Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 293532, 6 pages
http://dx.doi.org/10.1155/2013/293532

Research Article

Some Identities on the Generalized q-Bernoulli, q-Euler, and q-Genocchi Polynomials

Daeyeoul Kim,[1] Burak Kurt,[2] and Veli Kurt[2]

[1] *National Institute for Mathematical Sciences, Yuseong-daero 1689-gil, Yuseong-gu, Daejeon 305-811, Republic of Korea*
[2] *Department of Mathematics, Akdeniz University, 07058 Antalya, Turkey*

Correspondence should be addressed to Veli Kurt; vkurt@akdeniz.edu.tr

Received 13 September 2013; Accepted 12 November 2013

Academic Editor: Junesang Choi

Mahmudov (2012, 2013) introduced and investigated some q-extensions of the q-Bernoulli polynomials $\mathscr{B}_{n,q}^{(\alpha)}(x, y)$ of order α, the q-Euler polynomials $\mathscr{E}_{n,q}^{(\alpha)}(x, y)$ of order α, and the q-Genocchi polynomials $\mathscr{G}_{n,q}^{(\alpha)}(x, y)$ of order α. In this paper, we give some identities for $\mathscr{B}_{n,q}^{(\alpha)}(x, y)$, $\mathscr{G}_{n,q}^{(\alpha)}(x, y)$, and $\mathscr{E}_{n,q}^{(\alpha)}(x, y)$ and the recurrence relations between these polynomials. This is an analogous result to the q-extension of the Srivastava-Pintér addition theorem in Mahmudov (2013).

1. Introduction, Definitions, and Notations

Throughout this paper, we always make use of the following notation: $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{C}$ denotes the set of complex numbers. The q-numbers and q-factorial are defined by

$$[a]_q = \frac{1 - q^a}{1 - q}, \quad q \neq 1,$$
$$[n]_q! = [n]_q [n-1]_q \cdots [2]_q [1]_q, \tag{1}$$

respectively, where $[0]_q! = 1$, $n \in \mathbb{N}$, and $a \in \mathbb{C}$. The q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q:q)_n}{(q:q)_{n-k}(q:q)_k}, \tag{2}$$

where $(q:q)_n = (1-q)\cdots(1-q^n)$. The q-analogue of the function $(x+y)_q^n$ is defined by

$$(x+y)_q^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{(k(k-1))/2} x^{n-k} y^k. \tag{3}$$

The q-binomial formula is known as

$$\begin{aligned} (n;q)_a &= (1-a)_q^n \\ &= \prod_{j=0}^{n-1} \left(1 - q^j a\right) \\ &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{(k(k-1))/2} (-1)^k a^k. \end{aligned} \tag{4}$$

The q-exponential functions are given by

$$\begin{aligned} e_q(z) &= \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} \\ &= \prod_{k=0}^{\infty} \frac{1}{(1-(1-q)q^k z)}, \quad 0 < |q| < 1, \\ &\qquad |z| < \frac{1}{|1-q|}, \end{aligned}$$

$$E_q(z) = \sum_{n=0}^{\infty} q^{(n(n-1))/2} \frac{z^n}{[n]_q!} = \prod_{k=0}^{\infty} \left(1 + (1-q) q^k z\right), \quad 0 < |q| < 1, \ z \in \mathbb{C}. \tag{5}$$

From these forms, we easily see that $e_q(z)E_q(-z) = 1$. Moreover, $D_q e_q(z) = e_q(z)$ and $D_q E_q(z) = E_q(qz)$, where D_q is defined by

$$D_q f(z) = \frac{f(qz) - f(z)}{qz - z}, \quad 0 < |q| < 1, \ 0 \neq z \in \mathbb{C}. \tag{6}$$

The previous q-standard notation can be found in [1, 2]. Carlitz firstly extended the classical Bernoulli numbers and polynomials and Euler numbers and polynomials [3, 4]. There are numerous recent investigations on this subject by many other authors. Among them are Cenkci et al. [5, 6], Choi et al. [1], Cheon [7], Kim [8], Kurt [9], Kurt [10], Luo and Srivastava [11–13], Srivastava et al. [14, 15], Natalini and Bernardini [16], Tremblay et al. [17, 18], Gaboury and Kurt [19], Mahmudov [2, 20, 21], Araci et al. [22], and Kupershmidt [23].

