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Research Article

(L^2, H^1)-Random Attractors for Stochastic Reaction-Diffusion Equation on Unbounded Domains

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We study the random dynamical system generated by a stochastic reaction-diffusion equation with additive noise on the whole space $\mathbb{R}^n$ and prove the existence of an (L^2, H^1)-random attractor for such a random dynamical system. The nonlinearity f is supposed to satisfy the growth of arbitrary order $p-1$ $(p \geq 2)$. The (L^2, H^1)-asymptotic compactness of the random dynamical system is obtained by using an extended version of the tail estimate method introduced by Wang (1999) and the cut-off technique.

1. Introduction

In this paper, we consider the asymptotic behavior of solutions to the following stochastic reaction-diffusion equation (SRDE) with additive noise in the entire space $\mathbb{R}^n$:

$$du + (\lambda u - \Delta u)\, dt = \left(f(x,u) + g(x)\right) dt + \sum_{j=1}^{m} h_j dw_j, \quad x \in \mathbb{R}^n,\ t > 0, \tag{1}$$

with the initial condition

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^n, \tag{2}$$

where λ is a positive constant, g is a given function in $L^2(\mathbb{R}^n)$, for each $j = 1, \ldots, m, h_j \in H^2(\mathbb{R}^n) \bigcap W^{2,p}(\mathbb{R}^n) \bigcap L^\infty(\mathbb{R}^n)$ for some $p \geq 2$, $\{w_j\}_{j=1}^m$ are independent two-sided real-valued Wiener processes on a probability space which will be specified below, and f is a nonlinear function satisfying the following conditions (see, e.g., [1, 2]). For all $x \in \mathbb{R}^n$ and $s \in \mathbb{R}$,

$$f(x,s)\, s \leq -\alpha_1 |s|^p + \psi_1(x), \tag{3}$$

$$\left| f(x,s) \right| \leq \alpha_2 |s|^{p-1} + \psi_2(x), \tag{4}$$

$$\frac{\partial f}{\partial s}(x,s) \leq \beta, \tag{5}$$

$$\left| \frac{\partial f}{\partial x}(x,s) \right| \leq \psi_3(x), \tag{6}$$

where α_1, α_2 and β are positive constants, $\psi_1 \in L^1(\mathbb{R}^n) \bigcap L^\infty(\mathbb{R}^n)$, $\psi_2 \in L^2(\mathbb{R}^n) \bigcap L^p(\mathbb{R}^n) \bigcap L^4(\mathbb{R}^n)$, and $\psi_3 \in L^2(\mathbb{R}^n)$.

As we know, the asymptotic behavior of a random dynamical system (RDS) is characterized by random attractors, which were first introduced by Crauel and Flandoli [3] and Schmalfuss [4] and then developed in [1, 2, 5–12] and among others. Recently, the existence of random attractors of the RDS associated with problem (1)-(2) was studied by many authors. For example, in [1, 2] the authors proved the existence of $(L^2(\mathbb{R}^n), L^2(\mathbb{R}^n))$-random attractor and $(L^2(\mathbb{R}^n), L^p(\mathbb{R}^n))$-random attractor, respectively, in the case of additive noise. Wang and Zhou obtained $(L^2(\mathbb{R}^n), L^2(\mathbb{R}^n))$-random attractor in [12] and Li et al. proved the existence of $(L^2(D), L^p(D))$-random attractor in bounded domains in [10] in the case of multiplicative noise. A necessary and sufficient condition for the existence of random attractors for the so-called quasicontinuous RDS was established in [9], and in the most recent papers [13, 14], the author employed this result to prove the existence of random attractors for some reaction-diffusion equations with additive noise and multiplicative noise on H^1, respectively, when the domain is bounded. In this paper, we study the existence of $(L^2(\mathbb{R}^n), H^1(\mathbb{R}^n))$-random attractor with additive noise for the same problem in the entire space $\mathbb{R}^n$.

For our problem, there are two difficulties when we consider the existence of (L^2, H^1)-random attractor. The first is the lack of compactness of Sobolev embeddings when the domain is unbounded. It is worth mentioning that in deterministic case differential equations of this type were extensively studied in both autonomous and nonautonomous cases and in both bounded domains and unbounded domains [15–29]. In the case of unbounded domains the difficulty of noncompact embeddings can be overcome by the energy equation approach introduced by Ball in [30, 31] and other methods. We are interested in the method used in [22] for the deterministic version of the initial problem (1)-(2) on $\mathbb{R}^n$. In [22] the author approached $\mathbb{R}^n$ by a bounded ball and found that the approximation error of the norm of solutions is arbitrary small uniformly for large time, and thus they proved asymptotic compactness by passing the limit of the energy equation. More recently, the idea of the tail estimate was used in [1] to prove the existence of random attractor in $L^2(\mathbb{R}^n)$ for the SRDE (1)-(2). In this paper, we use an extended version of the tail estimate described in [22] to overcome the difficulty of noncompact embeddings.

Another difficulty is that one can not differentiate the stochastic equation with respect to time t in usual sense. In the case of deterministic equation, by differentiating the reaction-diffusion equation with respect to t, one can prove the existence of $(L^2(\mathbb{R}^n), H^1(\mathbb{R}^n))$ or $(L^2(D), H_0^1(D))$ ($D \subset \mathbb{R}^n$ is bounded) attractors; see [24, 27, 29, 32] for autonomous equations and [23, 26, 33] for non-autonomous equations. But in stochastic case this idea breaks down, since, as we know, neither the Winner process nor the Ornstein-Uhlenbeck process is differentiable with respect to t in usual sense. However, this is only a matter of method or estimate. In [25], the author used a result for compactness in $H^1(\mathbb{R}^n)$ introduced in [17] to establish the asymptotic compactness in $H^1(\mathbb{R}^n)$ without differentiating the equation. Unfortunately, the growth order p is restricted in that case. In this paper, we overcome this drawback by using an appropriate estimate motivated by the works in [19] and the estimate is accurate enough so that we needn't differentiate the equation as usual.

This paper is organized as follows. In Section 2, we recall some basic notions of bispaces random attractors for RDS. In Section 3, we transform the problem (1)-(2) into a parameterized evolution equation and obtain the corresponding RDS. In Section 4, we give some uniform estimates of the solutions as $t \to \infty$. In Section 5, we prove the asymptotic compactness and the existence of an $(L^2(\mathbb{R}^n), H^1(\mathbb{R}^n))$-random attractor.

Throughout this paper, we denote by $\|\cdot\|_X$ the norm of Banach space X and by $(\cdot,\cdot)_H$ the inner product in Hilbert space H. The inner product and norm of $L^2(\mathbb{R}^n)$ are written as $(\cdot,\cdot)$ and $\|\cdot\|$, respectively. We also use $\|u\|_r$ to denote the norm of $u \in L^r(\mathbb{R}^n)$ ($r \geq 1, r \neq 2$) and $|u|$ to denote the modular of u. The letter c denotes any positive constant which may be different from line to line or even in the same line (sometimes for special case, we also denote the different positive constants by c_i $(i = 1, 2, \ldots)$).

2. Preliminaries and Abstract Results

In this section, we first recall some basic concepts related to random attractors for RDS (see [1, 3, 5–8, 34] for more detail) and then give some abstract results on the existence of $(L^2(\mathbb{R}^n), H^1(\mathbb{R}^n))$-random attractors.

2.1. Preliminaries. Let X, Y be two Banach spaces with Borel σ-algebra $\mathscr{B}(X)$ and $\mathscr{B}(Y)$, respectively, and let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space.

