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Research Article

New Exact Solitary Wave Solutions of a Coupled Nonlinear Wave Equation

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By using the theory of planar dynamical systems to a coupled nonlinear wave equation, the existence of bell-shaped solitary wave solutions, kink-shaped solitary wave solutions, and periodic wave solutions is obtained. Under the different parametric values, various sufficient conditions to guarantee the existence of the above solutions are given. With the help of three different undetermined coefficient methods, we investigated the new exact explicit expression of all three bell-shaped solitary wave solutions and one kink solitary wave solutions with nonzero asymptotic value for a coupled nonlinear wave equation. The solutions cannot be deduced from the former references.

1. Introduction

For the investigation of traveling wave solutions to nonlinear partial differential equations, which have been the subject of study in various branches of mathematical physical sciences such as physics, biology, and chemistry, many effective methods (see, e.g., [1–12]) have been presented, such as inverse scattering transform method [1], Hirota's method [2], Backlund and Darboux transformation method [5], the Jacobi (Weierstrass) elliptic function method [7], undetermined coefficient method [9, 10], G'/G-expansion method and others.

Following the work of Hirota and Satsuma in [13], recently, considerable attention has been focused on the study of coupled nonlinear partial differential equation (see, e.g., [14–19]) that can be solved exactly. However, less work on the coupled version of the higher KdV equation seems to have been reported.

In [20], Guha-Roy has been presented a system of coupled nonlinear wave equations as follows

$$u_t + \beta u^2 u_x + \alpha v^2 v_x + \lambda u u_x + \gamma u_{xxx} = 0,$$
$$v_t + \delta (uv)_x + e v v_x = 0, \quad (1)$$

where the subscripts refer to partial differentiations with respect to the indicated variables, and α, β, γ, δ, λ, e are arbitrary parameters. Equation (1) is the coupled version of combined form of the higher (modified) KdV equation and KdV equation. It is interesting to point out that, as is outlined in Wadati [21], (1) shares properties with the KdV and the modified KdV equation, under certain conditions.

Guha-Roy [22] supposed that $|\xi| = |x - ct| \rightarrow +\infty$ with $u(\xi)$, $u'(\xi)$, $u''(\xi) \rightarrow 0$ by transforming to $u(\xi) = 1/\phi(\xi)$ that Weierstrauss elliptic function, some exact solitary wave solutions in special conditions to (1) are obtained. Lu et al. [23] obtain the kink-antikink solitary wave solutions of (1) by using a truncated expansion. However, there has not been found any literature on the general analysis of the existences of the solitary wave solution and the exact expressions of the solitary wave solution with nonzero asymptotic value for (1).

In this work, we investigate generally the existence of bell-shaped, kink-shaped solitary wave solutions and periodic wave solutions by using the theory of planar dynamical system under different parameter conditions. It follows that we obtain the new exact expressions of all three bell-shaped solitary wave solutions, and one kink-shaped solitary wave solution with nonzero asymptotic value for (1) by undetermined coefficient methods. These solutions obtained here cannot be deduced from [20, 23].

2. Existence of the Bounded Traveling Wave Solutions to (1)

By introducing an analogue of the stream function, Guha-Roy et al. [18] have shown that if one of the solutions of some coupled nonlinear equations is of the traveling wave type, then the other must also exhibit the same form. Keeping this in mind, we choose a new variable $\xi = x - ct$, where c is the wave speed, such that $u(x,t) = \varphi(\xi)$ and $v(x,t) = \psi(\xi)$, substituting them to (1) yields

$$\begin{gathered}-c\varphi' + \beta\varphi^2\varphi' + \alpha\psi^2\psi' + \lambda\varphi\varphi' + \gamma\varphi''' = 0,\\ -c\psi' + \delta(\phi\varphi)' + e\psi\psi' = 0,\end{gathered} \tag{2}$$

where the prime denotes the derivative with respect to ξ.

Integrating the second equation of (2), we have

$$\varphi = \frac{k}{\psi} + \frac{c}{\delta} - \frac{e\psi}{2\delta}, \tag{3}$$

where k is the integration constant treated as an arbitrary parameters. In order to have a regular φ everywhere, we have to impose $k = 0$. It may be noted that $\varphi(\xi)$ satisfies the following boundary conditions:

$$\lim_{x\to+\infty}\varphi(\xi) = D_+, \qquad \lim_{x\to-\infty}\varphi(\xi) = D_-, \tag{4}$$

$$\varphi'(\xi) \longrightarrow 0, \qquad \varphi''(\xi) \longrightarrow 0, \tag{5}$$

as $|\xi| \to \infty$. Condition (4) is different from those considered by Guha-Roy [20], in Guha-Roy's, $\varphi(\xi)$ was found to be vanished in the infinity.