Mahmudov defined and studied the properties of the following generalized q-Bernoulli polynomials $\mathscr{B}_{n,q}^{(\alpha)}(x,y)$ of order α and q-Euler polynomials $\mathscr{E}_{n,q}^{(\alpha)}(x,y)$ of order α as follows [2].

Let $q \in \mathbb{C}$, $\alpha \in \mathbb{N}$, and $0 < |q| < 1$. The q-Bernoulli numbers $\mathscr{B}_{n,q}^{(\alpha)}$ and polynomials $\mathscr{B}_{n,q}^{(\alpha)}(x,y)$ in x and y of order α are defined by means of the generating functions:

$$\sum_{n=0}^{\infty} \mathscr{B}_{n,q}^{(\alpha)} \frac{t^n}{[n]_q!} = \left(\frac{t}{e_q(t) - 1}\right)^{\alpha}, \quad |t| < 2\pi, \tag{7}$$

$$\sum_{n=0}^{\infty} \mathscr{B}_{n,q}^{(\alpha)}(x,y) \frac{t^n}{[n]_q!} = \left(\frac{t}{e_q(t) - 1}\right)^{\alpha} e_q(tx) E_q(ty), \quad |t| < 2\pi. \tag{8}$$

The q-Euler numbers $\mathscr{E}_{n,q}^{(\alpha)}$ and polynomials $\mathscr{E}_{n,q}^{(\alpha)}(x,y)$ in x and y of order α are defined by means of the generating functions:

$$\sum_{n=0}^{\infty} \mathscr{E}_{n,q}^{(\alpha)} \frac{t^n}{[n]_q!} = \left(\frac{2}{e_q(t) + 1}\right)^{\alpha}, \quad |t| < \pi, \tag{9}$$

$$\sum_{n=0}^{\infty} \mathscr{E}_{n,q}^{(\alpha)}(x,y) \frac{t^n}{[n]_q!} = \left(\frac{2}{e_q(t) + 1}\right)^{\alpha} e_q(tx) E_q(ty), \quad |t| < \pi. \tag{10}$$

The q-Genocchi numbers $\mathscr{G}_{n,q}^{(\alpha)}$ and polynomials $\mathscr{G}_{n,q}^{(\alpha)}(x,y)$ in x and y of order α are defined by means of the generating functions:

$$\sum_{n=0}^{\infty} \mathscr{G}_{n,q}^{(\alpha)} \frac{t^n}{[n]_q!} = \left(\frac{2t}{e_q(t) + 1}\right)^{\alpha}, \quad |t| < \pi, \tag{11}$$

$$\sum_{n=0}^{\infty} \mathscr{G}_{n,q}^{(\alpha)}(x,y) \frac{t^n}{[n]_q!} = \left(\frac{2t}{e_q(t) + 1}\right)^{\alpha} e_q(tx) E_q(ty), \quad |t| < \pi. \tag{12}$$

The familiar q-Stirling numbers $S_{2,q}(n,k)$ of the second kind are defined by

$$\frac{\left(e_q(t) - 1\right)^k}{[k]_q!} = \sum_{n=0}^{\infty} S_{2,q}(n,k) \frac{t^n}{[n]_q!}. \tag{13}$$

