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*Research Article*

# Quasilinearization for the Boundary Value Problem of Second-Order Singular Differential System

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We study the boundary value problems of second-order singular differential equations. At first, we reduce the BVPs to initial value problems of first-order singular integrodifferential equations and then we employ the quasilinearization method in studying the IVPs and obtain two monotone iterative sequences, which converge uniformly and quadratically to the unique solution of the IVPs. Finally, we get the similar result for the given BVPs.

## 1. Introduction

It is well known that quasilinearization method is a powerful tool for proving the existence of approximate solutions of nonlinear systems and the converge quadratically to the unique solution of the given problems (see [1]). Recently, Abd-Ellateef Kamar et al. investigated the first-order singular systems of differential equations with initial value problem [2]. In [3], Wang and Liu developed monotone iterative technique combined with the method of upper and lower solutions for studying the second-order singular systems with boundary value problems (BVPs).

In this paper, we extend quasilinearization method to study the second-order singular systems with the boundary conditions:

$$A\ddot{X} = f\left(t, X, \dot{X}\right), \qquad \dot{X}(0) = X_0, \qquad X(b) = X_1, \quad (1)$$

where $A \in R^{n\times n}$ is a singular matrix, $f \in C[I \times R^n \times R^n, R^n]$, $I = [0, b]$, and $X_0$ and $X_1$ are two constant vectors. By using the existence result [4] for linear singular systems and the comparison result [5],we investigate two monotone iterative sequences which converge uniformly and quadratically to the solution of the problem.

## 2. Preliminaries

Consider the following initial value problem:

$$-A\dot{U} = f(t, SU, -U), \qquad U(0) = -X_0, \qquad (2)$$

where $A$ is a singular $n \times n$ matrix, $f \in C[I \times R^n \times R^n, R^n]$, $I = [0, b]$, $SU = X_1 + \int_t^b U(t)ds$ is an increasing operator, and $X_0$ is a constant vector.

In order to obtain two monotone sequences, we introduce an existence result for the corresponding linear singular systems and a comparison result.

The existence of the solution of the linear initial value problem of the form

$$-A\dot{U} + M(t)U = g(t), \quad U(0) = -X_0 \qquad (3)$$

is well known and is given by the following lemma.

**Lemma 1** (see [4]). *Assume that the nonhomogeneous linear system* (3) *exists and if*

(i) *there exists a* $\lambda \in R$ *such that* $(-\lambda A + M(t))^{-1}$ *exists,*

(ii) $X_0$ *is the solution of* $(I-\widehat{A}\widehat{A}^D)(-X_0-\widehat{W}(0)) = 0$, *where*

$$\begin{aligned} \widehat{A} &= (-A + M(t))^{-1}(-A), \\ \widehat{M} &= (-A + M(t))^{-1} M(t), \\ \widehat{g}(t) &= (-A + M(t))^{-1} g(t), \\ \widehat{W}(t) &= \left(I - \widehat{A}^D \widehat{A}\right) \widehat{M}^D \widehat{g}(t), \end{aligned} \tag{4}$$

*and* $\widehat{A}^D$, $\widehat{M}^D$ *are the Drazin inverses of* $\widehat{A}$, $\widehat{M}(t)$, *respectively, then the solution is given by*

$$\begin{aligned} U(t) &= e^{-\widehat{A}^D \widehat{M} t} \widehat{A}^D \widehat{A}(-X_0) + e^{-\widehat{A}^D \widehat{M} t} \int_0^t e^{\widehat{A}^D \widehat{M} s} \widehat{A}^D \widehat{g}(s)\, ds \\ &\quad + \left(I - \widehat{A}^D \widehat{A}\right) \widehat{M}^D \widehat{g}(t). \end{aligned} \tag{5}$$

*Now one gives the following assumptions for convenience.*

$(H_{2.1})$ *Let $A$ and $M(t)$ be matrices such that $(-\lambda A + M(t))^{-1}$ exists and is nonnegative for some $\lambda \in R^1$. Suppose further that $T, T^{-1}$ exist and are nonnegative, such that*

$$T^{-1}\widehat{A}T = \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix}, \qquad T^{-1}\widehat{M}T = \begin{pmatrix} I_1 - \lambda C & 0 \\ 0 & I_2 \end{pmatrix}, \tag{6}$$

*where $\widehat{A} = (-\lambda A + M(t))^{-1} A$, $\widehat{M} = (-\lambda A + M(t))^{-1} M(t)$, and $C$ is a diagonal square matrix with $C^{-1} > 0$ and $C^{-1}(I_1 - \lambda C) > 0$.*

$(H_{2.2})$ *There exist $\alpha_0(t), \beta_0(t) \in C^1[I, R^n]$ with $\alpha_0(t) \le \beta_0(t)$ on $I$, such that*

$$\begin{aligned} -A\dot{\alpha}_0 &\le f(t, S\alpha_0, -\alpha_0), \quad \alpha_0(0) \le -X_0, \\ -A\dot{\beta}_0 &\ge f(t, S\beta_0, -\beta_0), \quad \beta_0(0) \ge -X_0. \end{aligned} \tag{7}$$

$(H_{2.3})$ *All second-order derivatives of $f(t, X, U)$ exist and are bounded, $f(t, X, U)$ is convex in $X$ for each $(t, U)$, $f(t, X, U)$ is convex in $U$ for each $(t, X)$, $f_x(t, X, U)$ is nonincreasing in $U$ for each $(t, X)$, and $f_y(t, X, U)$ is nonincreasing in $X$ for each $(t, U)$.*

$(H_{2.4})$ *Moreover $M(t) = -[f_x(t, S\eta, -\eta)b - f_y(t, S\eta, -\eta)]$.*

*To obtain the results, one needs the following comparison theorem.*

**Lemma 2.** *Let $-A\dot{P} + M(t)P \le 0$ such that $A$ and $M(t)$ satisfy assumptions $(H_{2.1})$ and $(H_{2.4})$. Then $P(0) \le 0$ implies $P(t) \le 0$ on $I$.*

*The proof is similar to [5] and one omits it.*

## 3. Main Results

Firstly, we develop the following result which is important for the final result.