Thus, (3) reduces to

$$\psi = \frac{2c - 2\delta\varphi}{e}. \tag{6}$$

This shows that $\psi(\xi)$ is directly related to $\varphi(\xi)$. By (6) and the first equation of (2), we get, after rearrangement,

$$\gamma\varphi'' + \frac{b_3}{3}\varphi^3 + \frac{b_2}{2}\varphi^2 - b_1\varphi = g, \tag{7}$$

where $b_1 = c + (8\alpha\delta c^2/e^3)$, $b_2 = \lambda + (16\alpha\delta^2 c/e^3)$, $b_3 = \beta - (8\alpha\delta^3/e^3)$, and g is an arbitrary integration constant.

After the translation $\overline{\varphi} = \varphi + (b_2/2b_3)$, we rewrite (7) to

$$\gamma\overline{\varphi}'' + \frac{b_3}{3}\left(\overline{\varphi}^3 + p\overline{\varphi} + q\right) = 0, \tag{8}$$

where $p = -(12b_1b_3 + 3b_2^2)/4b_3^2$ and $q = (b_2^3 + 6b_1b_2b_3 - 12b_3^2 g)/4b_3^3$.

It follows $x = \overline{\varphi}(\xi)$ and $y = \overline{\varphi}'(\xi)$ that (8) is equivalent to the two dimensional system

$$\begin{aligned}\frac{dx}{d\xi} &= y,\\ \frac{dy}{d\xi} &= -\frac{b_3}{3\gamma}\left(x^3 + px + q\right),\end{aligned} \tag{9}$$

which has the first integral

$$H(x,y) = \frac{y^2}{2} + \frac{b_3}{3\gamma}\left(qx + \frac{p}{2}x^2 + \frac{1}{4}x^4\right) = h. \tag{10}$$

System (9) is a four-parameter planar dynamical system depending on the parameter group (γ, b_3, p, q). Because of the phase orbits defined by the vector fields of system (9) that determine all traveling wave solutions of (8), we will investigate the phase portraits of (9) in the phase plane (x, y) as the parameters γ, b_3, p, q are changed.

We point out that here we are considering a physical model where only bounded traveling waves solutions are meaningful, so that we only pay attention to the bounded solutions of system (9).

To investigate the equilibrium points of system (9), we need to find all real zeros of the function $f(x) = x^3 + px + q$. Suppose that $p < 0$ and $\Delta = (q/2)^2 + (p/3)^3 \le 0$, clearly, $f(x)$ has three real zeros at most, denoted by x_1, x_2, and x_3. Therefore, system (9) has three equilibrium points at $p_i(x_i, 0)$, $i = 1, 2, 3$, at most.

Let

$$J(x_i, 0) = \begin{pmatrix} 0 & 1 \\ -\frac{\beta_2}{3\gamma}f'(x_i) & 0 \end{pmatrix}, \tag{11}$$

where $f'(x_i) = 3x_i^2 + p$, $i = 1, 2, 3$, is the coefficient matrix of the linearized system of (9) at equilibrium point $p_i(x_i, 0)$, $i = 1, 2, 3$. At this equilibrium point, we obtain the determinant of matrix $J(x_i, 0)$ which is