It is obvious that

$$\begin{gathered}
\mathscr{B}_{n,q}^{(1)}(x,y) := \mathscr{B}_{n,q}(x,y), \qquad \mathscr{E}_{n,q}^{(1)}(x,y) := \mathscr{E}_{n,q}(x,y), \\
\mathscr{G}_{n,q}^{(1)}(x,y) := \mathscr{G}_{n,q}(x,y), \qquad \mathscr{B}_{n,q}(0,0) := \mathscr{B}_{n,q}, \\
\mathscr{E}_{n,q}(0,0) := \mathscr{E}_{n,q}, \qquad \mathscr{G}_{n,q}(0,0) := \mathscr{G}_{n,q}, \\
\mathscr{B}_{n,q}^{(\alpha)} = \mathscr{B}_{n,q}^{(\alpha)}(0,0), \\
\lim_{q \to 1^-} \mathscr{B}_{n,q}^{(\alpha)}(x,y) = \mathscr{B}_{n}^{(\alpha)}(x+y), \\
\lim_{q \to 1^-} \mathscr{B}_{n,q}^{(\alpha)} = \mathscr{B}_{n}^{(\alpha)}, \qquad \mathscr{E}_{n,q}^{(\alpha)} = \mathscr{E}_{n,q}^{(\alpha)}(0,0), \\
\lim_{q \to 1^-} \mathscr{E}_{n,q}^{(\alpha)}(x,y) = \mathscr{E}_{n}^{(\alpha)}(x+y), \qquad \lim_{q \to 1^-} \mathscr{E}_{n,q}^{(\alpha)} = \mathscr{E}_{n}^{(\alpha)}, \\
\mathscr{G}_{n,q}^{(\alpha)} = \mathscr{G}_{n,q}^{(\alpha)}(0,0), \qquad \lim_{q \to 1^-} \mathscr{G}_{n,q}^{(\alpha)}(x,y) = \mathscr{G}_{n}^{(\alpha)}(x+y), \\
\lim_{q \to 1^-} \mathscr{G}_{n,q}^{(\alpha)} = \mathscr{G}_{n}^{(\alpha)}.
\end{gathered} \tag{14}$$

From (8) and (10), it is easy to check that

$$\begin{aligned}
\mathscr{B}_{n,q}^{(\alpha)}(x,y) &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathscr{B}_{n-k,q}(x,0) \mathscr{B}_{k,q}^{(\alpha-1)}(0,y), \\
\mathscr{E}_{n,q}^{(\alpha)}(x,y) &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathscr{E}_{n-k,q}^{(\alpha-1)}(x,0) \mathscr{E}_{k,q}(0,y).
\end{aligned} \tag{15}$$

In this work, we give some identities for the q-Bernoulli polynomials. Also, we give some relations between the q-Bernoulli polynomials and q-Euler polynomials and the q-Genocchi polynomials and q-Bernoulli polynomials. Furthermore, we give a different form of the analogue of the Srivastava-Pintér addition theorem. More precisely, we prove the following theorems.

Theorem 1. *There are the following relations between the q-Bernoulli polynomials and q-Stirling numbers of the second kind:*

$$\mathscr{B}_{n,q}^{(\alpha)}(x,y) = \frac{[k]_q![n]_q!}{[n+k]_q!} \times \sum_{l=0}^{n+k} \begin{bmatrix} n+k \\ l \end{bmatrix}_q \mathscr{B}_{l,q}^{(\alpha+k)}(x,y) \times S_{2,q}(n+k-l,k), \quad (16)$$

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathscr{B}_{n-k,q}^{(\alpha)}(x,y)[\alpha]_q! S_{2,q}(k,\alpha) = \sum_{l=0}^{n-\alpha} \begin{bmatrix} n-\alpha \\ l \end{bmatrix}_q \frac{[n]_q!}{[n-\alpha]_q!} x^{n-\alpha-l} y^l q^{\binom{l}{2}}, \quad (17)$$

where $q \in \mathbb{C}$, $\alpha, n \in \mathbb{N}$, *and* $0 < |q| < 1$.

Theorem 2. *The q-Stirling numbers of the second kind satisfy the following relations:*

$$\mathscr{E}_{n,q}^{(\alpha)}(x,y) = \sum_{j=0}^{\infty} \binom{-\alpha}{j} \frac{1}{2^j}[j]_q! \times \sum_{p=0}^{n} \begin{bmatrix} n \\ p \end{bmatrix}_q S_{2,q}(n-p,j) \times \sum_{l=0}^{p} \begin{bmatrix} p \\ l \end{bmatrix}_q x^{p-l} y^l q^{\binom{l}{2}}, \quad (18)$$

$$\mathscr{B}_{n,q}^{(\alpha)} = [\alpha]_q! \sum_{j=0}^{\infty} \binom{-\alpha}{j} \times \sum_{k=0}^{j} \binom{j}{k} [k]_q! \frac{S_{2,q}(n+k,k)}{[n+k]_q!} [k]_q! (-1)^{j-k},$$

$$\mathscr{B}_{n,q}^{(-\alpha)}(x,y) = [\alpha]_q! \sum_{m=0}^{n+\alpha} \begin{bmatrix} n+\alpha \\ m \end{bmatrix}_q S_{2,q}(m,\alpha) \times (x+y)_q^{n+\alpha-m} \frac{[n]_q!}{[n+\alpha]_q!}, \quad (19)$$

where $q \in \mathbb{C}$, $\alpha, n \in \mathbb{N}$, *and* $0 < |q| < 1$.