**Theorem 3.** *Suppose that assumptions $(H_{2.1})$–$(H_{2.4})$ hold and, for $X_0 \in R^n$,*

(i) $(I - \widehat{A}\widehat{A}^D)(-X_0 - \widehat{W}(0)) = 0$, *where* $\widehat{W}(t) = (I - \widehat{A}^D\widehat{A})\widehat{M}^D\widehat{g}(t)$.

*Then there exist two monotone sequences $\{\alpha_n\}$ and $\{\beta_n\}$, which converge uniformly on $I$ to the unique solution of problem* (2) *and the convergence rate is quadratic.*

*Proof.* From $(H_{2.3})$ we find that $f_{xx}(t, X, U) \ge 0$, and

$$f(t, X_1, U) \ge f(t, X_2, U) + f_x(t, X_2, U)(X_1 - X_2), \tag{8}$$

for all $X_1, X_2, U \in R^n$ and $t \in I$. The convexity of $f(t, X, U)$ in $U$ implies that

$$f(t, X, U_1) \ge f(t, X, U_2) + f_y(t, X, U_2)(U_1 - U_2), \tag{9}$$

for $U_1, U_2 \in R^n$, $t \in I$, and $X \in R^n$.

Now consider the following linear problems. For $k = 0, 1, 2, 3 \ldots$,

$$\begin{aligned} -A\dot{\alpha}_{k+1} &= f(t, S\alpha_k, -\alpha_k) + f_x(t, S\alpha_k, -\alpha_k)(S\alpha_{k+1} - S\alpha_k) \\ &\quad + f_y(t, S\alpha_k, -\alpha_k)(-\alpha_{k+1} + \alpha_k), \quad \alpha_{k+1}(0) = -X_0, \end{aligned} \tag{10}$$

$$\begin{aligned} -A\dot{\beta}_{k+1} &= f(t, S\beta_k, -\beta_k) + f_x(t, S\alpha_k, -\alpha_k)(S\beta_{k+1} - S\beta_k) \\ &\quad + f_y(t, S\alpha_k, -\alpha_k)(-\beta_{k+1} + \beta_k), \quad \beta_{k+1}(0) = -X_0. \end{aligned} \tag{11}$$

For (10), set

$$\begin{aligned} g(t, Sx, -x) \equiv{}& f(t, S\alpha_0, -\alpha_0) + f_x(t, S\alpha_0, -\alpha_0)(Sx - S\alpha_0) \\ &+ f_y(t, S\alpha_0, -\alpha_0)(-x + \alpha_0). \end{aligned} \tag{12}$$

Then, we obtain that $-A\dot{\alpha}_0 \le f(t, S\alpha_0, -\alpha_0) = g(t, S\alpha_0, -\alpha_0)$. Furthermore, we can get

$$\begin{aligned} -A\dot{\beta}_0 &\ge f(t, S\beta_0, -\beta_0) \\ &\ge f(t, S\alpha_0, -\alpha_0) + f_x(t, S\alpha_0, -\alpha_0)(S\beta_0 - S\alpha_0) \\ &\quad + f_y(t, S\alpha_0, -\alpha_0)(-\beta_0 + \alpha_0) \\ &= g(t, S\beta_0, -\beta_0). \end{aligned} \tag{13}$$

Hence $\alpha_0$ and $\beta_0$ are lower and upper solutions of (12), respectively. Thus (12) has a solution $\alpha_1$ on $I$, and we have $\alpha_0 \le \alpha_1 \le \beta_0$. Similarly, set

$$\begin{aligned} g^*(t, Sx, -x) \equiv{}& f(t, S\beta_0, -\beta_0) + f_x(t, S\alpha_0, -\alpha_0)(Sx - S\beta_0) \\ &+ f_y(t, S\alpha_0, -\alpha_0)(-x + \beta_0). \end{aligned} \tag{14}$$

Then we obtain that (14) has a solution $\beta_1$ on $I$, and we have $\alpha_0 \le \beta_1 \le \beta_0$.