Definition 1. $(\Omega, \mathscr{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ is called a metric dynamical system (MDS) if $\theta : \mathbb{R} \times \Omega \to \Omega$ is $(\mathscr{B}(\mathbb{R}) \times \mathscr{F}, \mathscr{F})$-measurable, and θ_0 is the identity on Ω, $\theta_{s+t} = \theta_s \circ \theta_t$ for all $s, t \in \mathbb{R}$ and $\theta_t \mathbb{P} = \mathbb{P}$ for all $t \in \mathbb{R}$.

Definition 2. An RDS on X over an MDS $(\Omega, \mathscr{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ is a mapping $\phi : \mathbb{R}^+ \times \Omega \times X \to X$, $(t, \omega, x) \to \phi(t, \omega, x)$ which is $(\mathscr{B}(\mathbb{R}^+) \times \mathscr{F} \times \mathscr{B}(X), \mathscr{B}(X))$-measurable and satisfies that, for $\mathbb{P}$-a.e. $\omega \in \Omega$,

(i) $\phi(0, \omega, \cdot) = \text{id}$ on X;

(ii) $\phi(t+s, \omega, \cdot) = \phi(t, \theta_s\omega, \cdot) \circ \phi(s, \omega, \cdot)$ (cocycle property) on X for all $s, t \in \mathbb{R}^+$.

An RDS is said to be continuous on X if $\phi(t, \omega) : X \to X$ is continuous for all $t \in \mathbb{R}^+$ and $\mathbb{P}$-a.e. $\omega \in \Omega$.

Definition 3. A random set is a set-valued map $A : \Omega \to 2^X$, $\omega \mapsto A(\omega)$, which satisfies that, for each $x \in X$, the map $\omega \mapsto d(x, A(\omega))$ is measurable. A random set $\{A(\omega)\}$ is called a random closed (compact) set if $A(\omega)$ is closed (compact) for all $\omega \in \Omega$. A random set $\{A(\omega)\}$ is called a random bounded set if there exist $x_0 \in X$ and a random variable $r(\omega) > 0$ such that, for all $\omega \in \Omega$,

$$A(\omega) \subset \{x \in X : d(x, x_0) \leq r(\omega)\}. \tag{7}$$

Definition 4. (1) A random bounded set $\{B(\omega)\}_{\omega\in\Omega}$ of X is called tempered with respect to $(\theta_t)_{t\in\mathbb{R}}$ if, for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\lim_{t\to\infty} e^{-\beta t} d(B(\theta_{-t}\omega)) = 0, \quad \forall \beta > 0, \tag{8}$$

where

$$d(B) = \sup_{x\in B} \|x\|_X. \tag{9}$$

(2) A random variable $r(\omega) \geq 0$ is called tempered with respect to $(\theta_t)_{t\in\mathbb{R}}$ if, for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\lim_{t\to\infty} e^{-\beta t} r(\theta_{-t}\omega) = 0, \quad \forall \beta > 0. \tag{10}$$

Next, we introduce some notions about the bi-spaces random attractors which are motivated by the works in [2, 20, 25, 35]. We assume that ϕ is an RDS on X and Y over an MDS $(\Omega, \mathscr{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$, respectively. Let $\mathscr{P}(X)$ denote the family of all nonempty subsets of X and $\mathcal{S}_X$ the class of all families $\widehat{D} = \{D(\omega)\}_{\omega\in\Omega} \subset \mathcal{P}(X)$. $\mathcal{P}(Y)$, and $\mathcal{S}_Y$ can be defined in the same way. We consider the given nonempty subclasses $\mathcal{D}_X, \mathcal{D}_Y$, where $\mathcal{D}_X \subset \mathcal{S}_X$, $\mathcal{D}_Y \subset \mathcal{S}_Y$.

Definition 5. A family $\widehat{B} = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}_Y$ is said to be (X,Y)-random absorbing for ϕ if, for every $\widehat{D} = \{D(\omega)\}_{\omega\in\Omega} \in \mathcal{D}_X$, there exists $T_{\widehat{D}}(\omega) > 0$ such that, for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\phi(t, \theta_{-t}\omega, D(\theta_{-t}\omega)) \subset B(\omega), \quad \forall t \geq T_{\widehat{D}}(\omega). \tag{11}$$

Definition 6. A family $\widehat{C} = \{C(\omega)\}_{\omega\in\Omega} \in \mathcal{D}_Y$ is said to be (X,Y)-random attracting for ϕ if, for every $\widehat{D} = \{D(\omega)\}_{\omega\in\Omega} \in \mathcal{D}_X$, we have, for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\lim_{t\to\infty} d_Y(\phi(t, \theta_{-t}\omega, D(\theta_{-t}\omega)), C(\omega)) = 0, \tag{12}$$

where $d_Y(C_1, C_2)$ denotes the Hausdorff semi-distance between C_1 and C_2 in Y; that is,

$$d_Y(C_1, C_2) = \sup_{x\in C_1} \inf_{y\in C_2} \|x - y\|_Y \quad \text{for } C_1, C_2 \subset Y. \tag{13}$$

Definition 7. The RDS ϕ is said to be (X,Y)-asymptotically compact if, for $\mathbb{P}$-a.e. $\omega \in \Omega$, $\{\phi(t_n, \theta_{-t_n}\omega, x_n)\}_{n=1}^{\infty}$ has a convergent subsequence in Y whenever $t_n \to \infty$ and $x_n \in D(\theta_{-t_n}\omega)$ with $\widehat{D} = \{D(\omega)\}_{\omega\in\Omega} \in \mathcal{D}_X$.

Definition 8. A random set $\widehat{A} = \{A(\omega)\}_{\omega\in\Omega} \in \mathcal{S}_Y$ is said to be an (X,Y)-random attractor if the following conditions are satisfied for $\mathbb{P}$-a.e. $\omega \in \Omega$,

(i) $A(\omega)$ is closed in X and compact in Y;

(ii) $\widehat{A} = \{A(\omega)\}_{\omega\in\Omega}$ is invariant; that is, $\phi(t, \omega, A(\omega)) = A(\theta_t\omega)$ for all $t \geq 0$;

(iii) $\widehat{A} = \{A(\omega)\}_{\omega\in\Omega}$ attracts every random set in $\mathcal{D}_X$ in the norm topology of Y in the sense of (12).

2.2. Abstract Results. Now, we present the main abstract results. Recall that a collection $\mathcal{D}$ of random subsets is called *inclusion closed* if whenever $\{E(\omega)\}_{\omega\in\Omega}$ is an arbitrary random set and $\{F(\omega)\}_{\omega\in\Omega}$ is in $\mathcal{D}$ with $E(\omega) \subset F(\omega)$ for all $\omega \in \Omega$, then $\{E(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$.

The following theorem is an adaptation of a result of [25] to the case of RDS. The proof is similar to that of [25], and here we omit it.

Theorem 9. *Let ϕ be a continuous RDS on X and an RDS on Y over $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$, respectively, and $\mathcal{D}_X$ and $\mathcal{D}_Y$ are inclusion closed.*

(i) *Case 1 ($X = Y$) (see [1]). Assume that the family $\widehat{B}_0 = \{B_0(\omega)\}_{\omega\in\Omega} \in \mathcal{D}_X$ is a closed (X,X)-random absorbing set for ϕ and ϕ is (X,X)-asymptotically compact. Then ϕ has a unique (X,X)-random attractor $\widehat{A} = \{A(\omega)\}_{\omega\in\Omega}$ which is given by*

$$A(\omega) = \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} \phi(t, \theta_{-t}\omega, B_0(\theta_{-t}\omega))}^{X}, \tag{14}$$

where $\overline{A}^X$ denotes the closure of A with respect to the norm topology in X.