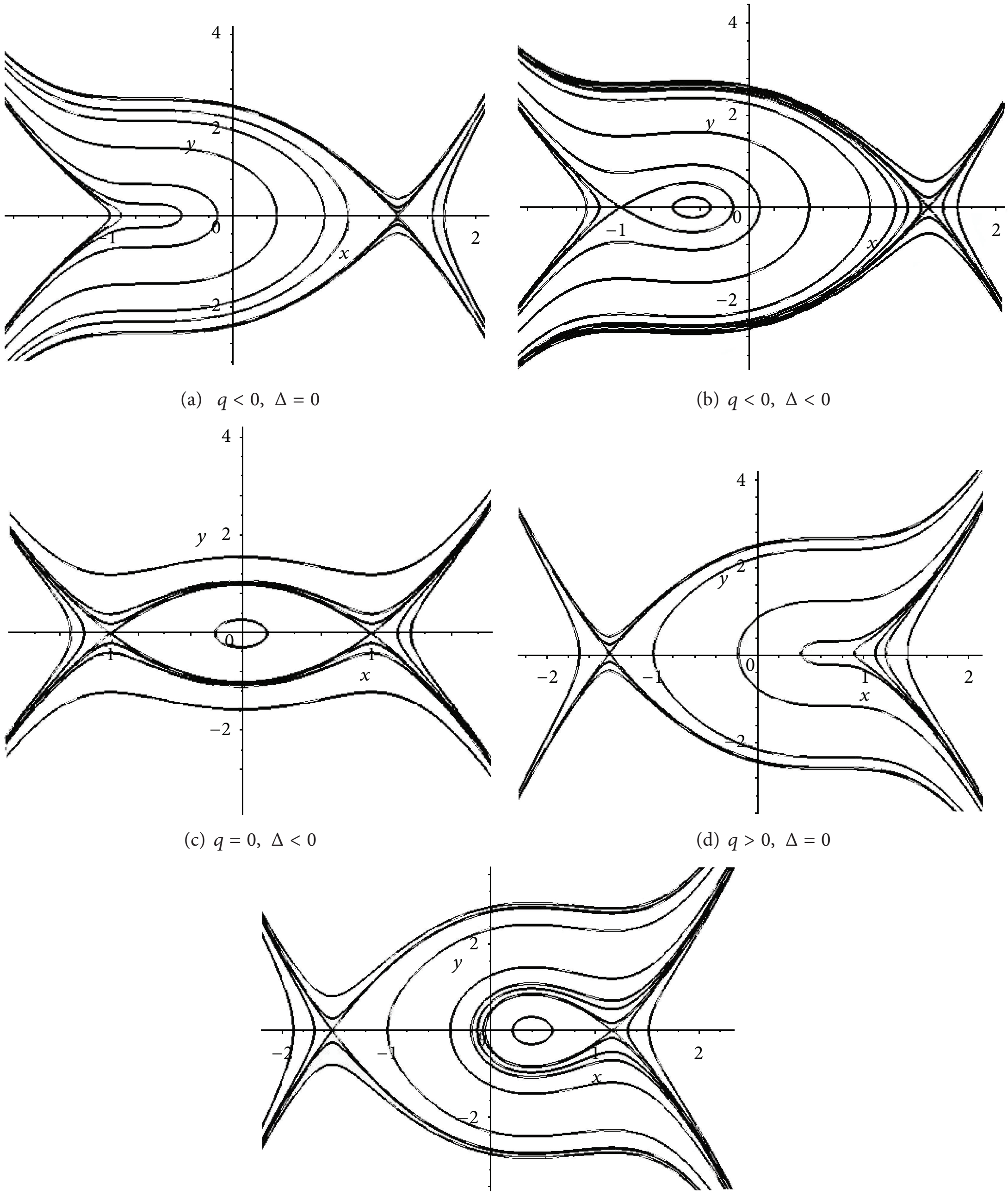
$$\det J(x_i, 0) = \frac{b_3}{3\gamma}f'(x_i), \quad i = 1, 2, 3. \tag{12}$$

By the theory of planar dynamical systems [24–26] for an equilibrium point of a planar dynamical (Hamiltonian) system, if $\det J(x_i, 0) < 0$, then the equilibrium point p_i is a saddle point; if $\det J(x_i, 0) > 0$, then the equilibrium point p_i is a center point; and if $\det J(x_i, 0) = 0$ and the Poincare index of the equilibrium point is zero, then the equilibrium point p_i is a cusp point. So, we have

(1) for $b_3 > 0$ and $\Delta < 0$, there exists three equilibrium points of system (9) at $p_i(x_i, 0)$, $i = 1, 2, 3$, with $x_1 < x_2 < x_3$. The points p_1 and p_3 are center points, p_2 is a saddle point. There is two homoclinic orbits to the saddle point p_2, in which there exists a family of periodic orbits surrounding the center p_1 and p_3. The phase portrait is shown in Figures 1(b), 1(c), and 1(e).

(2) For $b_3 > 0$ and $\Delta = 0$, there exists two equilibrium points of system (9) at $p_i(x_i, 0)$. If $q < 0$, the point $p_1 = p_2$ is a cusp point, p_3 is a center point; if $q > 0$, the point p_1 is a center point, $p_3 = p_2$ is a cusp point. There is a homoclinic orbit to the cusp point p_2, in which there exists a family of periodic orbits surrounding the center p_1 (or p_3). The phase portrait is shown in Figures 1(a) and 1(d).

(3) For $b_3 < 0$ and $\Delta < 0$, there exists three equilibrium points of system (9) at $p_i(x_i, 0)$ with $x_1 < x_2 < x_3$.