Now we claim that

$$\alpha_0 \le \alpha_1 \le \beta_1 \le \beta_0 \quad \text{on } I. \tag{15}$$

Let $p = \alpha_1 - \beta_1$. We get that

$$\begin{aligned}
-A\dot{p} &= -A\dot{\alpha}_1 - \left(-A\dot{\beta}_1\right) \\
&= f\left(t, S\alpha_0, -\alpha_0\right) + f_x\left(t, S\alpha_0, -\alpha_0\right)\left(S\alpha_1 - S\alpha_0\right) \\
&\quad + f_y\left(t, S\alpha_0, -\alpha_0\right)\left(-\alpha_1 + \alpha_0\right) \\
&\quad - f\left(t, S\beta_0, -\beta_0\right) - f_x\left(t, S\alpha_0, -\alpha_0\right)\left(S\beta_1 - S\beta_0\right) \\
&\quad - f_y\left(t, S\alpha_0, -\alpha_0\right)\left(-\beta_1 + \beta_0\right) \\
&= f\left(t, S\alpha_0, -\alpha_0\right) - f\left(t, S\beta_0, -\alpha_0\right) \\
&\quad + f\left(t, S\beta_0, -\alpha_0\right) - f\left(t, S\beta_0, -\beta_0\right) \\
&\quad + f_x\left(t, S\alpha_0, -\alpha_0\right)\left(S\alpha_1 - S\alpha_0\right) \\
&\quad + f_y\left(t, S\alpha_0, -\alpha_0\right)\left(-\alpha_1 + \alpha_0\right) \\
&\quad - f_x\left(t, S\alpha_0, -\alpha_0\right)\left(S\beta_1 - S\beta_0\right) \\
&\quad - f_y\left(t, S\alpha_0, -\alpha_0\right)\left(-\beta_1 + \beta_0\right) \\
&\le f_x\left(t, S\alpha_0, -\alpha_0\right)\left(S\alpha_0 - S\beta_0\right) \\
&\quad + f_y\left(t, S\alpha_0, -\alpha_0\right)\left(-\alpha_0 + \beta_0\right) \\
&\quad + f_x\left(t, S\alpha_0, -\alpha_0\right)\left(S\alpha_1 - S\alpha_0\right) \\
&\quad + f_y\left(t, S\alpha_0, -\alpha_0\right)\left(-\alpha_1 + \alpha_0\right) \\
&\quad - f_x\left(t, S\beta_0, -\beta_0\right)\left(S\beta_1 - S\beta_0\right) \\
&\quad - f_y\left(t, S\beta_0, -\beta_0\right)\left(-\beta_1 + \beta_0\right) \\
&\le f_x\left(t, S\alpha_0, -\alpha_0\right) \\
&\quad \times\left(S\alpha_0 - S\beta_0 + S\alpha_1 - S\alpha_0 - S\beta_1 + S\beta_0\right) \\
&\quad + f_y\left(t, S\alpha_0, -\alpha_0\right) \\
&\quad \times\left(-\alpha_0 + \beta_0 - \alpha_1 + \alpha_0 + \beta_1 - \beta_0\right) \\
&= f_x\left(t, S\alpha_0, -\alpha_0\right)\left(S\alpha_1 - S\beta_1\right) \\
&\quad - f_y\left(t, S\alpha_0, -\alpha_0\right)\left(\alpha_1 - \beta_1\right) \le -M(t)\left(\alpha_1 - \beta_1\right).
\end{aligned} \tag{16}$$

Noticing that $p(0) = \alpha_1(0) - \beta_1(0) \le 0$, we get $\alpha_1(t) \le \beta_1(t)$, on $I$.

Now, assume that, for $n = 0, 1, 2, \ldots, k$, (10) and (11) admit solutions $\alpha_n$ and $\beta_n$, respectively, such that

$$\alpha_0 \le \alpha_1 \le \cdots \le \alpha_k \le \beta_k \le \cdots \le \beta_1 \le \beta_0, \quad \text{on } I. \tag{17}$$

Then setting $n = k$ in (10) and (11), we observe that the assumptions in Theorem 3 are satisfied. Thus, there exist solutions $\alpha_{k+1}(t)$ and $\beta_{k+1}(t)$ for (10) and (11), respectively, and we now will show that the following relation

$$\alpha_k(t) \le \alpha_{k+1}(t) \le \beta_{k+1}(t) \le \beta_k(t), \quad t \in I, \tag{18}$$

holds. Firstly, we can easily know that $\alpha_k(t) \le \alpha_{k+1}(t) \le \beta_k(t)$ and $\alpha_k(t) \le \beta_{k+1}(t) \le \beta_k(t)$.

To prove that $\alpha_{k+1} \le \beta_{k+1}$, consider $p = \alpha_{k+1} - \beta_{k+1}$. Then