(ii) *Case 2 ($X \neq Y$). If the assumptions in (i) are satisfied, moreover, we assume that $\widehat{B}_1 = \{B_1(\omega)\}_{\omega\in\Omega} \in \mathcal{D}_Y$ is (X,Y)-random absorbing and ϕ is (X,Y)-asymptotically compact. Then ϕ has an (X,Y)-random attractor $\widehat{A'} = \{A'(\omega)\}_{\omega\in\Omega}$ which is given by*

$$\begin{aligned} A'(\omega) &= \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} \phi\left(t, \theta_{-t}\omega, B_0(\theta_{-t}\omega) \bigcap B_1(\theta_{-t}\omega)\right)}^{X} \\ &= \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} \phi\left(t, \theta_{-t}\omega, B_0(\theta_{-t}\omega) \bigcap B_1(\theta_{-t}\omega)\right)}^{Y}, \end{aligned} \tag{15}$$

where $\widehat{B}_0 = \{B_0(\omega)\}_{\omega\in\Omega} \in \mathcal{D}_X$ is the (X,X)-random absorbing set in (i).

In the following of this paper we only consider $X = L^2(\mathbb{R}^n)$, $Y = H^1(\mathbb{R}^n)$, and $\mathcal{D}_X = \mathcal{D}_2$, $\mathcal{D}_Y = \mathcal{D}_v$, where $\mathcal{D}_2$ and $\mathcal{D}_v$ denote the collections of all tempered random subsets of $L^2(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$, respectively.

Theorem 10. *Assume that ϕ is an RDS on $L^2(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$, respectively, and then ϕ is $(L^2(\mathbb{R}^n), H^1(\mathbb{R}^n))$-asymptotically compact if*

(i) *for every $\widehat{D} = \{D(\omega)\}_{\omega\in\Omega} \in \mathcal{D}_2$, $\mathbb{P}$-a.e. $\omega \in \Omega$ and every $\varepsilon > 0$, there exist $k_0 = k_0(\omega, \varepsilon) > 0$ and $T_{\widehat{D}}(\omega, \varepsilon) > 0$ such that, for all $t \geq T_{\widehat{D}}(\omega, \varepsilon)$,*

$$\left\|\chi_{Q_{k_0}^c} \cdot \phi(t, \theta_{-t}\omega, x)\right\|_{H^1(Q_{k_0}^c)} \leq \varepsilon, \quad \forall x \in D(\theta_{-t}\omega), \tag{16}$$

(ii) *$\chi_{Q_k} \cdot \phi$ is $(L^2(\mathbb{R}^n), H^1(Q_k))$-asymptotically compact, $\forall k \geq 1$, where $Q_{k_0} = \{x \in \mathbb{R}^n : |x| \leq k_0\}$, $Q_{k_0}^c = \mathbb{R}^n \setminus Q_{k_0}$, $Q_k = \{x \in \mathbb{R}^n : |x| \leq k\}$, and χ_A is the identical function on A.*

Proof. It suffices to check that, for all $\widehat{D} = \{D(\omega)\}_{\omega\in\Omega} \in \mathcal{D}_2$ and $\mathbb{P}$-a.e. $\omega \in \Omega$, we can extract a Cauchy subsequence $\{\phi(t_{n'}, \theta_{-t_{n'}}\omega, x_{n'})\}$ from $\{\phi(t_n, \theta_{-t_n}\omega, x_n)\}$, whenever $t_n \to \infty$ and $x_n \in D(\theta_{-t_n}\omega)$. We assume that there is $\widetilde{\Omega} \subset \Omega$ of full $\mathbb{P}$-measure such that assumption (i) holds for every $\omega \in \widetilde{\Omega}$. We now fix $\omega \in \widetilde{\Omega}$ and $\varepsilon > 0$, and then by (i) there exist $k_0 = k_0(\omega, \varepsilon) > 0$ and $T_{\widehat{D}}(\omega, \varepsilon) > 0$ such that for all $t \geq T_{\widehat{D}}(\omega, \varepsilon)$,

$$\left\|\chi_{Q_{k_0}^c} \cdot \phi(t, \theta_{-t}\omega, x)\right\|_{H^1(Q_{k_0}^c)} \leq \varepsilon, \quad \forall x \in D(\theta_{-t}\omega). \tag{17}$$

On the other hand, by (ii), $\chi_{Q_k} \cdot \phi$ is $(L^2(\mathbb{R}^n), H^1(Q_k))$-asymptotically compact, for all $k \geq 1$. For the above k_0, there is a subsequence $\{\phi(t_{n'}, \theta_{-t_{n'}}\omega, x_{n'})\}$ such that $\{\chi_{Q_{k_0}} \cdot \phi(t_{n'}, \theta_{-t_{n'}}\omega, x_{n'})\}$ is convergent in $H^1(Q_{k_0})$. Therefore, there exists an integer $N(\widehat{D}, \omega, \varepsilon) > 0$ such that for all $n', m' \geq N(\widehat{D}, \omega, \varepsilon)$, we have

$$\begin{aligned}
&\left\|\phi\left(t_{n'}, \theta_{-t_{n'}}\omega, x_{n'}\right) - \phi\left(t_{m'}, \theta_{-t_{m'}}\omega, x_{m'}\right)\right\|^2_{H^1(\mathbb{R}^n)} \\
&= \left\|\chi_{Q_{k_0}} \cdot \left(\phi\left(t_{n'}, \theta_{-t_{n'}}\omega, x_{n'}\right) - \phi\left(t_{m'}, \theta_{-t_{m'}}\omega, x_{m'}\right)\right)\right\|^2_{H^1(Q_{k_0})} \\
&\quad + \left\|\chi_{Q^c_{k_0}} \cdot \left(\phi\left(t_{n'}, \theta_{-t_{n'}}\omega, x_{n'}\right) - \phi\left(t_{m'}, \theta_{-t_{m'}}\omega, x_{m'}\right)\right)\right\|^2_{H^1\left(Q^c_{k_0}\right)} \\
&\leq \left\|\chi_{Q_{k_0}} \cdot \phi\left(t_{n'}, \theta_{-t_{n'}}\omega, x_{n'}\right) - \chi_{Q_{k_0}} \cdot \phi\left(t_{m'}, \theta_{-t_{m'}}\omega, x_{m'}\right)\right\|^2_{H^1(Q_{k_0})} \\
&\quad + 2\left\|\chi_{Q^c_{k_0}} \cdot \phi\left(t_{n'}, \theta_{-t_{n'}}\omega, x_{n'}\right)\right\|^2_{H^1\left(Q^c_{k_0}\right)} \\
&\quad + 2\left\|\chi_{Q^c_{k_0}} \cdot \phi\left(t_{m'}, \theta_{-t_{m'}}\omega, x_{m'}\right)\right\|^2_{H^1\left(Q^c_{k_0}\right)} \leq 5\varepsilon.
\end{aligned} \tag{18}$$

The proof is complete. □

Remark 11. If we replace $H^1(\mathbb{R}^n)$ by other Banach spaces in Theorem 10, such as $L^2(\mathbb{R}^n)$, $L^p(\mathbb{R}^n)$ and $H^2(\mathbb{R}^n)$, the corresponding results also hold true. In particular, in the deterministic case, it is the exact method used in [22] when $H^1(\mathbb{R}^n)$ is replaced by $L^2(\mathbb{R}^n)$.