Theorem 3. *The generalized q-Euler polynomials satisfy the following relation:*

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathscr{E}_{k,q}(x,y) = 2(x+y)_q^n - \mathscr{E}_{n,q}(x,y), \quad (20)$$

where $q \in \mathbb{C}$, α, $n \in \mathbb{N}$, *and* $0 < |q| < 1$.

Theorem 4. *The polynomials* $B_{n,q}(x,y)$ *and* $\mathscr{G}_{n,q}(x,y)$ *satisfy the following difference relationships:*

$$\mathscr{B}_{n,q}(x,y) = \sum_{\substack{l=0 \\ l \neq n}}^{n+1} \begin{bmatrix} n+1 \\ l \end{bmatrix}_q \frac{1}{[n+1]_q} \mathscr{G}_{l,q}(x,y) \mathscr{B}_{n+1-l,q}, \quad (21)$$

$$\mathscr{G}_{n,q}(x,y) = -2 \sum_{\substack{l=0 \\ l \neq n}}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \frac{1}{[l+1]_q} \mathscr{G}_{l+1,q} \mathscr{B}_{n-l,q}(x,y), \quad (22)$$

where $q \in \mathbb{C}$, $\alpha, n \in \mathbb{N}$, *and* $0 < |q| < 1$.

Theorem 5. *There is the following relation between the generalized q-Euler polynomials and generalized q-Bernoulli polynomials:*

$$\mathscr{E}_{n,q}^{(\alpha)}(x,y) = \left\{ \sum_{s=0}^{n+1} \begin{bmatrix} n+1 \\ s \end{bmatrix}_q \sum_{l=0}^{s} \begin{bmatrix} s \\ l \end{bmatrix}_q \mathscr{B}_{s-l,q}(mx,0) - \sum_{l=0}^{n+1} \begin{bmatrix} n+1 \\ l \end{bmatrix}_q \mathscr{B}_{n+1-l,q}(mx,0) \right\} \times \frac{m}{[n+1]_q!} \mathscr{E}_{l,q}^{(\alpha)}(0,y) m^{l-n-1}, \quad (23)$$

where $q \in \mathbb{C}$, $\alpha, n \in \mathbb{N}$, *and* $0 < |q| < 1$.

2. Proof of the Theorems

Lemma 6. *The generalized q-Bernoulli polynomials, q-Euler polynomials, and q-Genocchi polynomials satisfy the following relations:*

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathscr{B}_{k,q}^{(\alpha)}(x,y) \mathscr{B}_{n-k,q}^{(-\alpha)} = (x+y)_q^n,$$

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathscr{B}_{k,q}^{(\alpha)}(0,y) \mathscr{B}_{n-k,q}^{(-\alpha)} = q^{(n(n-1))/2} y^n,$$

$$\mathscr{B}_{n,q}^{(\alpha)}(x,y) = \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \mathscr{B}_{n-l,q}^{(\alpha)}(0,y) \times \sum_{k=0}^{l} \begin{bmatrix} l \\ k \end{bmatrix}_q \mathscr{E}_{k,q}^{(\alpha)}(x,0) \mathscr{E}_{l-k,q}^{(-\alpha)},$$

$$\mathscr{E}_{n,q}^{(\alpha)}(x,y)=\sum_{l=0}^{n}\begin{bmatrix} n\\ l\end{bmatrix}_q \mathscr{E}_{n-l,q}^{(\alpha)}(0,y)\times\sum_{k=0}^{l}\begin{bmatrix} l\\ k\end{bmatrix}_q \mathscr{B}_{k,q}^{(\alpha)}(x,0)\mathscr{B}_{l-k,q}^{(-\alpha)},$$

$$\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q \mathscr{G}_{k,q}(x,y)+\mathscr{G}_{n,q}(x,y)=2[n]_q(x+y)_q^{n-1},$$

$$\mathscr{G}_{n,q}^{(\alpha-\beta)}(x,y)=\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q \mathscr{G}_{k,q}^{(\alpha)}(x,0)\mathscr{G}_{n-k,q}^{(-\beta)}(0,y). \tag{24}$$