$$\begin{aligned}
-A\dot{p} &= -A\dot{\alpha}_{k+1} - \left(-A\dot{\beta}_{k+1}\right) \\
&= f\left(t, S\alpha_k, -\alpha_k\right) + f_x\left(t, S\alpha_k, -\alpha_k\right)\left(S\alpha_{k+1} - S\alpha_k\right) \\
&\quad + f_y\left(t, S\alpha_k, -\alpha_k\right)\left(-\alpha_{k+1} + \alpha_k\right) \\
&\quad - f\left(t, S\beta_k, -\beta_k\right) - f_x\left(t, S\beta_k, -\beta_k\right)\left(S\beta_{k+1} - S\beta_k\right) \\
&\quad - f_y\left(t, S\beta_k, -\beta_k\right)\left(-\beta_{k+1} + \beta_k\right) \\
&= f\left(t, S\alpha_k, -\alpha_k\right) - f\left(t, S\beta_k, -\alpha_k\right) \\
&\quad + f\left(t, S\beta_k, -\alpha_k\right) - f\left(t, S\beta_k, -\beta_k\right) \\
&\quad + f_x\left(t, S\alpha_k, -\alpha_k\right)\left(S\alpha_{k+1} - S\alpha_k\right) \\
&\quad + f_y\left(t, S\alpha_k, -\alpha_k\right)\left(-\alpha_{k+1} + \alpha_k\right) \\
&\quad - f_x\left(t, S\beta_k, -\beta_k\right)\left(S\beta_{k+1} - S\beta_k\right) \\
&\quad - f_y\left(t, S\beta_k, -\beta_k\right)\left(-\beta_{k+1} + \beta_k\right) \\
&\le f_x\left(t, S\alpha_k, -\alpha_k\right)\left(S\alpha_k - S\beta_k\right) \\
&\quad + f_y\left(t, S\beta_k, -\alpha_k\right)\left(-\alpha_k + \beta_k\right) \\
&\quad + f_x\left(t, S\alpha_k, -\alpha_k\right)\left(S\alpha_{k+1} - S\alpha_k\right) \\
&\quad + f_y\left(t, S\alpha_k, -\alpha_k\right)\left(-\alpha_{k+1} + \alpha_k\right) \\
&\quad - f_x\left(t, S\beta_k, -\beta_k\right)\left(S\beta_{k+1} - S\beta_k\right) \\
&\quad - f_y\left(t, S\beta_k, -\beta_k\right)\left(-\beta_{k+1} + \beta_k\right) \\
&\le f_x\left(t, S\alpha_k, -\alpha_k\right) \\
&\quad \times\left(S\alpha_k - S\beta_k + S\alpha_{k+1} - S\alpha_k - S\beta_{k+1} + S\beta_k\right) \\
&\quad + f_y\left(t, S\alpha_k, -\alpha_k\right) \\
&\quad \times\left(-\alpha_k + \beta_k - \alpha_{k+1} + \alpha_k - \beta_{k+1} + \beta_k\right) \\
&= f_x\left(t, S\alpha_k, -\alpha_k\right)\left(S\alpha_{k+1} - S\beta_{k+1}\right) \\
&\quad + f_y\left(t, S\alpha_k, -\alpha_k\right)\left(-\alpha_{k+1} + \beta_{k+1}\right) = -M(t)\,p.
\end{aligned} \tag{19}$$

We know that $p(0) = \alpha_{k+1}(0) - \beta_{k+1}(0) \le 0$. Hence, from Lemma 2, we deduce that $\alpha_{k+1}(t) \le \beta_{k+1}(t)$, on $I$. Thus, we have monotone sequences $\{\alpha_n\}$, $\{\beta_n\}$ such that

$$\alpha_0 \le \alpha_1 \le \cdots \le \alpha_n \le \beta_n \le \cdots \le \beta_1 \le \beta_0, \quad \text{on } I. \tag{20}$$

Now, employing Ascoli-Arzela's theorem we conclude that the sequences converge uniformly and monotonically to the unique solution $U(t)$ of (2) on $I$.

To show that the convergence rate is quadratic, we begin with $p_{n+1} = x - \alpha_{n+1}$. Then

$$\begin{aligned}
-A\dot{p}_{n+1} &= -A\dot{x} - (-A\dot{\alpha}_{n+1}) \\
&= f(t, Sx, -x) - f(t, S\alpha_n, -\alpha_n) \\
&\quad - f_x(t, S\alpha_n, -\alpha_n)(S\alpha_{n+1} - S\alpha_n) \\
&\quad - f_y(t, S\alpha_n, -\alpha_n)(-\alpha_{n+1} + \alpha_n) \\
&= f(t, Sx, -x) - f(t, S\alpha_n, -\alpha_n) \\
&\quad - f_x(t, S\alpha_n, -\alpha_n)(S\alpha_{n+1} - Sx + Sx - S\alpha_n) \\
&\quad - f_y(t, S\alpha_n, -\alpha_n)(-\alpha_{n+1} + x - x + \alpha_n) \\
&= f(t, Sx, -x) - f(t, S\alpha_n, -\alpha_n) \\
&\quad + f_x(t, S\alpha_n, -\alpha_n) Sp_{n+1} - f_x(t, S\alpha_n, -\alpha_n) Sp_n \\
&\quad - f_y(t, S\alpha_n, -\alpha_n) p_{n+1} + f_y(t, S\alpha_n, -\alpha_n) p_n \\
&= f_x(t, S\alpha_n, -\alpha_n) Sp_{n+1} - f_y(t, S\alpha_n, -\alpha_n) p_{n+1} \\
&\quad + f(t, Sx, -x) - f(t, S\alpha_n, -\alpha_n) \\
&\quad - f_x(t, S\alpha_n, -\alpha_n) Sp_n + f_y(t, S\alpha_n, -\alpha_n) p_n \\
&\le -M(t) p + f(t, Sx, -x) \\
&\quad - f(t, S\alpha_n, -\alpha_n) - f_x(t, S\alpha_n, -\alpha_n) Sp_n \\
&\quad + f_y(t, S\alpha_n, -\alpha_n) p_n \\
&= -M(t) p + f(t, Sx, -x) - f(t, S\alpha_n, -x) \\
&\quad + f(t, S\alpha_n, -x) - f(t, S\alpha_n, -\alpha_n) \\
&\quad - f_x(t, S\alpha_n, -\alpha_n) Sp_n + f_y(t, S\alpha_n, -\alpha_n) p_n \\
&= -M(t) p \\
&\quad + \int_0^1 f_x(t, \sigma Sx + (1-\sigma) S\alpha_n, -x)(Sx - S\alpha_n)\, d\sigma \\
&\quad + \int_0^1 f_y(t, S\alpha_n, \sigma(-x) + (1-\sigma)(-\alpha_n)) \\
&\qquad \times (-x + \alpha_n)\, d\sigma \\
&\quad - f_x(t, S\alpha_n, -\alpha_n) Sp_n + f_y(t, S\alpha_n, -\alpha_n) p_n \\
&= -M(t) p + \int_0^1 [f_x(t, \sigma Sx + (1-\sigma) S\alpha_n, -x) \\
&\qquad - f_x(t, S\alpha_n, -\alpha_n)] Sp_n d\sigma \\
&\quad - \int_0^1 \left[f_y(t, S\alpha_n, \sigma(-x) + (1-\sigma)(-\alpha_n))\right. \\
&\qquad \left. - f_y(t, S\alpha_n, -\alpha_n)\right] p_n d\sigma \\
&= -M(t) p + A_1 + B_1,
\end{aligned} \tag{21}$$