3. The Reaction-Diffusion Equation on $\mathbb{R}^n$ with Additive Noise

We consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where

$$\Omega = \left\{\omega = (\omega_1, \omega_2, \ldots, \omega_m) \in C\left(\mathbb{R}, \mathbb{R}^m\right) : \omega(0) = 0\right\}, \tag{19}$$

$\mathcal{F}$ is the Borel σ-algebra induced by the compact-open topology of Ω, and $\mathbb{P}$ the corresponding Wiener measure on $(\Omega, \mathcal{F})$. Then we will identify ω with

$$W(t) \equiv \left(w_1(t), w_2(t), \ldots, w_m(t)\right) = \omega(t) \quad \text{for } t \in \mathbb{R}. \tag{20}$$

Define the time shift by

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad \omega \in \Omega,\ t \in \mathbb{R}, \tag{21}$$

and then $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ is an MDS.

We now translate the stochastic equation (1)-(2) into a deterministic equation with a random parameter.

To this end, we consider the one-dimensional Ornstein-Uhlenbeck process given by

$$z_j(t) = z_j\left(\theta_t\omega_j\right) \equiv -\lambda \int_{-\infty}^0 e^{\lambda\tau}\left(\theta_t\omega_j\right)(\tau)\,d\tau, \quad \forall t \in \mathbb{R}, \tag{22}$$

which solves the Itô differential equation

$$dz_j + \lambda z_j dt = d\omega_j(t). \tag{23}$$

Note that the random variable $|z_j(\omega_j)|$ is tempered and $z_j(\theta_t\omega_j)$ is $\mathbb{P}$-a.e. continuous in $t \in \mathbb{R}$. Therefore, it follows from the Proposition 4.3.3 in [34] that there exists a tempered function $r(\omega) > 0$ such that

$$\begin{aligned}
p(\omega) := \sum_{j=1}^m \Big(&\left|z_j\left(\omega_j\right)\right|^2 + \left|z_j\left(\omega_j\right)\right|^4 + \left|z_j\left(\omega_j\right)\right|^p \\
&+ \left|z_j\left(\omega_j\right)\right|^{2p} + \left|z_j\left(\omega_j\right)\right|^{2p-2}\Big) \leq r(\omega),
\end{aligned} \tag{24}$$

where $r(\omega)$ satisfies that for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$r\left(\theta_t\omega\right) \leq e^{(\lambda/2)|t|} r(\omega), \quad \forall t \in \mathbb{R}. \tag{25}$$

Therefore, for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$p\left(\theta_t\omega\right) \leq e^{(\lambda/2)|t|} r(\omega), \quad \forall t \in \mathbb{R}. \tag{26}$$

Putting

$$z\left(\theta_t\omega\right) = \sum_{j=1}^m h_j z_j\left(\theta_t\omega_j\right), \tag{27}$$

then by (23) we have

$$dz + \lambda z dt = \sum_{j=1}^m h_j d\omega_j. \tag{28}$$

Remark 12. From (24) and (27), we can easily show that the sum

$$\begin{aligned}
&\|z(\omega)\|^2 + \|z(\omega)\|_p^p + \|z(\omega)\|_{2p}^{2p} \\
&+ \|z(\omega)\|_{2p-2}^{2p-2} + \|z(\omega)\|_4^4 + \|\nabla z(\omega)\|^2 \\
&+ \|\nabla z(\omega)\|^2 + \|\Delta z(\omega)\|_p^p,
\end{aligned} \tag{29}$$

is bounded by $p(\omega)$ with a deterministic positive constant c_0. In the following of this paper, we use the symbols $p(\omega)$ and $r(\omega)$ to denote the random variables in (24).

In order to show that the initial problem (1)-(2) generates an RDS, we set $v(t) = u(t) - z(\theta_t\omega)$. Then we can consider the following evolution equation with random parameter but without white noise:

$$\frac{\partial v}{\partial t} + \lambda v - \Delta v = f\left(x, v + z\left(\theta_t\omega\right)\right) + g + \Delta z\left(\theta_t\omega\right), \tag{30}$$

with initial value condition

$$v(x, 0) = v_0(\omega) = u_0 - z(\omega). \tag{31}$$

From [1, 2], we see that for $\mathbb{P}$-a.e. $\omega \in \Omega$ and all $v_0 \in L^2(\mathbb{R}^n)$, the parameterized evolution equation (30)-(31) with conditions (3)–(6) has a unique solution

$$\begin{aligned}
v\left(\cdot, \omega, v_0\right) \in\ & C\left([0, \infty); L^2\left(\mathbb{R}^n\right)\right) \\
& \bigcap L^p\left((0, T); L^p\left(\mathbb{R}^n\right)\right) \\
& \bigcap L^2\left((0, T); H^1\left(\mathbb{R}^n\right)\right),
\end{aligned} \tag{32}$$

for every $T > 0$. Furthermore, $v(t, \omega, v_0)$ is continuous with respect to v_0 in $L^2(\mathbb{R}^n)$, for all $t \geq 0$.

As

$$u(t, \omega, u_0) = v(t, \omega, u_0 - z(\omega)) + z(\theta_t \omega), \tag{33}$$

the process u is the solution to the problem (1)-(2) in a certain sense. We now define a mapping $\phi : \mathbb{R}^+ \times \Omega \times L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ by

$$\phi(t, \omega, u_0) = u(t, \omega, u_0) = v(t, \omega, u_0 - z(\omega)) + z(\theta_t \omega), \tag{34}$$

for all $(t, \omega, u_0) \in \mathbb{R}^+ \times \Omega \times L^2(\mathbb{R}^n)$. Then ϕ is a continuous RDS on $L^2(\mathbb{R}^n)$ and an RDS on $H^1(\mathbb{R}^n)$ respectively associated with the initial value problem of SRDE (1)-(2) on $\mathbb{R}^n$.

Theorem 13 (see [1, 2]). *Assume that $g \in L^2(\mathbb{R}^n)$ and (3)–(6) hold. Then the RDS ϕ generated by (1)-(2) has a unique $(L^2(\mathbb{R}^n), L^2(\mathbb{R}^n))$-random attractor $\widehat{A}_2$ and has a unique $(L^2(\mathbb{R}^n), L^p(\mathbb{R}^n))$-random attractor $\widehat{A}_p$; furthermore, we have $\widehat{A}_2 = \widehat{A}_p$.*

4. Uniform Estimates of Solutions

4.1. $(L^2(\mathbb{R}^n), H^1(\mathbb{R}^n))$-Random Absorbing Set and Some Useful Estimates. The next lemma shows that ϕ has a tempered $(L^2(\mathbb{R}^n), H^1(\mathbb{R}^n))$-random absorbing set.

Lemma 14 (see [1]). *Assume that $g \in L^2(\mathbb{R}^n)$ and (3)–(6) hold. Let $\widehat{D} = \{D(\omega)\}_{\omega \in \Omega} \in \mathscr{D}_2$ and $u_0(\omega) \in D(\omega)$. Then, for $\mathbb{P}$-a.e. $\omega \in \Omega$, there exists $T_{\widehat{D}}(\omega) > 0$ such that, for all $t \geq T_{\widehat{D}}(\omega)$,*

$$\left\| u(t, \theta_{-t}\omega, u_0(\theta_{-t}\omega)) \right\|^2_{H^1(\mathbb{R}^n)} \leq c(1 + r(\omega)), \tag{35}$$

where c is a constant independent of t, ω, and u_0.

We now give some new estimates for the solution $v(t, \omega, v_0)$ of (30)-(31).