Proof. The proof of this lemma can be found from (7)–(12). □

Proof of Theorem 1. By (8) and (13) we have

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathscr{B}_{n,q}^{(\alpha)}(x,y)\frac{t^n}{[n]_q!}
&=\left(\frac{t}{e_q(t)-1}\right)^{\alpha}e_q(tx)E_q(ty)\\
&\quad\times\frac{[k]_q!}{\left(e_q(t)-1\right)^k}\frac{\left(e_q(t)-1\right)^k}{[k]_q!}\\
&=[k]_q!\frac{t^{\alpha}}{\left(e_q(t)-1\right)^{\alpha+k}}e_q(tx)E_q(ty)\\
&\quad\times\sum_{m=0}^{\infty}S_{2,q}(m,k)\frac{t^m}{[m]_q!}\\
&=[k]_q!t^{-k}\sum_{n=0}^{\infty}\sum_{l=0}^{n}\begin{bmatrix} n\\ l\end{bmatrix}_q\mathscr{B}_{l,q}^{(\alpha+k)}\\
&\quad\times(x,y)S_{2,q}(n-l,k)\frac{t^n}{[n]_q!}\\
&=[k]_q!\sum_{n=0}^{\infty}\sum_{l=0}^{n}\begin{bmatrix} n\\ l\end{bmatrix}_q\mathscr{B}_{l,q}^{(\alpha+k)}\\
&\quad\times(x,y)S_{2,q}(n-l,k)\frac{t^{n-k}}{[n]_q!}\\
&=[k]_q!\sum_{n=-k}^{\infty}\sum_{l=0}^{n+k}\begin{bmatrix} n+k\\ l\end{bmatrix}_q\mathscr{B}_{l,q}^{(\alpha+k)}\\
&\quad\times(x,y)S_{2,q}(n+k-l,k)\frac{t^{n-k}}{[n]_q!}.
\end{aligned}\tag{25}$$

Equating the coefficients of $(t^n/[n]_q!)$, we obtain (16).

Similarly, we have (17). □

Proof of Theorem 2. Combining (10) and (13), we obtain

$$\begin{aligned}
\left(\frac{2}{e_q(t)+1}\right)^{\alpha}&=\left(1+\frac{e_q(t)-1}{2}\right)^{(-\alpha)}\\
&=\sum_{j=0}^{\infty}\binom{-\alpha}{j}\left(\frac{e_q(t)-1}{2}\right)^{(j)},\\
\sum_{n=0}^{\infty}\mathscr{E}_{n,q}^{(\alpha)}(x,y)\frac{t^n}{[n]_q!}
&=\sum_{j=0}^{\infty}\binom{-\alpha}{j}\left(\frac{e_q(t)-1}{2}\right)^{(j)}e_q(tx)E_q(ty)\\
&=\sum_{j=0}^{\infty}\binom{-\alpha}{j}\frac{1}{2^j}[j]_q!\sum_{n=0}^{\infty}S_{2,q}(n,j)\frac{t^n}{[n]_q!}\\
&\quad\times\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q x^{n-k}y^kq^{\binom{k}{2}}\frac{t^n}{[n]_q!}\\
&=\sum_{j=0}^{\infty}\binom{-\alpha}{j}\frac{1}{2^j}\\
&\quad\times\sum_{n=0}^{\infty}\sum_{p=0}^{n}\begin{bmatrix} n\\ p\end{bmatrix}_q[j]_q!S_{2,q}(n-p,j)\\
&\quad\times\sum_{l=0}^{p}\begin{bmatrix} p\\ l\end{bmatrix}_q x^{p-l}y^lq^{\binom{l}{2}}\frac{t^n}{[n]_q!}.
\end{aligned}\tag{26}$$

Comparing the coefficients of $(t^n/[n]_q!)$, we find (18). Similarly, we have (19). □

Proof of Theorem 3. It is obvious that

$$\frac{-2}{\left(e_q(t)+1\right)e_q(t)}=\frac{2}{\left(e_q(t)+1\right)}-\frac{2}{e_q(t)}. \tag{27}$$