where

$$\begin{aligned}
A_1 &= \int_0^1 [f_x(t, \sigma Sx + (1-\sigma) S\alpha_n, -x) \\
&\qquad - f_x(t, S\alpha_n, -\alpha_n)] Sp_n d\sigma, \\
B_1 &= -\int_0^1 \left[f_y(t, S\alpha_n, \sigma(-x) + (1-\sigma)(-\alpha_n))\right. \\
&\qquad \left. - f_y(t, S\alpha_n, -\alpha_n)\right] p_n d\sigma.
\end{aligned} \tag{22}$$

Set $\xi_1 = \sigma Sx + (1-\sigma)S\alpha_n$. Then

$$\begin{aligned}
A_1 &= \int_0^1 [f_x(t, \xi_1, -x) - f_x(t, S\alpha_n, -\alpha_n)] Sp_n d\sigma \\
&= \int_0^1 [f_x(t, \xi_1, -x) - f_x(t, S\alpha_n, -x) + f_x(t, S\alpha_n, -x) \\
&\qquad - f_x(t, S\alpha_n, -\alpha_n)] Sp_n d\sigma \\
&= \iint_0^1 [f_{xx}(t, \tau\xi_1 + (1-\tau) S\alpha_n, -x)(\xi_1 - S\alpha_n) \\
&\qquad + f_{xy}(t, S\alpha_n, \tau(-x) + (1-\tau)(-\alpha_n)) \\
&\qquad \times (-x + \alpha_n)] Sp_n d\tau\, d\sigma \\
&= \iint_0^1 \left[f_{xx}(t, \xi_2, -x) \sigma (Sx - S\alpha_n)\right. \\
&\qquad \left. - f_{xy}(t, S\alpha_n, \xi_3)(x - \alpha_n)\right] Sp_n d\tau\, d\sigma \\
&= \frac{1}{2} N_1 Sp_n^2 + N_2 p_n Sp_n \le \left(\frac{1}{2} N_1 b^2 + N_2 b\right) p_n^2,
\end{aligned} \tag{23}$$

where $\xi_2 = \tau\xi_1 + (1-\tau)S\alpha_n$, $\xi_3 = \tau(-x) + (1-\tau)(-\alpha_n)$, $f_{xx}(t, U, X) \le N_1$, and $-f_{xy}(t, U, X) \le N_2$.

Next, set $\eta_1 = \sigma(-x) + (1-\sigma)(-\alpha_n)$. Then

$$\begin{aligned}
B_1 &= -\int_0^1 \left[f_y(t, S\alpha_n, \eta_1) - f_y(t, S\alpha_n, -\alpha_n)\right] p_n d\sigma \\
&= -\iint_0^1 \left[f_{yy}(t, S\alpha_n, \tau\eta_1 + (1-\tau)(-\alpha_n))\right. \\
&\qquad \left. \times (\eta_1 + \alpha_n)\right] p_n d\sigma\, d\tau \\
&= \iint_0^1 f_{yy}(t, S\alpha_n, \eta_2) \sigma p_n^2 d\sigma\, d\tau \le \frac{1}{2} N_3 p_n^2,
\end{aligned} \tag{24}$$

where $f_{yy}(t, U, X) \le N_3$ and $\eta_2 = \tau\eta_1 + (1-\tau)(-\alpha_n)$.

Furthermore, we have that

$$-A\dot{p}_{n+1} \le -M(t) p_{n+1} + N_4 p_n^2, \tag{25}$$

where $N_4 = (1/2)N_1 b^2 + N_2 b + (1/2)N_3$.

Using Lemma 2, we show that $p_{n+1}(t) \le U(t)$ on $I$, where $U(t)$ is the solution of

$$-A\dot{U} + M(t) U = N_4 p_n^2, \quad U(0) = 0. \tag{26}$$

Thus, using Lemma 1, the solution of the previously mentioned equation is given as

$$U(t) = e^{-\widehat{A}^D\widehat{M}t}\int_0^t e^{\widehat{A}^D\widehat{M}s}\widehat{A}^D(-\lambda A + M(s))^{-1}N_4 p_n^2(s)\,ds + \left(I - \widehat{A}^D\widehat{A}\right)\widehat{M}^D(-\lambda A + M(t))^{-1}N_4 p_n^2(t). \tag{27}$$

After taking suitable estimates, we obtain

$$p_{n+1} \le K_1 p_n^2, \tag{28}$$

where $K_1 = e^{-\widehat{A}^D\widehat{M}t}\int_0^t e^{\widehat{A}^D\widehat{M}s}\widehat{A}^D(-\lambda A + M(s))^{-1}N_4 ds + (I - \widehat{A}^D\widehat{A})\widehat{M}^D(-\lambda A + M(t))^{-1}N_4$.