Lemma 15. *Assume that $g \in L^2(\mathbb{R}^n)$ and (3)–(6) hold. Let $\widehat{D} = \{D(\omega)\}_{\omega \in \Omega} \in \mathscr{D}_2$ and $u_0(\omega) \in D(\omega)$. Then, for $\mathbb{P}$-a.e. $\omega \in \Omega$, there exists $T_{\widehat{D}}(\omega) \geq 2$, such that the solution $v(t, \omega, v_0(\omega))$ of (30)-(31) satisfies that, for all $t \geq T_{\widehat{D}}(\omega)$ and for all $s \in [t-1, t+1]$,*

$$\left\| v(s, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega)) \right\|^2 \leq c(1 + r(\omega)), \tag{36}$$

$$\left\| v(s, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega)) \right\|^p_p \leq c(1 + r(\omega)), \tag{37}$$

$$\left\| \nabla v(s, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega)) \right\|^2 \leq c(1 + r(\omega)), \tag{38}$$

where c is a constant independent of s, t, and ω.

Proof. The first assertion was proved in [2] in the case of $s \in [t, t+1]$, and the case for $s \in [t-1, t+1]$ can be obtained by slightly modifying the proof of Lemma 4.4 in [2] (in fact, for $s \in [t-2, t+1]$, (36) also holds true, and we will use this result in (55)), and here we omit it.

Now, we prove the second assertion. Multiplying (30) with $|v|^{p-2}v$ and integrating over $\mathbb{R}^n$, we get

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\|v\|^p_p + \lambda\|v\|^p_p + \left(-\Delta v, |v|^{p-2}v\right) \\ &\quad = \int_{\mathbb{R}^n} f(x, v + z(\theta_t\omega))|v|^{p-2}v\,dx \\ &\qquad + \left(g + \Delta z(\theta_t\omega), |v|^{p-2}v\right). \end{aligned} \tag{39}$$

Since $p \geq 2$, we have

$$\begin{aligned} \left(-\Delta v, |v|^{p-2}v\right) &= \left(\nabla v, \nabla\left(|v|^{p-2}v\right)\right) \\ &= \left(\nabla v, (p-2)|v|^{p-4}v^2\nabla v\right) \\ &\quad + \left(\nabla v, |v|^{p-2}\nabla v\right) \geq 0. \end{aligned} \tag{40}$$

For the nonlinearity, similar to (4.6) and (4.8) in [2], we have

$$\begin{aligned} &f(x, v + z(\theta_t\omega))v \\ &\quad \leq -\frac{\alpha_1}{2^p}|v|^p + c\left(|z(\theta_t\omega)|^p + |z(\theta_t\omega)|^2\right) + \psi_1 + \frac{1}{2}\psi_2^2, \end{aligned} \tag{41}$$

$$\begin{aligned} &\int_{\mathbb{R}^n} f(x, v + z(\theta_t\omega))|v|^{p-2}v\,dx \\ &\quad \leq -\frac{\alpha_1}{2^{p+1}}\|v\|^{2p-2}_{2p-2} + \lambda\|v\|^p_p \\ &\qquad + c\left(\|z(\theta_t\omega)\|^{2p-2}_{2p-2} + \|z(\theta_t\omega)\|^p_p\right) \\ &\qquad + c\left(\|\psi_1\|^{p/2}_{p/2} + \|\psi_2\|^p_p\right), \end{aligned} \tag{42}$$

$$\begin{aligned} &\left(g + \Delta z(\theta_t\omega), |v|^{p-2}v\right) \\ &\quad \leq \|g\|\,\|v\|^{p-1}_{2p-2} + \|\Delta z(\theta_t\omega)\|\,\|v\|^{p-1}_{2p-2} \\ &\quad \leq \frac{\alpha_1}{2^{p+2}}\|v\|^{2p-2}_{2p-2} + c\|\Delta z(\theta_t\omega)\|^2 + c\|g\|^2. \end{aligned} \tag{43}$$

From (39)-(40) and (42)-(43), we get

$$\begin{aligned} &\frac{d}{dt}\|v\|^p_p + c\|v\|^{2p-2}_{2p-2} \\ &\quad \leq c_1\left(\|z(\theta_t\omega)\|^{2p-2}_{2p-2} + \|z(\theta_t\omega)\|^p_p + \|\Delta z(\theta_t\omega)\|^2\right) + c_1 \\ &\quad \leq c_2 p(\theta_t\omega) + c_2. \end{aligned} \tag{44}$$

On the other hand, multiplying (30) by v and integrating over $\mathbb{R}^n$, we get the results in [1]:

$$\frac{d}{dt}\|v\|^2 + \lambda\|v\|^2 + \|\nabla v\|^2 + \alpha_1\|u\|_p^p \le cp(\theta_t\omega) + c, \tag{45}$$

$$\|v(t,\omega,v_0(\omega))\|^2 \le e^{-\lambda t}\|v_0(\omega)\|^2 + c\int_0^t e^{\lambda(\tau-t)}p(\theta_\tau\omega)\,d\tau + c. \tag{46}$$

By Hölder inequality, $\|u\|_p^p \ge 2^{1-p}\|v\|_p^p - \|z(\theta_t\omega)\|_p^p$, we can convert (45) into

$$\frac{d}{dt}\|v\|^2 + \lambda\|v\|^2 + \|\nabla v\|^2 + \alpha_1 2^{1-p}\|v\|_p^p \le cp(\theta_t\omega) + c. \tag{47}$$

For any $t \ge 0$, integrating (47) over $(t, t+1)$ and using (46), we get

$$\begin{aligned}\int_t^{t+1}\|v(s)\|_p^p ds &\le c\left(\|v(t)\|^2 + \int_t^{t+1}p(\theta_\tau\omega)\,d\tau + 1\right)\\ &\le c\left(e^{-\lambda t}\|v_0(\omega)\|^2 + \int_0^t e^{\lambda(\tau-t)}p(\theta_\tau\omega)\,d\tau + \int_t^{t+1}p(\theta_\tau\omega)\,d\tau + 1\right).\end{aligned} \tag{48}$$

Next, fix $s \in (t, t+1)$ and integrate (44) over $(s, t+1)$ to get

$$\begin{aligned}\|v(t+1,\omega,v_0(\omega))\|_p^p &\le \|v(s,\omega,v_0(\omega))\|_p^p + c_2\int_s^{t+1}p(\theta_\tau\omega)\,d\tau + c_2(t+1-s)\\ &\le \|v(s,\omega,v_0(\omega))\|_p^p + c_2\int_t^{t+1}p(\theta_\tau\omega)\,d\tau + c_2.\end{aligned} \tag{49}$$

Integrating the above inequality with respect to s over $(t, t+1)$ and using (48), we obtain, for all $t \ge 0$,

$$\begin{aligned}\|v(t+1,\omega,v_0(\omega))\|_p^p &\le \int_t^{t+1}\|v(s,\omega,v_0(\omega))\|_p^p ds + c_2\int_t^{t+1}p(\theta_s\omega)\,ds + c_2\\ &\le c\left(e^{-\lambda t}\|v_0(\omega)\|^2 + \int_0^t e^{\lambda(\tau-t)}p(\theta_\tau\omega)\,d\tau + \int_t^{t+1}p(\theta_\tau\omega)\,d\tau + 1\right)\\ &\le c\left(e^{-\lambda(t+1)}\|v_0(\omega)\|^2 + \int_0^{t+1}e^{\lambda(\tau-t-1)}p(\theta_\tau\omega)\,d\tau + \int_t^{t+1}p(\theta_\tau\omega)\,d\tau + 1\right).\end{aligned} \tag{50}$$