(a) $q < 0,\ \Delta = 0$

(b) $q < 0,\ \Delta < 0$

(c) $q = 0,\ \Delta < 0$

(d) $q > 0,\ \Delta = 0$

(e) $q > 0,\ \Delta < 0$

Figure 1: The phase portraits of system (9) for $b_3 > 0$.

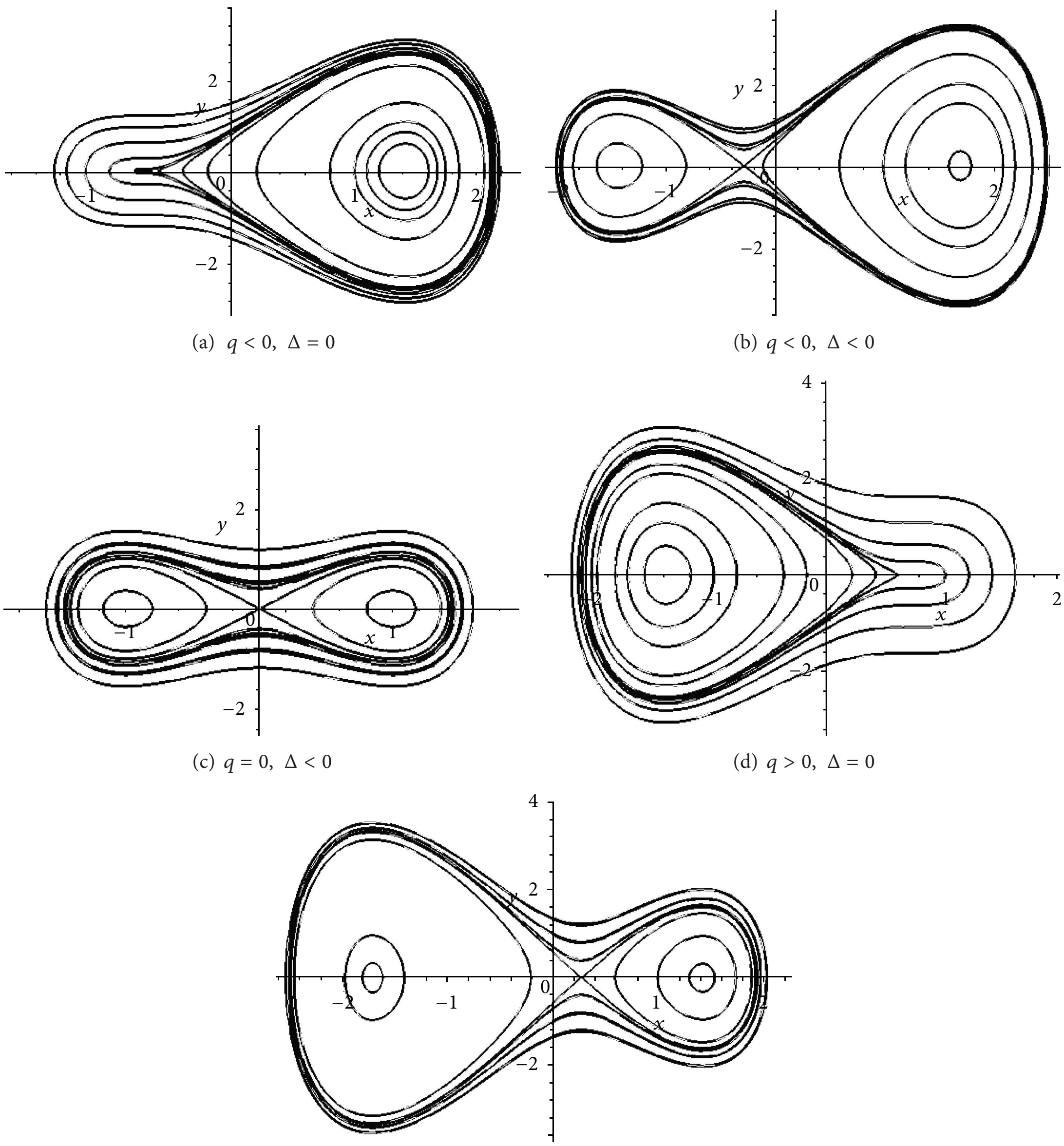
The points p_1 and p_3 are saddle points, p_2 is a center point. If $q \neq 0$, There is a homoclinic orbit to the saddle points p_1 and p_3, respectively, if $q = 0$. There are two heteroclinic orbits connecting the saddle point p_1 and p_3, in which there exists a family of periodic orbits surrounding the center p_2. The phase portrait is shown in Figures 2(b), 2(c), and 2(e).