Set $q_{n+1} = \beta_{n+1} - x$. Then we can get

$$\begin{aligned}
-A\dot{q}_{n+1} &= -A\dot{\beta}_{n+1} - (-A\dot{x})\\
&= f(t, S\beta_n, -\beta_n) + f_x(t, S\alpha_n, -\alpha_n)(S\beta_{n+1} - S\beta_n)\\
&\quad + f_y(t, S\alpha_n, -\alpha_n)(-\beta_{n+1} + \beta_n) - f(t, Sx, -x)\\
&= f_x(t, S\alpha_n, -\alpha_n)(S\beta_{n+1} - Sx + Sx - S\beta_n)\\
&\quad + f_y(t, S\alpha_n, -\alpha_n)(-\beta_{n+1} + x - x + \beta_n)\\
&\quad + f(t, S\beta_n, -\beta_n) - f(t, Sx, -x)\\
&= f_x(t, S\alpha_n, -\alpha_n)(S\beta_{n+1} - Sx)\\
&\quad - f_y(t, S\alpha_n, -\alpha_n)(\beta_{n+1} - x) + f(t, S\beta_n, -\beta_n)\\
&\quad - f(t, Sx, -x) - f_x(t, S\alpha_n, -\alpha_n)(S\beta_n - Sx)\\
&\quad + f_y(t, S\alpha_n, -\alpha_n)(\beta_n - x)\\
&\le -M(t)q_{n+1} + f(t, S\beta_n, -\beta_n) - f(t, Sx, -\beta_n)\\
&\quad + f(t, Sx, -\beta_n) - f(t, Sx, -x)\\
&\quad - f_x(t, S\alpha_n, -\alpha_n)(S\beta_n - Sx)\\
&\quad + f_y(t, S\alpha_n, -\alpha_n)(\beta_n - x)\\
&= -M(t)q_{n+1}\\
&\quad + \int_0^1 f_x(t, \sigma S\beta_n + (1-\sigma)Sx, -\beta_n)(S\beta_n - Sx)\,d\sigma\\
&\quad + \int_0^1 f_y(t, Sx, \sigma(-\beta_n) + (1-\sigma)(-x))\\
&\qquad \times (-\beta_n + x)\,d\sigma\\
&\quad - f_x(t, S\alpha_n, -\alpha_n)(S\beta_n - Sx)\\
&\quad + f_y(t, S\alpha_n, -\alpha_n)(\beta_n - x)\\
&= -M(t)q_{n+1} + \int_0^1 [f_x(t, \sigma S\beta_n + (1-\sigma)Sx, -\beta_n)\\
&\qquad - f_x(t, S\alpha_n, -\alpha_n)]\,Sq_n d\sigma\\
&\quad - \int_0^1 \left[f_y(t, Sx, \sigma(-\beta_n) + (1-\sigma)(-x))\right.\\
&\qquad \left. - f_y(t, S\alpha_n, -\alpha_n)\right] q_n d\sigma\\
&= -M(t)q_{n+1} + A_2 + B_2,
\end{aligned} \tag{29}$$

in which

$$\begin{aligned}
A_2 &= \int_0^1 [f_x(t, \sigma S\beta_n + (1-\sigma)Sx, -\beta_n)\\
&\qquad - f_x(t, S\alpha_n, -\alpha_n)]\,Sq_n d\sigma,\\
B_2 &= -\int_0^1 \left[f_y(t, Sx, \sigma(-\beta_n) + (1-\sigma)(-x))\right.\\
&\qquad \left. - f_y(t, S\alpha_n, -\alpha_n)\right] q_n d\sigma.
\end{aligned} \tag{30}$$

Set $\theta_1 = \sigma S\beta_n + (1-\sigma)Sx$. Then we get that

$$\begin{aligned}
A_2 &= \int_0^1 [f_x(t, \theta_1, -\beta_n) - f_x(t, S\alpha_n, -\alpha_n)]\,Sq_n d\sigma\\
&= \int_0^1 [f_x(t, \theta_1, -\beta_n) - f_x(t, S\alpha_n, -\beta_n)\\
&\qquad + f_x(t, S\alpha_n, -\beta) - f_x(t, S\alpha_n, -\alpha_n)]\,Sq_n d\sigma\\
&= \iint_0^1 [f_{xx}(t, \tau\theta_1 + (1-\tau)S\alpha_n, -\beta_n)(\theta_1 - S\alpha_n)\\
&\qquad + f_{xy}(t, S\alpha_n, \tau(-\beta_n) + (1-\tau)(-\alpha_n))\\
&\qquad \times (-\beta_n + \alpha_n)]\,Sq_n d\sigma\, d\tau\\
&= \iint_0^1 \left[f_{xx}(t, \theta_2, -\beta_n)(\sigma q_n + p_n)\right.\\
&\qquad \left. - f_{xy}(t, S\alpha_n, \theta_3)(q_n + p_n)\right] Sq_n d\sigma\, d\tau\\
&\le N_1\sigma q_n S q_n + N_1 p_n S q_n + N_2 q_n S q_n + N_2 p_n S q_n\\
&\le \frac{1}{2}N_1 q_n^2 b + bN_1\frac{1}{2}\left(p_n^2 + q_n^2\right) + N_2 b q_n^2 + N_2 b\frac{1}{2}\left(p_n^2 + q_n^2\right)\\
&= \left(N_1 + \frac{3}{2}N_2\right) b q_n^2 + \frac{1}{2}b(N_1 + N_2)\,p_n^2,
\end{aligned} \tag{31}$$

where $\theta_2 = \tau\theta_1 + (1-\tau)S\alpha_n$ and $\theta_3 = \tau(-\beta_n) + (1-\tau)(-\alpha_n)$.