Replacing $t+1$ by s first, then substituting $\theta_{-t-1}\omega$ for ω in the aforementioned inequality, and noting that $s \in [t-1, t+1]\,(t \ge 2)$, we have

$$\begin{aligned}&\|v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|_p^p\\ &\le c\left[e^{-\lambda s}\|v_0(\theta_{-t-1}\omega)\|^2 + \int_0^s e^{\lambda(\tau-s)}p(\theta_{\tau-t-1}\omega)\,d\tau + \int_{s-1}^s p(\theta_{\tau-t-1}\omega)\,d\tau + 1\right]\\ &\le c\left[e^{\lambda(t+1-s)}e^{-\lambda(t+1)}\|v_0(\theta_{-t-1}\omega)\|^2 + \int_0^{t+1}e^{\lambda(t+1-s)}e^{\lambda(\tau-t-1)}p(\theta_{\tau-t-1}\omega)\,d\tau + \int_{t-2}^{t+1}p(\theta_{\tau-t-1}\omega)\,d\tau + 1\right]\\ &\le c\left[e^{-\lambda(t+1)}\|v_0(\theta_{-t-1}\omega)\|^2 + \int_0^{t+1}e^{\lambda(\tau-t-1)}p(\theta_{\tau-t-1}\omega)\,d\tau + \int_{t-2}^{t+1}p(\theta_{\tau-t-1}\omega)\,d\tau + 1\right]\\ &= c\left[e^{-\lambda(t+1)}\|v_0(\theta_{-t-1}\omega)\|^2 + \int_{-t-1}^0 e^{\lambda\tau}p(\theta_\tau\omega)\,d\tau + \int_{-3}^0 p(\theta_\tau\omega)\,d\tau + 1\right]\\ &\le c\left[e^{-\lambda(t+1)}\|v_0(\theta_{-t-1}\omega)\|^2 + r(\omega)\int_{-t-1}^0 e^{(\lambda/2)\tau}d\tau + r(\omega)\int_{-3}^0 e^{-(\lambda/2)\tau}d\tau + 1\right],\end{aligned} \tag{51}$$

where we have used (26) in the pervious inequality. Noting that $u_0(\omega) \in D(\omega)$ with $\widehat{D} = \{D(\omega)\}_{\omega\in\Omega} \in \mathscr{D}_2$ and $v_0(\omega) = u_0(\omega) - z(\omega)$, we get from (51) that there exists $T_{\widehat{D}}(\omega) \ge 2$ such that, for all $t \ge T_{\widehat{D}}(\omega)$ and for all $s \in [t-1, t+1]$,

$$\|v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|_p^p \le c(1 + r(\omega)). \tag{52}$$

That is, (37) holds true.

To prove (38), we take the inner product of (30) with $-\Delta v$ in $L^2(\Omega)$, and using (4.31) in [1], we get

$$\frac{d}{dt}\|\nabla v\|^2 \le c\left(\|\nabla u\|^2 + \|u\|_p^p\right) + c\left(\left\|\Delta z\left(\theta_t\omega\right)\right\|^2 + \left\|\Delta z\left(\theta_t\omega\right)\right\|_p^p + 1\right). \tag{53}$$

This implies that

$$\frac{d}{dt}\|\nabla v\|^2 \le c\left(\|\nabla v\|^2 + \|v\|_p^p\right) + c\left(p\left(\theta_t\omega\right) + 1\right). \tag{54}$$

Integrating (47) over $(t-2, t+1)$ and substituting $\theta_{-t-1}\omega$ for ω, then from (36), we get

$$\begin{aligned}\int_{t-2}^{t+1} &\left\|\nabla v\left(s, \theta_{-t-1}\omega, v_0\left(\theta_{-t-1}\omega\right)\right)\right\|^2 ds \\ &\le \left\|v\left(t-2, \theta_{-t-1}\omega, v_0\left(\theta_{-t-1}\omega\right)\right)\right\|^2 \\ &\quad + c\left(\int_{t-2}^{t+1} p\left(\theta_{\tau-t-1}\omega\right) d\tau + 1\right) \\ &\le c\left(1 + r\left(\omega\right)\right), \quad \forall t \ge T_{\widehat{D}}\left(\omega\right).\end{aligned} \tag{55}$$

Obviously, from (51) we can easily see that (37) also holds for $s \in [t-2, t+1]$; then, by (37), (47), (54)-(55), and a similar procedure as the proof of (37), one can show that, for all $t \ge T_{\widehat{D}}(\omega)$, and for all $s \in [t-1, t+1]$,

$$\left\|\nabla v\left(s, \theta_{-t-1}\omega, v_0\left(\theta_{-t-1}\omega\right)\right)\right\|^2 \le c\left(1 + r\left(\omega\right)\right). \tag{56}$$

The proof is complete. □

Lemma 16. *Assume that* $g \in L^2(\mathbb{R}^n)$ *and* (3)–(6) *hold. Let* $\widehat{D} = \{D(\omega)\}_{\omega\in\Omega} \in \mathscr{D}_2$ *and* $u_0(\omega) \in D(\omega)$. *Then, for* $\mathbb{P}$-*a.e.* $\omega \in \Omega$, *there exists* $T_{\widehat{D}}(\omega) > 0$, *such that the solution* $v(t, \omega, v_0(\omega))$ *of* (30)-(31) *satisfies that for all* $t \ge T_{\widehat{D}}(\omega)$,

$$\int_t^{t+1} \left\|v\left(s, \theta_{-t-1}\omega, v_0\left(\theta_{-t-1}\omega\right)\right)\right\|_{2p-2}^{2p-2} ds \le c\left(1 + r\left(\omega\right)\right), \tag{57}$$

$$\int_t^{t+1} \left\|v_s\left(s, \theta_{-t-1}\omega, v_0\left(\theta_{-t-1}\omega\right)\right)\right\|^2 ds \le c\left(1 + r\left(\omega\right)\right). \tag{58}$$

Proof. Integrating (44) over $(t, t+1)$, we get

$$\int_t^{t+1} \|v\left(s\right)\|_{2p-2}^{2p-2} ds \le c\left(\|v\left(t\right)\|_p^p + \int_t^{t+1} p\left(\theta_\tau\omega\right) d\tau + 1\right). \tag{59}$$

Replacing ω by $\theta_{-t-1}\omega$ in the aforementioned inequality, it yields that

$$\begin{aligned}\int_t^{t+1} &\left\|v\left(s, \theta_{-t-1}\omega, v_0\left(\theta_{-t-1}\omega\right)\right)\right\|_{2p-2}^{2p-2} ds \\ &\le c\left(\left\|v\left(t, \theta_{-t-1}\omega, v_0\left(\theta_{-t-1}\omega\right)\right)\right\|_p^p + \int_t^{t+1} p\left(\theta_{\tau-t-1}\omega\right) d\tau + 1\right) \\ &= c\left(\left\|v\left(t, \theta_{-t-1}\omega, v_0\left(\theta_{-t-1}\omega\right)\right)\right\|_p^p + \int_{-1}^{0} p\left(\theta_\tau\omega\right) d\tau + 1\right) \\ &\le c\left(\left\|v\left(t, \theta_{-t-1}\omega, v_0\left(\theta_{-t-1}\omega\right)\right)\right\|_p^p + r\left(\omega\right)\int_{-1}^{0} e^{-(\lambda/2)\tau} d\tau + 1\right).\end{aligned} \tag{60}$$

Lemma 15 and the aforementioned inequality imply that there exists $T_{\widehat{D}}(\omega) > 0$, $\forall t \ge T_{\widehat{D}}(\omega)$, such that (57) holds.

