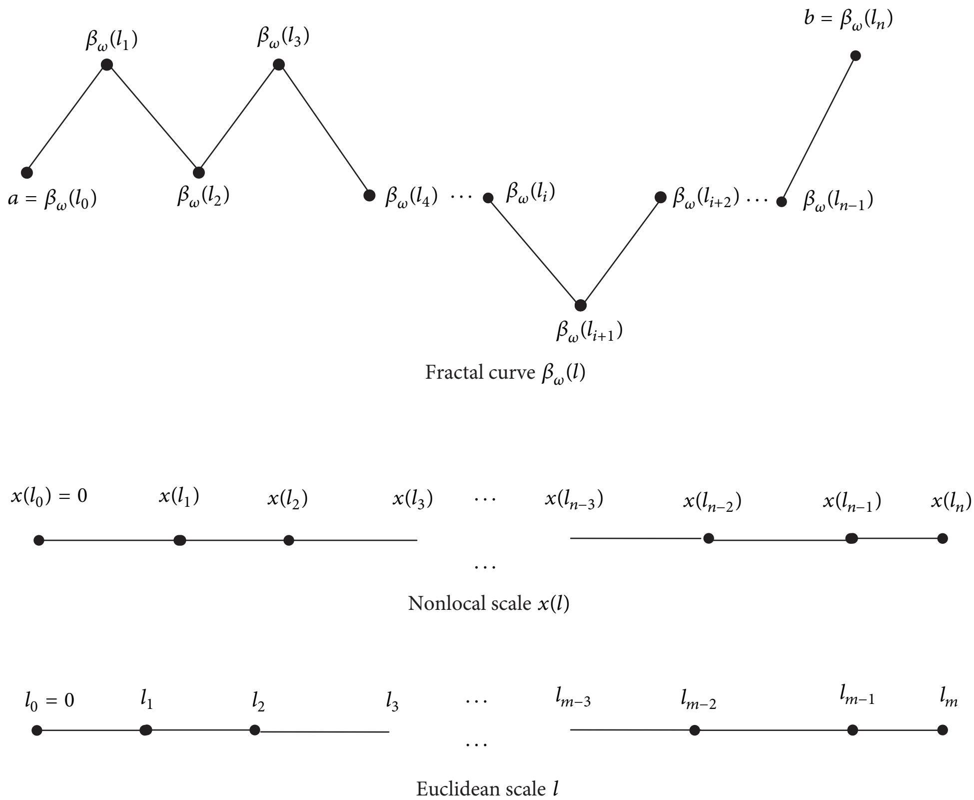

Figure 3: The fractal curve $\beta_\omega(l)$ consists of the union $\bigcup_{i=1}^{n}[\beta_\omega(l_{i-1}), \beta_\omega(l_i)]$, where $n = n(m)$. The dimension of fractal curve $\beta_\omega(l)$ equals $\omega = \lim_{m\to\infty}(\ln n(m)/\ln m)$. Since $\beta_\omega(l)$ is a fractal curve, the distance between points $\beta_\omega(l_0)$ and $\beta_\omega(l_n)$ cannot be measured using the Euclidean scale (ruler) l; otherwise, we will have $d[\beta_\omega(l_0), \beta_\omega(l_n)] = \lim_{m\to\infty} n(m)((l_m - l_0)/m) = \infty$ or $=0$. However, the fractal curve $\beta_\omega(l)$ can be measured using the nonlocal scale (ruler) $x(l)$; for the way of measure see the formula (A.25).

Proof. Using formulas (A.29), (B.2), and (B.3) we have

$$\begin{aligned}\frac{{}_xD_m f(x)}{{}_xD_m x} &= \frac{d^mC/dx^m}{d^mx/dx^m} = \frac{(C/\Gamma(1-m))\,x^{-m}}{(\Gamma(2)/\Gamma(2-m))\,x^{1-m}} \\ &= C\frac{\Gamma(2-m)}{\Gamma(1-m)}\cdot\frac{1}{x}.\end{aligned} \tag{B.6}$$

Considering the property of Gamma function

$$\Gamma(x+1) = x\Gamma(x), \tag{B.7}$$

we have

$$1 - m = \frac{\Gamma(2-m)}{\Gamma(1-m)}. \tag{B.8}$$

Substituting (B.8) into (B.6) we arrive at

$$\frac{{}_xD_m C}{{}_xD_m x} = C(1-m)\cdot\frac{1}{x}. \tag{B.9}$$

The proof is complete. □

C. Quantum Behavior and Nonlocal Distance

Now we investigate the connection between quantum behaviour and nonlocal distance. For simplicity, we still consider the Cantor set in Figure 1, where the nonlocal distance between points $x_2^{(n)}$ and $x_1^{(n)}$ is equal to

$$\begin{aligned}\Pi\left(x_2^{(n)}, x_1^{(n)}\right) &= \left|\Delta_{m=D}\left[x_2^{(n)}, x_1^{(n)}\right]\right| \\ &= \left|\sum_{j=1}^{2^n}\frac{D(D-1)\cdots(D-j+1)(-1)^j}{j!}x_j^{(n)}\right|.\end{aligned} \tag{C.1}$$

Clearly, the output value of nonlocal distance (C.1) depends on the value of each element in the set $\{x_i^{(n)}\}$, where i runs from 1 to 2^n. If $n \to \infty$, we will have $\Pi(x_2^{(n)}, x_1^{(n)}) \to 0$; however, the number of elements in the set $\{x_i^{(n)}\}$ would tend to infinity, too. This means that if we want to precisely measure the distance of smaller scale (e.g., $\lim_{n\gg 1}\Pi(x_2^{(n)}, x_1^{(n)})$), correspondingly we will need to collect the points $\{x_i^{(n)}\}$. As a result, if we want to precisely measure the nonlocal distance between points $\lim_{n\to\infty} x_2^{(n)}$ and $\lim_{n\to\infty} x_1^{(n)}$, we will need to collect a set of infinite points, that is, $\{x_i^{(\infty)}\}_{i=1}^{\infty}$. Unfortunately, we must fail to arrive at this purpose on an actual measurement. In other words, we cannot precisely measure the distance of microscopic scale; this fact is consistent with the "Heisenberg Uncertainty Principle." (In fact, the connection between quantum mechanics and fractal has been noticed in some earlier papers [14–17].)

On the other hand, the nonlocal distance (C.1) between points $x_2^{(n)}$ and $x_1^{(n)}$ depends clearly on the position of $x_i^{(n)}$ $(i = 1, 2, \ldots, 2^n)$; for example, any displacement of point $x_{2^n}^{(n)}$ would influence the output value of nonlocal distance (C.1). It is a clearly nonlocal correlation (correlations span arbitrarily distances) and similar to quantum entanglement.

The above two facts imply that the nonlocal description may be an intrinsic way of describing quantum behavior. (In references [18, 19], we have shown that the local description is not a way of completely describing physical reality.)

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