Similarly, set $\vartheta_1 = \sigma(-\beta_n) + (1-\sigma)(-x)$. Then

$$\begin{aligned}
B_2 &= -\int_0^1 \left[f_y\left(t, Sx, \vartheta_1\right) - f_y\left(t, S\alpha_n, -\alpha_n\right)\right] q_n d\sigma \\
&= -\int_0^1 \left[f_y\left(t, Sx, \vartheta_1\right) - f_y\left(t, S\alpha_n, \vartheta_1\right)\right. \\
&\qquad \left. + f_y\left(t, S\alpha_n, \vartheta_1\right) - f_y\left(t, S\alpha_n, -\alpha_n\right)\right] q_n d\sigma \\
&= -\iint_0^1 \left[f_{yx}\left(t, \tau Sx + (1-\tau) S\alpha_n, \vartheta_1\right)\left(Sx - S\alpha_n\right)\right. \\
&\qquad + f_{yy}\left(t, S\alpha_n, \tau\vartheta_1 + (1-\tau)\left(-\alpha_n\right)\right) \\
&\qquad \left. \times \left(\vartheta_1 + \alpha_n\right)\right] q_n d\sigma\, d\tau \\
&= -\iint_0^1 \left[f_{yx}\left(t, \vartheta_2, \vartheta_1\right) S p_n + f_{yy}\left(t, S\alpha_n, \vartheta_3\right)\right. \\
&\qquad \left. \times \left(\sigma\left(-\beta_n + x\right) - \left(x - \alpha_n\right)\right)\right] q_n d\sigma\, d\tau \\
&= \iint_0^1 \left[-f_{yx}\left(t, \vartheta_2, \vartheta_1\right) S p_n\right. \\
&\qquad \left. + f_{yy}\left(t, S\alpha_n, \vartheta_3\right)\left(\sigma q_n + p_n\right)\right] q_n d\sigma\, d\tau \\
&\le N_5 b q_n p_n + \frac{1}{2} N_3 q_n^2 + N_3 p_n q_n \\
&\le \left(\frac{1}{2} b N_5 + N_3\right) q_n^2 + \frac{1}{2}\left(b N_5 + N_3\right) p_n^2,
\end{aligned} \tag{32}$$

where $\vartheta_2 = \tau Sx + (1-\tau)S\alpha_n$, $\vartheta_3 = \tau\vartheta_1 + (1-\tau)(-\alpha_n)$, and $-f_{yx}(t, X, U) \le N_5$.

Then we get that

$$-A\dot{q}_{n+1} \le -M(t) q_{n+1} + N_6 q_n^2 + N_7 p_n^2, \tag{33}$$

where $N_6 = (N_1 + (3/2)N_2)b + ((1/2)bN_5 + N_3)$ and $N_7 = (1/2)b(N_1 + N_2) + (1/2)(bN_5 + N_3)$. Thus we have that

$$\begin{aligned}
& q_{n+1}(t) \\
&\quad \le U(t) = e^{-\widehat{A}^D \widehat{M} t} \int_0^t e^{\widehat{A}^D \widehat{M} s} \widehat{A}^D (-\lambda A + M(s))^{-1} \\
&\qquad \times \left(N_6 q_n(s)^2 + N_7 p_n(s)^2\right) ds \\
&\qquad + \left(I - \widehat{A}^D \widehat{A}\right) \widehat{M}^D (-\lambda A + M(t))^{-1} N_6 q_n(t)^2 \\
&\qquad + N_7 p_n(t)^2.
\end{aligned} \tag{34}$$

Furthermore, we obtain after taking suitable estimates

$$p_{n+1} \le K_2 q_n^2 + K_3 p_n^2, \tag{35}$$

where $K_2 = e^{-\widehat{A}^D \widehat{M} t} \int_0^t e^{\widehat{A}^D \widehat{M} s} \widehat{A}^D (-\lambda A + M(s))^{-1} N_6 ds + (I - \widehat{A}^D \widehat{A}) \widehat{M}^D (-\lambda A + M(t))^{-1} N_6$ and $K_3 = e^{-\widehat{A}^D \widehat{M} t} \int_0^t e^{\widehat{A}^D \widehat{M} s} \widehat{A}^D (-\lambda A + M(s))^{-1} N_7 ds + (I - \widehat{A}^D \widehat{A}) \widehat{M}^D (-\lambda A + M(t))^{-1} N_7$.

Hence we proved that the convergence rate is quadratic. □

Next, we consider singular differential systems BVPs and prove the following main result.