(4) For $b_3 < 0$ and $\Delta = 0$, there exists two equilibrium points of system (9). If $q < 0$, the point $p_1 = p_2$ is a cusp point, p_3 is a saddle point; if $q > 0$, the point p_1 is a saddle point, $p_3 = p_2$ is a cusp point. There does not exist bounded orbits. The phase portraits are shown in Figures 2(a) and 2(d).

Because orbits cannot be changed by the transformation $\overline{\varphi} = \varphi + (b_2/2b_3)$, it follows that (6), (7), (8), and the above discussion the following.

Theorem 1. *Suppose that $b_3 > 0$, wave speed c, and integration constant g satisfy $4b_1b_3 + b_2^2 > 0$; then*

(1) *when $\Delta < 0$, (1) has two bell-shaped solitary wave solutions and uncountable infinite many periodic traveling*



(a) $q < 0,\ \Delta = 0$

(b) $q < 0,\ \Delta < 0$

(c) $q = 0,\ \Delta < 0$

(d) $q > 0,\ \Delta = 0$

(e) $q > 0,\ \Delta < 0$

FIGURE 2: The phase portraits of system (9) for $b_3 < 0$.

wave solutions in the case of $q > 0$, $q = 0$, *and* $q < 0$, *respectively, (see Figures 1(b), 1(c), and 1(e)).*

(2) *When* $\Delta = 0$, (1) *has one bell-shaped solitary wave solution and uncountably infinite many periodic traveling wave solutions in the case of* $q > 0$ *and* $q < 0$, *respectively, (see Figures 1(a), and 1(d)).*

Theorem 2. *Suppose that* $b_3 < 0$, *wave speed c, and integration constant g satisfy* $4b_1b_3 + b_2^2 > 0$; *then*

(1) *when* $\Delta < 0$, (1) *has one bell-shaped solitary wave solution and uncountably infinite many periodic traveling wave solutions in the case of* $q > 0$ *and* $q < 0$, *respectively, and two kink-shaped solitary wave solutions and uncountably infinite many periodic traveling wave solutions in the case of* $q = 0$ *(see Figures 2(b), 2(c), and 2(e)).*

(2) *When* $\Delta = 0$, (1) *does not exist bounded traveling wave solutions (see Figures 2(a), and 2(d)).*

3. Exact Explicit Representations of Bell-Shaped and Kink-Shaped Solitary Wave Solutions

According to the discussion in Section 2, we assume that (7) has solution with the following form:

$$\begin{aligned}\varphi(\xi) &= \frac{Ae^{m(\xi+\xi_0)}}{\left(1+e^{m(\xi+\xi_0)}\right)^2 + Be^{m(\xi+\xi_0)}} + D\\ &= \frac{A\,\mathrm{sech}^2\left(\left(m\left(\xi+\xi_0\right)\right)/2\right)}{4 + BA\,\mathrm{sech}^2\left(m\left(\xi+\xi_0\right)/2\right)} + D,\end{aligned} \tag{13}$$

where A, B, D, and m are undetermined real parameters and ξ_0 is arbitrary constant.

Substituting (13) and $\varphi'(\xi)$, $\varphi''(\xi)$ into (7), by using the linear independence of $e^{km(\xi+\xi_0)}$, $k = 0, 1, \ldots, 6$, we obtain that the following algebraic equations with A, B, D, and m:

$$\begin{gathered} 2b_3D^3 + 3b_2D^2 - 6b_1D - 6g = 0, \\ \gamma m^2 + b_2D + b_3D^2 - b_1 = 0, \\ (b_2 + 2b_3D)A - 6\gamma m^2(2 + B) = 0, \\ 2b_3A^2 + (2 + B)(3b_2 + 6b_3D)A - 6m^2(2 + B)^2 - 48m^2 = 0. \end{gathered} \tag{14}$$

Suppose that D satisfy

$$2b_3D^3 + 3b_2D^2 - 6b_1D - 6g = 0. \tag{15}$$

By solving the above equations; we have the following.