We write it as

$$\begin{aligned}
\frac{-2}{e_q(t)+1}\frac{e_q(tx)E_q(ty)}{e_q(t)}&=\frac{2}{e_q(t)+1}e_q(tx)E_q(ty)\\
&\quad-\frac{2}{e_q(t)}e_q(tx)E_q(ty),\\
\frac{-2}{e_q(t)+1}e_q(tx)E_q(ty)&=\frac{2}{e_q(t)+1}e_q(tx)E_q(ty)\\
&\quad-2e_q(tx)E_q(ty)
\end{aligned}$$

$$- \sum_{n=0}^{\infty} \mathscr{E}_{n,q}(x,y) \frac{t^n}{[n]_q!}$$

$$= \sum_{n=0}^{\infty} \mathscr{E}_{n,q}(x,y) \frac{t^n}{[n]_q!} \times \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} - 2\sum_{n=0}^{\infty} (x+y)_q^n \frac{t^n}{[n]_q!}. \tag{28}$$

Using the Cauchy product and comparing the coefficients of $(t^n/[n]_q!)$, we have

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathscr{E}_{k,q}(x,y) = 2(x+y)_q^n - \mathscr{E}_{k,q}(x,y). \tag{29}$$

□

Finally, we consider the interesting relationships between the q-Bernoulli polynomials and q-Genocchi polynomials and the q-Euler polynomials and q-Bernoulli polynomials. These relations are q-analogues to the Srivastava-Pintér addition theorems.

Proof of Theorem 4. It follows immediately that

$$\sum_{n=0}^{\infty} \mathscr{B}_{n,q}(x,y) \frac{t^n}{[n]_q!}$$

$$= \frac{1}{2} \frac{2te_q(tx) E_q(ty)}{e_q(t)+1} + \frac{1}{t}\left(\frac{t}{e_q(t)-1}\right) \frac{2t}{e_q(t)+1} e_q(tx) E_q(ty)$$

$$= \frac{1}{2}\sum_{n=0}^{\infty} \mathscr{G}_{n,q}(x,y) \frac{t^n}{[n]_q!} + \frac{1}{t} \times \sum_{n=0}^{\infty} \mathscr{B}_{n,q} \frac{t^n}{[n]_q!} \sum_{n=0}^{\infty} \mathscr{G}_{n,q}(x,y) \frac{t^n}{[n]_q!}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty} \mathscr{G}_{n,q}(x,y) \frac{t^n}{[n]_q!} + \sum_{n=0}^{\infty}\sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \frac{1}{[n]_q} \mathscr{G}_{l,q}(x,y) \times \mathscr{B}_{n-l,q} \frac{t^{n-1}}{[n-1]_q!}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty} \mathscr{G}_{n,q}(x,y) \frac{t^n}{[n]_q!} + \sum_{n=0}^{\infty}\left(-\frac{1}{2}\mathscr{G}_{n,q}(x,y) + \sum_{l=0}^{n+1} \begin{bmatrix} n+1 \\ l \end{bmatrix}_q \frac{1}{[n+1]_q} \times \mathscr{G}_{l,q}(x,y) \mathscr{B}_{n+1-l,q}\right) \frac{t^n}{[n]_q!}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{\substack{l=0 \\ l\neq n}}^{n+1} \begin{bmatrix} n+1 \\ l \end{bmatrix}_q \frac{1}{[n+1]_q} \times \mathscr{G}_{l,q}(x,y) \mathscr{B}_{n+1-l,q}\right) \frac{t^n}{[n]_q!}. \tag{30}$$

Equating the coefficients of $(t^n/[n]_q!)$, we have (21).