Next, taking the inner product of (30) with v_t in $L^2(\mathbb{R}^n)$, and using (4), we obtain

$$\begin{aligned}\left\|v_t\right\|^2 &+ \frac{\lambda}{2}\frac{d}{dt}\|v\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla v\|^2 \\ &= \int_{\mathbb{R}^n} f\left(x, u\right) v_t\, dx + \int_{\mathbb{R}^n} g v_t\, dx + \int_{\mathbb{R}^n} \Delta z\left(\theta_t\omega\right) v_t\, dx \\ &\le \alpha_2 \int_{\mathbb{R}^n} |u|^{p-1}\left|v_t\right| dx + \int_{\mathbb{R}^n} \left|\psi_2\right|\left|v_t\right| dx + \int_{\mathbb{R}^n} |g|\left|v_t\right| dx \\ &\quad + \int_{\mathbb{R}^n} \left|\Delta z\left(\theta_t\omega\right)\right|\left|v_t\right| dx \\ &\le \frac{1}{2}\left\|v_t\right\|^2 + c\left(\|u\|_{2p-2}^{2p-2} + \left\|\psi_2\right\|^2 + \|g\|^2 + \left\|\Delta z\left(\theta_t\omega\right)\right\|^2\right) \\ &\le \frac{1}{2}\left\|v_t\right\|^2 + c\left(\|v\|_{2p-2}^{2p-2} + \left\|\psi_2\right\|^2 + \|g\|^2 \right. \\ &\qquad \left. + \left\|\Delta z\left(\theta_t\omega\right)\right\|^2 + \left\|z\left(\theta_t\omega\right)\right\|_{2p-2}^{2p-2}\right),\end{aligned} \tag{61}$$

that is,

$$\begin{aligned}\left\|v_t\right\|^2 &+ \lambda\frac{d}{dt}\|v\|^2 + \frac{d}{dt}\|\nabla v\|^2 \\ &\le c\left(\|v\|_{2p-2}^{2p-2} + \left\|\psi_2\right\|^2 + \|g\|^2 + \left\|\Delta z(\theta_t\omega)\right\|^2 + \left\|z(\theta_t\omega)\right\|_{2p-2}^{2p-2}\right).\end{aligned} \tag{62}$$

We now integrate (62) over $(t, t+1)$ to obtain

$$\begin{aligned}\int_t^{t+1} &\left\|v_s\left(s, \omega, v_0\left(\omega\right)\right)\right\|^2 ds \\ &\le c\left(\left\|v\left(t, \omega, v_0\left(\omega\right)\right)\right\|^2 + \left\|\nabla v\left(t, \omega, v_0\left(\omega\right)\right)\right\|^2\right) \\ &\quad + c\int_t^{t+1} \left(\left\|z\left(\theta_\tau\omega\right)\right\|_{2p-2}^{2p-2} + \left\|\Delta z\left(\theta_\tau\omega\right)\right\|^2\right) d\tau \\ &\quad + c\int_t^{t+1} \left\|v\left(\tau, \omega, v_0\left(\omega\right)\right)\right\|_{2p-2}^{2p-2} d\tau \\ &\quad + c\left(\left\|\psi_2\right\|^2 + \|g\|^2\right).\end{aligned} \tag{63}$$

Replacing ω by $\theta_{-t-1}\omega$, we get

$$\begin{aligned}\int_t^{t+1} &\|v_s(s,\theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))\|^2 ds \\ &\le c\left(\|v(t,\theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))\|^2 \right.\\ &\quad \left.+\|\nabla v(t,\theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))\|^2\right) \\ &\quad + c\int_t^{t+1}\left(\|z(\theta_{\tau-t-1}\omega)\|_{2p-2}^{2p-2} + \|\Delta z(\theta_{\tau-t-1}\omega)\|^2\right)d\tau \\ &\quad + c\int_t^{t+1}\|v(s,\theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))\|_{2p-2}^{2p-2} ds \\ &\quad + c\left(\|\psi_2\|^2 + \|g\|^2\right).\end{aligned} \tag{64}$$

Equations (36), (38) and (57) together imply that (58) is also true. The proof is complete. □

4.2. Tail Estimate in $H^1(\mathbb{R}^n)$. We next estimate "the tail" of the solution to the problem (1)-(2) in $H^1(\mathbb{R}^n)$.

Lemma 17. *Assume that $g \in L^2(\mathbb{R}^n)$ and (3)–(6) hold. Let $\widehat{D} = \{D(\omega)\}_{\omega\in\Omega} \in \mathscr{D}_2$ and $u_0(\omega) \in D(\omega)$. Then, for $\mathbb{P}$-a.e. $\omega \in \Omega$ and for every $\varepsilon > 0$, there exist $T^* = T^*_{\widehat{D}}(\omega,\varepsilon) > 0$ and $R^* = R^*(\omega,\varepsilon)$, such that the solution $v(t,\omega,v_0(\omega))$ of (30)-(31) satisfies that $\forall t \ge T^*$,*

$$\begin{aligned}\int_{|x|\ge R^*}&\left(|v(t,\theta_{-t}\omega, v_0(\theta_{-t}\omega))|^2\right. \\ &\left.+|\nabla v(t,\theta_{-t}\omega, v_0(\theta_{-t}\omega))|^2\right)dx \le \varepsilon.\end{aligned} \tag{65}$$

Proof. By Lemma 4.6 in [1], it suffices to prove that

$$\int_{|x|\ge R^*} |\nabla v(t,\theta_{-t}\omega, v_0(\theta_{-t}\omega))|^2 dx \le \varepsilon. \tag{66}$$

Let φ be a smooth function defined on $\mathbb{R}^+$ such that $0 \le \varphi(s) \le 1$, for all $s \in \mathbb{R}^+$, and

$$\varphi(s) = \begin{cases} 0, & 0 \le s \le 1, \\ 1, & s \ge 2. \end{cases} \tag{67}$$

Then there is a positive constant c such that $|\varphi'(s)| + |\varphi''(s)| \le c$ for all $s \in \mathbb{R}^+$.

Multiplying (30) by $-\varphi(|x|^2/k^2)\Delta v$ and integrating with respect to x over $\mathbb{R}^n$, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&\int_{\mathbb{R}^n}\varphi\left(\frac{|x|^2}{k^2}\right)|\nabla v|^2 dx + \int_{\mathbb{R}^n}\left(\nabla\varphi\left(\frac{|x|^2}{k^2}\right)\cdot\nabla v\right)v_t\, dx \\ &+ \lambda\int_{\mathbb{R}^n}\varphi\left(\frac{|x|^2}{k^2}\right)|\nabla v|^2 dx \\ &+ \lambda\int_{\mathbb{R}^n}\left(\nabla\varphi\left(\frac{|x|^2}{k^2}\right)\cdot\nabla v\right)v\, dx \\ &+ \int_{\mathbb{R}^n}\varphi\left(\frac{|x|^2}{k^2}\right)|\Delta v|^2 dx \\ &= -\int_{\mathbb{R}^n}\varphi\left(\frac{|x|^2}{k^2}\right)f(x,u)\,\Delta v\, dx \\ &\quad -\int_{\mathbb{R}^n}\varphi\left(\frac{|x|^2}{k^2}\right)(g+\Delta z(\theta_t\omega))\,\Delta v\, dx.\end{aligned} \tag{68}$$

The second term of the left-hand side is bounded by

$$\begin{aligned}\int_{\mathbb{R}^n}&\left|\nabla\varphi\left(\frac{|x|^2}{k^2}\right)\right||\nabla v|\,|v_t|\, dx \\ &= \int_{\mathbb{R}^n}\left|\varphi'\left(\frac{|x|^2}{k^2}\right)\right|\left|\frac{2x}{k^2}\right||\nabla v|\,|v_t|\, dx \\ &\le \frac{c}{k}\int_{k\le|x|\le\sqrt{2}k}|\nabla v|\,|v_t|\, dx \\ &\le \frac{c}{k}\left(\|v_t\|^2 + \|\nabla v\|^2\right).\end{aligned} \tag{69}$$