Case 1. For $b_2 + 2b_3D \neq 0$, we have

$$\begin{gathered} m = \pm\sqrt{-\frac{b_3D^2 + b_2D - b_1}{\gamma}}, \qquad A = \frac{6\gamma m^2(2 + B)}{b_2 + 2b_3D}, \\ 2 + B = \pm\frac{\left|b_2 + 2b_3D\right|}{\sqrt{b_2^2 + 6b_3b_1 - 2b_3^2D^2 - 2b_2b_3D}}. \end{gathered} \tag{16}$$

Case 2. For $b_2 + 2b_3D = 0$, $b_3 > 0$, $b_1 > b_2D/2$, we have

$$\begin{gathered} m = \pm\sqrt{\frac{b_1}{\gamma} - \frac{b_2D}{2\gamma}}, \qquad A = \pm\sqrt{\frac{12}{b_3}(2b_1 - b_2D)}, \\ B = -2. \end{gathered} \tag{17}$$

By (6), (13), and the above conclusions, we can write the exact solutions of (1).

Theorem 3. *For $\gamma > 0$, $\xi = x - ct$, ξ_0 is an arbitrary constant; suppose that D is real and satisfies $2b_3D^3 + 3b_2D^2 - 6b_1D - 6g = 0$; then*

(1) *when $A_1 = b_2 + 2b_3D \neq 0$, $A_2 = b_3D^2 + b_2D - b_1 < 0$, and $A_3 = b_2^2 + 6b_3b_1 - 2b_3^2D^2 - 2b_2b_3D > 0$, (1) has the following bell-shaped solitary wave solutions:*

$$\begin{gathered} \varphi_1^{\pm}(\xi) = \frac{\pm\left(6A_2\left|A_1\right|/\sqrt{A_3}A_1\right)\operatorname{sech}^2\eta}{2 + \left(-1 \pm \left(\left|A_1\right|/\sqrt{A_3}\right)\right)\operatorname{sech}^2\eta} + D, \\ \psi_1^{\pm}(\xi) = \frac{2c - 2\delta\varphi_1^{\pm}}{e}, \end{gathered} \tag{18}$$

where $\eta = (1/2)\sqrt{-(b_3D^2 + b_2D - b_1)/\gamma}(\xi + \xi_0)$.

(2) *When $A_1 = b_2 + 2b_3D = 0$, $b_3 > 0$ and $b_1 > (b_2/2)D$, (1) has the following bell-shaped solitary wave solutions*

$$\begin{aligned} &\varphi_2^{\pm}(\xi) \\ &= \pm\sqrt{\frac{3}{b_3}(2b_1 - b_2D)}\operatorname{sech}\left(\sqrt{\frac{2b_1 - b_2D}{2}}(\xi + \xi_0)\right) + D, \\ &\psi_2^{\pm}(\xi) = \frac{2c - 2\delta\varphi_2^{\pm}}{e}. \end{aligned} \tag{19}$$

Remark 4. (1) (φ_i^+, ψ_i^+), $i = 1, 2$, denotes the solitary wave solutions taking "+" in expression of (16) and (17), (φ_i^-, ψ_i^-), $i = 1, 2$, is similar.

(2) Substituting $D = 0$, $\beta = 8\alpha\delta^3/e^3$ into solutions (φ_1^+, ψ_1^+) of expressions (18) yields

$$\begin{gathered} \varphi_1^+(\xi) = \frac{3b_1}{b_2}\operatorname{sech}^2\frac{1}{2}\sqrt{\frac{b_1}{\gamma}}(\xi + \xi_0), \\ \psi_1^+(\xi) = \frac{2c - 2\delta\varphi_1^+}{e}, \end{gathered} \tag{20}$$

where $b_1 > 0$ and $b_2 > 0$ are denoted by (7). Solution (20) is the solitary wave solution (17) and (7) of Guha-Roy [20].

(3) Substituting $D = 0$ and $\lambda = -16\alpha\delta^2c/e^3$ into solutions $(\varphi_2^{\pm}, \psi_2^{\pm})$ of expressions (19) yields

$$\begin{gathered} \varphi_2^{\pm}(\xi) = \pm\sqrt{\frac{6b_1}{b_3}}\operatorname{sech}\sqrt{\frac{b_1}{\gamma}}(\xi + \xi_0), \\ \psi_2^{\pm}(\xi) = \frac{2c - 2\delta\varphi_2^{\pm}}{e}, \end{gathered} \tag{21}$$

where $b_1 > 0$ and $b_3 > 0$ are denoted by (7). Solution (21) is the solitary wave solution (20) and (7) of Guha-Roy [20].

(4) Solutions (18) and (19) cannot be obtained by the method used in Guha-Roy [20].