In a similar fashion, (12) yields

$$\sum_{n=0}^{\infty} \mathscr{G}_{n,q}(x,y) \frac{t^n}{[n]_q!}$$

$$= \frac{1}{t}\left(\frac{2t}{e_q(t)+1}\left(e_q(t)-1\right)\right)\left(\frac{te_q(tx) E_q(ty)}{e_q(t)-1}\right)$$

$$= \frac{1}{t}\left(2t - 2\frac{2t}{e_q(t)+1}\right)\left(\frac{t}{e_q(t)-1} e_q(tx) E_q(ty)\right)$$

$$= \frac{1}{t}\left(2t - 2\sum_{n=0}^{\infty} \mathscr{G}_{n,q} \frac{t^n}{[n]_q!}\right)\left(\sum_{n=0}^{\infty} \mathscr{B}_{n,q}(x,y) \frac{t^n}{[n]_q!}\right)$$

$$= \frac{1}{t}\left(-2\sum_{l=0}^{\infty} \frac{1}{[l+1]_q!} \mathscr{G}_{l+1,q} \frac{t^{l+1}}{[l]_q!}\right)\left(\sum_{n=0}^{\infty} \mathscr{B}_{n,q}(x,y) \frac{t^n}{[n]_q!}\right)$$

$$= \sum_{n=1}^{\infty}\left(-2\sum_{\substack{l=0 \\ l\neq n}}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \frac{\mathscr{G}_{l+1,q}}{[l+1]_q} \mathscr{B}_{n-l,q}(x,y)\right) \frac{t^n}{[n]_q!}. \tag{31}$$

Comparing the coefficients of $(t^n/[n]_q!)$, we have (22). □

Proof of Theorem 5. By (10), we write

$$\sum_{n=0}^{\infty} \mathscr{E}_{n,q}^{(\alpha)}(x,y) \frac{t^n}{[n]_q!}$$

$$= \left(\frac{2}{e_q(t)+1}\right)^{\alpha} \times E_q(ty) \frac{e_q(t/m)-1}{(t/m)} \frac{(t/m)}{e_q(t/m) \;-1} e_q((t/m)mx)$$

$$
\begin{aligned}
&= \frac{m}{t}\left\{\sum_{n=0}^{\infty}\mathscr{E}_{n,q}^{(\alpha)}(x,y)\frac{t^n}{[n]_q!}\right.\\
&\qquad\times\sum_{n=0}^{\infty}\mathscr{B}_{n,q}(mx,0)\frac{t^n}{m^n[n]_q!}\\
&\qquad\times\sum_{n=0}^{\infty}\frac{t^n}{m^n[n]_q!}-\sum_{n=0}^{\infty}\mathscr{E}_{n,q}^{(\alpha)}(0,y)\frac{t^n}{[n]_q!}\\
&\qquad\left.\times\sum_{n=0}^{\infty}\mathscr{B}_{n,q}(mx,0)\frac{t^n}{m^n[n]_q!}\right\}\\
&= m\sum_{n=-1}^{\infty}\frac{1}{[n+1]_q}\\
&\qquad\times\left\{\sum_{s=0}^{n+1}\begin{bmatrix}n+1\\ s\end{bmatrix}_q\sum_{l=0}^{s}\begin{bmatrix}s\\ l\end{bmatrix}_q\mathscr{B}_{s-l,q}(mx,0)\right.\\
&\qquad\left.-\sum_{l=0}^{n+1}\begin{bmatrix}n+1\\ l\end{bmatrix}_q\mathscr{B}_{n+1-l,q}(mx,0)\right\}\\
&\quad\times\frac{m}{[n+1]_q!}\mathscr{E}_{l,q}^{(\alpha)}(0,y)\,m^{l-n-1}\frac{t^n}{[n]_q!}.
\end{aligned}
\tag{32}
$$

By equating the coefficients of $(t^n/[n]_q!)$, we get the theorem. □

Remark 7. There are many different relationships which are analogues to the Srivastava-Pintér addition theorems at these polynomials.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

The present investigation was supported by the Scientific Research Project Administration of Akdeniz University.

References

[1] J. Choi, P. J. Anderson, and H. M. Srivastava, "Some q-extensions of the Apostol-Bernoulli and the Apostol-Euler polynomials of order n, and the multiple Hurwitz zeta function," *Applied Mathematics and Computation*, vol. 199, no. 2, pp. 723–737, 2008.

[2] N. I. Mahmudov, "On a class of q-Bernoulli and q-Euler polynomials," *Advances in Difference Equations*, vol. 2013, article 108, 2013.

[3] L. Carlitz, "q-Bernoulli numbers and polynomials," *Duke Mathematical Journal*, vol. 15, pp. 987–1050, 1948.

[4] L. Carlitz, "Expansions of q-Bernoulli numbers," *Duke Mathematical Journal*, vol. 25, pp. 355–364, 1958.