**Theorem 4.** *Let assumptions $(H_{2.1})$–$(H_{2.4})$ hold. Suppose further the following.*

$(H_{3.1})$ *There exist $V_0(t), W_0(t) \in C^2[I, R^n]$ with $V_0(t) \le W_0(t)$ and $\dot{V}_0(t) \ge \dot{W}_0(t)$ on $I$ and*

$$\begin{aligned}
A\ddot{V}_0 &\le f\left(t, V_0, \dot{V}_0\right), \quad \dot{V}_0(0) \ge X_0, \quad V_0(b) \le X_1, \\
A\ddot{W}_0 &\ge f\left(t, W_0, \dot{W}_0\right), \quad \dot{W}_0(0) \le X_0, \quad W_0(b) \ge X_1.
\end{aligned} \tag{36}$$

$(H_{3.2})$ *For $X_0 \in R^n$, $(I - \widehat{A}\widehat{A}^D)(-X_0 - \widehat{W}(0)) = 0$, where $\widehat{W}(t) = (I - \widehat{A}^D\widehat{A})\widehat{M}^D\widehat{g}(t)$.*

*Then there exist monotone sequences $\{V_n\}$ and $\{W_n\}$ which converge uniformly on $I$ to the unique solution of (1) and the convergence rate is quadratic.*

*Proof.* Using the transformation $U = -\dot{X}$, we have that

$$SU(t) = X(t) = X_1 + \int_t^b U(s)\, ds, \quad t \in I. \tag{37}$$

BVPs (1) can be transformed into IVPs of singular system:

$$-A\dot{U} = f(t, SU, -U), \quad U(0) = -X_0. \tag{38}$$

Let $\alpha_0(0) = -\dot{V}_0(0)$ and $\beta_0(0) = -\dot{W}_0(0)$. By using assumption $(H_{3.1})$, we get that

$$\alpha_0(t) \le -\dot{V}_0(t) \le -\dot{W}_0(t) = \beta_0(t) \tag{39}$$

and $\alpha_0(0) = -\dot{V}_0(0) \le -X_0 \le -\dot{W}_0(0) = \beta_0(0)$.

Noticing that $\alpha_0(0) = -\dot{V}_0(0)$, we have that

$$\begin{aligned}
V_0(t) &= V_0(b) + \int_t^b \alpha_0(s)\, ds, \quad t \in I, \\
\left[S\alpha_0\right](t) &= X_1 + \int_t^b \alpha_0(s)\, ds, \quad t \in I.
\end{aligned} \tag{40}$$

Then $V_0(t) \le [S\alpha_0](t)$ from $V_0(b) \le X_1$, and

$$\begin{aligned}
-A\dot{\alpha}_0 = A\ddot{V}_0 &\le f\left(t, V_0, \dot{V}_0\right) \le f\left(t, S\alpha_0, \dot{V}_0\right) \\
&= f\left(t, S\alpha_0, -\alpha_0\right).
\end{aligned} \tag{41}$$

A similar argument shows that

$$\begin{aligned}
-A\dot{\beta}_0 = A\ddot{W}_0 &\ge f\left(t, W_0, \dot{W}_0\right) \ge f\left(t, S\beta_0, \dot{W}_0\right) \\
&= f\left(t, S\beta_0, -\beta_0\right).
\end{aligned} \tag{42}$$

By Theorem 3, there exist monotone sequences $\{\alpha_n\}, \{\beta_n\}$ such that

$$\alpha_n \longrightarrow U \longleftarrow \beta_n \tag{43}$$

and the convergence rate is quadratic. Again set $V_n = S\alpha_n$ and $W_n = S\beta_n$. Then

$$V_0 \leq V_1 \leq V_2 \leq \cdots \leq V_n \leq W_n \leq \cdots \leq W_2 \leq W_1 \leq W_0,$$
$$V_n \longrightarrow U \longleftarrow W_n, \tag{44}$$

and also the convergence rate is quadratic. Noticing that

$$\begin{aligned} x(t) - V_{n+1}(t) &= X_1 + \int_t^b U(s)\,ds - \left(X_1 + \int_0^b \alpha_{n+1}(s)\,ds\right) \\ &= \int_0^b \left(U(s) - \alpha_{n+1}(s)\right) ds \\ &\leq \int_0^b \left[K_1\left(U(s) - \alpha_n(s)\right)^2\right] ds \\ &= K_1\left(x(t) - V_n(t)\right)^2, \end{aligned} \tag{45}$$

we can obtain the similar result that $W_n$ converges quadratically to the solution of (1). The proof is complete. □

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

## Authors' Contribution

All authors completed the paper together. All authors read and approved the final paper.

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## References

[1] V. Lakshmikantham and A. S. Vatsala, *Generalized Quasilinearization for Nonlinear Problems*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1998.

[2] A. R. Abd-Ellateef Kamar, G. M. Attia, K. Vajravelu, and M. Mosaad, "Generalized quasilinearization for singular system of differential equations," *Applied Mathematics and Computation*, vol. 114, no. 1, pp. 69–74, 2000.

[3] W. Wang and Y. Q. Liu, "Monotone iterative for boundary value problem of second order singular dierential system," *Journal of Systems Science and Systems Engineering*, vol. 4, no. 4, pp. 266–272, 1995.

[4] S. L. Campbell, *Singular Systems of Differential Equations*, Pitman Advanced Publishing Program, London, UK, 1980.

[5] A. S. Vatsala, "Monotone iterative technique for singular systems of differential equations," in *Nonlinear Analysis and Applications*, vol. 109 of *Lecture Notes in Pure and Applied Mathematics*, pp. 579–582, Dekker, New York, NY, USA, 1987.