(5) For $b_3 > 0$, $q < 0$, and $\Delta < 0$, the roots D_1, D_2, and D_3 of $2b_3D^3 + 3b_2D^2 - 6b_1D - 6g = 0$ satisfy $x_i = D_i + (b_2/2b_3)$, $i = 1, 2, 3$, where x_1, x_2, and x_3 are the roots of $f(x) = 0$. After computation, we know that $D_1 < D_2 < -(b_2/2b_3) < D_3$, $b_2 + 2b_3D_2 < 0$, and $b_3D_2^2 + b_2D_2 - b_1 < 0$, so, solution φ_1^+ in expression (18) is less than D_2 and φ_1^- in expression (18) is over D_2. Then, two homoclinic orbits at a saddle point $(D_2, 0)$ can be denoted by solutions $\varphi_1^{\pm}$ in (18), in here, φ_1^+ denotes the left hand homoclinic orbit and φ_1^- denotes the right hand homoclinic orbit in Figure 1(b). Similarly, we can explain that solutions $\varphi_1^{\pm}$ denote the two homoclinic orbits in Figure 1(e).

(6) For $b_3 < 0$, $q < 0$, and $\Delta < 0$, the roots D_1, D_2, and D_3 of $2b_3D^3 + 3b_2D^2 - 6b_1D - 6g = 0$ satisfy $x_i = D_i + (b_2/2b_3)$, $i = 1, 2, 3$, where x_1, x_2, and x_3 are the roots of $f(x) = 0$. $(D_1, 0)$ and $(D_3, 0)$ are saddle points and $(D_2, 0)$ is a center point. After computation, we know that $D_1 < D_2 < -(b_2/2b_3) < D_3$, $b_2 + 2b_3D_1 > 0$ and $b_3D_1^2 + b_2D_1 - b_1 < 0$, by $q = A_1A_3/4b_3^3$ yields $A_3 > 0$, $\varphi_1^+(\xi) > D_1$. It follows $b_2 + 2b_3D_3 < 0$ and $b_3D_3^2 + b_2D_3 - b_1 < 0$, $q = A_1A_3/4b_3^3$ that $A_3 < 0$, so $\varphi_1^-(\xi)$ is

unbounded. Therefore, the homoclinic orbit at a saddle point $(D_1, 0)$ in Figure 2(b) is denoted by φ_1^+ in expression (18), φ_1^- is unbounded. Similarly, we can explain that solutions φ_1^+ denote the homoclinic orbits in Figure 2(e).

(7) For $b_3 > 0$, $q = 0$ and $\Delta < 0$, point $(D_1, 0)$ and $(D_3, 0)$ are corresponding to the centers $p_1(-\sqrt{-p}, 0)$ and $p_3(\sqrt{-p}, 0)$ of system (9), respectively, points $(D_2, 0)$ is corresponding to a saddle point $p_2(0, 0)$ of system (9), where $D_2 = -b_2/2b_3$, $D_2 < 2b_1/b_2$; then $\varphi_2^\pm$ denotes the two homoclinic orbits in Figure 1(c).

If we suppose that (7) has solutions with the following form:

$$\varphi(x - ct) = \varphi(\xi) = \frac{Ae^{m(\xi+\xi_0)}}{1 + e^{m(\xi+\xi_0)}} + D, \tag{22}$$

where A, D, and m are undetermined real parameters and ξ_0 is arbitrary constant.

Substituting (22) into (7) and using the linear independence of $e^{k(\xi+\xi_0)}$, $k = 0, 1, 2, 3$ yields

$$\begin{gathered} 2b_3D^3 + 3b_2D^2 - 6b_1D - 6g = 0, \\ \gamma m^2 + b_2D + b_3D^2 - b_1 = 0, \\ 2b_3A^2 + (3b_2 + 6b_3D)A - 6\gamma m^2 = 0, \\ (3b_2 + 6b_3D)A - 18\gamma m^2 = 0. \end{gathered} \tag{23}$$

By solving the above equations, we obtain

$$\begin{gathered} m = \pm\sqrt{-\frac{b_2^2 + 4b_1b_3}{2\gamma b_3}}, \qquad A = \pm\sqrt{\frac{3b_2^2 + 12b_1b_3}{b_3^2}}, \\ D = -\frac{b_2}{2b_3} \pm \frac{\sqrt{3b_2^2 + 12b_1b_3}}{2b_3}. \end{gathered} \tag{24}$$