[5] M. Cenkci, M. Can, and V. Kurt, "q-extensions of Genocchi numbers," *Journal of the Korean Mathematical Society*, vol. 43, no. 1, pp. 183–198, 2006.

[6] M. Cenkci, V. Kurt, S. H. Rim, and Y. Simsek, "On (i, q) Bernoulli and Euler numbers," *Applied Mathematics Letters*, vol. 21, no. 7, pp. 706–711, 2008.

[7] G.-S. Cheon, "A note on the Bernoulli and Euler polynomials," *Applied Mathematics Letters*, vol. 16, no. 3, pp. 365–368, 2003.

[8] T. Kim, "Some formulae for the q-Bernoulli and Euler polynomials of higher order," *Journal of Mathematical Analysis and Applications*, vol. 273, no. 1, pp. 236–242, 2002.

[9] B. Kurt, "A further generalization of the Bernoulli polynomials and on the 2D-Bernoulli polynomials $B_{n,q}^{(\alpha)}$," *Applied Mathematical Sciences*, vol. 4, no. 47, pp. 2315–2322, 2010.

[10] V. Kurt, "A new class of generalized q-Bernoulli and q-Euler polynomials," in *Proceedings of the International Western Balkans Conference of Mathematical Sciences*, Elbasan, Albania, May 2013.

[11] Q.-M. Luo, "Some results for the q-Bernoulli and q-Euler polynomials," *Journal of Mathematical Analysis and Applications*, vol. 363, no. 1, pp. 7–18, 2010.

[12] Q.-M. Luo and H. M. Srivastava, "Some relationships between the Apostol-Bernoulli and Apostol-Euler polynomials," *Computers & Mathematics with Applications*, vol. 51, no. 3-4, pp. 631–642, 2006.

[13] Q.-M. Luo and H. M. Srivastava, "q-extensions of some relationships between the Bernoulli and Euler polynomials," *Taiwanese Journal of Mathematics*, vol. 15, no. 1, pp. 241–257, 2011.

[14] H. M. Srivastava and J. Choi, *Series Associated with the Zeta and Related Functions*, Kluwer Academic, London, UK, 2001.

[15] H. M. Srivastava and A. Pintér, "Remarks on some relationships between the Bernoulli and Euler polynomials," *Applied Mathematics Letters*, vol. 17, no. 4, pp. 375–380, 2004.

[16] P. Natalini and A. Bernardini, "A generalization of the Bernoulli polynomials," *Journal of Applied Mathematics*, no. 3, pp. 155–163, 2003.

[17] R. Tremblay, S. Gaboury, and B.-J. Fugère, "A new class of generalized Apostol-Bernoulli polynomials and some analogues of the Srivastava-Pintér addition theorem," *Applied Mathematics Letters*, vol. 24, no. 11, pp. 1888–1893, 2011.

[18] R. Tremblay, S. Gaboury, and B. J. Fegure, "Some new classes of generalized Apostol Bernoulli and Apostol-Genocchi polynomials," *International Journal of Mathematics and Mathematical Sciences*, vol. 2012, Article ID 182785, 14 pages, 2012.

[19] S. Gaboury and B. Kurt, "Some relations involving Hermite-based Apostol-Genocchi polynomials," *Applied Mathematical Sciences*, vol. 6, no. 81–84, pp. 4091–4102, 2012.

[20] N. I. Mahmudov, "q-analogues of the Bernoulli and Genocchi polynomials and the Srivastava-Pintér addition theorems," *Discrete Dynamics in Nature and Society*, vol. 2012, Article ID 169348, 8 pages, 2012.

[21] N. I. Mahmudov and M. E. Keleshteri, "On a class of generalized q-Bernoulli and q-Euler polynomials," *Advances in Difference Equations*, vol. 2013, article 115, 2013.

[22] S. Araci, J. J. Seo, and M. Acikgoz, "A new family of q-analogue of Genocchi polynomials of higher order," *Kyungpook Mathematical Journal*. In press.

[23] B. A. Kupershmidt, "Reflection symmetries of q-Bernoulli polynomials," *Journal of Nonlinear Mathematical Physics*, vol. 12, no. 1, pp. 412–422, 2005.