It follows (22), (24) and (6) that

Theorem 5. *Suppose that $b_2^2 + 4b_1b_3 > 0$, $b_3 < 0$, $\gamma > 0$, $\xi = x - ct$, ξ_0 is an arbitrary constant, and $D = -(b_2/2b_3) \pm (\sqrt{3b_2^2 + 12b_1b_3}/2b_3)$, (1) has the following kink-shaped solitary wave solutions:*

$$\begin{gathered} \varphi_3^\pm(\xi) \\ = \left(D + \frac{b_2}{2b_3}\right)\tanh\left(\frac{1}{2}\sqrt{-\frac{b_2^2 + 4b_1b_3}{2\gamma b_3}}(\xi + \xi_0)\right) - \frac{b_2}{2b_3}, \\ \psi_3^\pm(\xi) = \frac{2c - 2\delta\varphi_3^\pm}{e}. \end{gathered} \tag{25}$$

Remark 6. (1) In [20], expression (13) can be rewritten as follows:

$$u(\xi) = -\frac{3b_1}{b_2}\left(1 \pm \frac{1}{2}\sqrt{\frac{b_1}{6}}(\xi + \xi_0)\right), \tag{26}$$

where $b_2^2 = -6b_1b_3$. Comparing expression (26) with $\varphi_3^\pm$ in expression (25), we know that the solution (13) solved by Guha-Roy [20] is in accordance with solutions $\varphi_3^\pm$ of expression (25) in the case of $b_2^2 = -6b_1b_3$. But the general solutions (25) cannot be solved by Guha-Roy [20].

(2) For $b_3 < 0$, $q = 0$, and $\Delta < 0$, points $(D_1, 0)$ and $(D_3, 0)$ are two saddle points of system (9), and $(D_2, 0)$ is a center point, where $D_{1,3} = -(b_2/2b_3) \pm (\sqrt{3b_2^2 + 12b_1b_3}/2b_3)$, $D_2 = -b_2/2b_3$. According to $q = 0$ yields $b_2^2 + 6b_1b_3 - 2b_3^2D_i^2 - 2b_2b_3D_i = 0$, $i = 1, 3$; thus, $\varphi_3^+(\xi)$ and $\varphi_3^-(\xi)$ of expression (25) are corresponding to the two heteroclinic orbits in Figure 2(c).

If we suppose that (7) has solutions with the following form:

$$\varphi(\xi) = \frac{A}{B + m\xi^2} + D, \tag{27}$$

where A, B, D, and m are undetermined real parameters and ξ_0 is arbitrary constant.

By using (27) and its derivations, it follows (7) that

$$\begin{gathered} b_3D^2 + b_2D - b_1 = 0, \qquad A = -\frac{2B(b_2 + 2b_3D)}{b_3}, \\ m = \frac{B(b_2^2 + 4b_1b_3)}{6\gamma b_3}, \qquad g = \frac{b_1b_2 - 4b_1b_3D - b_2^2D}{6b_3}. \end{gathered} \tag{28}$$

Further, we obtain the following results.

Theorem 7. *Suppose that $b_3 > 0$, $\gamma > 0$, and $\xi = x - ct$, ξ_0 is an arbitrary constant, and D satisfies $b_3D^2 + b_2D - b_1 = 0$ integration constant $g = (b_1b_2 - 4b_1b_3D - b_2^2D)/6b_3$; then (1) has the following bell-shaped solitary wave solutions:*

$$\begin{gathered} \varphi_4(\xi) = -\frac{12(b_2 + 2b_3D)\gamma}{6\gamma b_3 + (b_2^2 + 4b_1b_3)(\xi + \xi_0)^2} + D, \\ \psi_4(\xi) = \frac{2c - 2\delta\varphi_4}{e}. \end{gathered} \tag{29}$$

Remark 8. (1) By the hypothesis $p < 0$ and $b_3D^2 + b_2D - b_1 = 0$, we know that $b_2^2 + 4b_1b_3 > 0$, and that $D_1 = -b_2/2b_3 + \sqrt{3b_2^2 + 12b_1b_3}/2b_3$, $D_2 = -b_2/2b_3 - \sqrt{3b_2^2 + 12b_1b_3}/2b_3$. If we take $D = D_1$ in solution (29), solution $\varphi_4(\xi)$ of expression (29) is corresponding to the homoclinic orbit in Figure 1(d). If we take $D = D_2$ in solution (29), solution $\varphi_4(\xi)$ of expression (29) is corresponding to the homoclinic orbit in Figure 1(a).

4. Discussion and Conclusion

In this paper, we obtain all the three bell-shaped and one kink-shaped solitary wave solutions of (1) by using three different undetermined coefficient methods. The conclusions have not been deduced from the method reported by Guha-Roy. The method is simple and can be applied to solve many couple nonlinear equations such as Ito equation, Ito-type equation, and coupled KdV equations.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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