Similarly, the forth term of the left-hand side of (68) is bounded by

$$\lambda\int_{\mathbb{R}^n}\left|\nabla\varphi\left(\frac{|x|^2}{k^2}\right)\right||\nabla v|\,|v|\, dx \le \frac{c}{k}\left(\|v\|^2 + \|\nabla v\|^2\right). \tag{70}$$

For the last term of the right-hand side of (68), we have

$$\begin{aligned}&\left|\int_{\mathbb{R}^n}\varphi\left(\frac{|x|^2}{k^2}\right)(g+\Delta z(\theta_t\omega))\,\Delta v\, dx\right| \\ &\quad\le \frac{1}{2}\int_{\mathbb{R}^n}\varphi\left(\frac{|x|^2}{k^2}\right)|\Delta v|^2 dx + \int_{\mathbb{R}^n}\varphi\left(\frac{|x|^2}{k^2}\right)|g|^2 dx \\ &\qquad + \int_{\mathbb{R}^n}\varphi\left(\frac{|x|^2}{k^2}\right)|\Delta z(\theta_t\omega)|^2 dx.\end{aligned} \tag{71}$$

We next consider the nonlinear term in (68). Since

$$\begin{aligned}-\int_{\mathbb{R}^n}&\varphi\left(\frac{|x|^2}{k^2}\right)f(x,u)\,\Delta v\, dx \\ &= -\int_{\mathbb{R}^n}\varphi\left(\frac{|x|^2}{k^2}\right)f(x,u)\,\Delta u\, dx \\ &\quad + \int_{\mathbb{R}^n}\varphi\left(\frac{|x|^2}{k^2}\right)f(x,u)\,\Delta z(\theta_t\omega)\, dx\end{aligned}$$

$$= \int_{\mathbb{R}^n} f(x,u)\left(\nabla\varphi\left(\frac{|x|^2}{k^2}\right)\cdot\nabla u\right)dx + \int_{\mathbb{R}^n}\varphi\left(\frac{|x|^2}{k^2}\right)\left(\frac{\partial}{\partial x}f(x,u)\cdot\nabla u\right)dx + \int_{\mathbb{R}^n}\varphi\left(\frac{|x|^2}{k^2}\right)\frac{\partial}{\partial u}f(x,u)|\nabla u|^2dx + \int_{\mathbb{R}^n}\varphi\left(\frac{|x|^2}{k^2}\right)f(x,u)\,\Delta z(\theta_t\omega)\,dx. \tag{72}$$

We now estimate each term in the right-hand side of (72). Using (4), the property of φ, and Cauchy's inequality, we see that the first term of the right-hand side of (72) is bounded by

$$\begin{aligned}\int_{\mathbb{R}^n}\left|\nabla\varphi\left(\frac{|x|^2}{k^2}\right)\right||f(x,u)||\nabla u|\,dx &\le \alpha_2\int_{\mathbb{R}^n}\left|\nabla\varphi\left(\frac{|x|^2}{k^2}\right)\right||u|^{p-1}|\nabla u|\,dx + \int_{\mathbb{R}^n}\left|\nabla\varphi\left(\frac{|x|^2}{k^2}\right)\right||\psi_2||\nabla u|\,dx\\ &\le c\left(\int_{\mathbb{R}^n}\left|\nabla\varphi\left(\frac{|x|^2}{k^2}\right)\right||u|^{2p-2}dx + \int_{\mathbb{R}^n}\left|\nabla\varphi\left(\frac{|x|^2}{k^2}\right)\right||\nabla u|^2dx\right) + \frac{c}{k}\int_{k\le|x|\le\sqrt{2}k}|\psi_2||\nabla u|\,dx\\ &\le \frac{c}{k}\left(\|u\|_{2p-2}^{2p-2} + \|\nabla u\|^2 + 1\right).\end{aligned} \tag{73}$$

By (6), we can estimate the second term of the right-hand side of (72) as follows:

$$\begin{aligned}\int_{\mathbb{R}^n}\left|\varphi\left(\frac{|x|^2}{k^2}\right)\right|\left|\frac{\partial}{\partial x}f(x,u)\right||\nabla u|\,dx &\le \int_{\mathbb{R}^n}\left|\varphi\left(\frac{|x|^2}{k^2}\right)\right||\psi_3||\nabla u|\,dx\\ &\le \frac{\lambda}{4}\int_{\mathbb{R}^n}\varphi\left(\frac{|x|^2}{k^2}\right)|\nabla u|^2dx + \frac{c}{\lambda}\int_{\mathbb{R}^n}\varphi\left(\frac{|x|^2}{k^2}\right)|\psi_3|^2dx\\ &\le \frac{\lambda}{2}\int_{\mathbb{R}^n}\varphi\left(\frac{|x|^2}{k^2}\right)|\nabla v|^2dx + c\int_{\mathbb{R}^n}\varphi\left(\frac{|x|^2}{k^2}\right)|\nabla z(\theta_t\omega)|^2dx + c\int_{\mathbb{R}^n}\varphi\left(\frac{|x|^2}{k^2}\right)|\psi_3|^2dx.\end{aligned} \tag{74}$$

For the third term of the right-hand side of (72), by using (5), we have

$$\begin{aligned}\int_{\mathbb{R}^n}\varphi\left(\frac{|x|^2}{k^2}\right)\frac{\partial}{\partial u}f(x,u)|\nabla u|^2dx &\le \beta\int_{\mathbb{R}^n}\varphi\left(\frac{|x|^2}{k^2}\right)|\nabla u|^2dx\\ &\le 2\beta\int_{\mathbb{R}^n}\varphi\left(\frac{|x|^2}{k^2}\right)|\nabla v|^2dx + 2\beta\int_{\mathbb{R}^n}\varphi\left(\frac{|x|^2}{k^2}\right)|\nabla z(\theta_t\omega)|^2dx.\end{aligned} \tag{75}$$

For the last term of the right-hand side of (72), by using (4) and Young's inequality, we find

$$\begin{aligned}\left|\int_{\mathbb{R}^n}\varphi\left(\frac{|x|^2}{k^2}\right)f(x,u)\,\Delta z(\theta_t\omega)\,dx\right| &\le \alpha_2\int_{\mathbb{R}^n}\varphi\left(\frac{|x|^2}{k^2}\right)|u|^{p-1}|\Delta z(\theta_t\omega)|\,dx + \int_{\mathbb{R}^n}\varphi\left(\frac{|x|^2}{k^2}\right)|\psi_2||\Delta z(\theta_t\omega)|\,dx\\ &\le c\int_{\mathbb{R}^n}\varphi\left(\frac{|x|^2}{k^2}\right)|u|^pdx + c\int_{\mathbb{R}^n}\varphi\left(\frac{|x|^2}{k^2}\right)|\Delta z(\theta_t\omega)|^pdx\\ &\quad + \frac{1}{2}\int_{\mathbb{R}^n}\varphi\left(\frac{|x|^2}{k^2}\right)|\psi_2|^2dx + \frac{1}{2}\int_{\mathbb{R}^n}\varphi\left(\frac{|x|^2}{k^2}\right)|\Delta z(\theta_t\omega)|^2dx.\end{aligned} \tag{76}$$

Putting (73)–(76) together into (72), it yields that

$$\begin{aligned}-\int_{\mathbb{R}^n}\varphi\left(\frac{|x|^2}{k^2}\right)f(x,u)\,\Delta v\,dx &\le \frac{\lambda+4\beta}{2}\int_{\mathbb{R}^n}\varphi\left(\frac{|x|^2}{k^2}\right)|\nabla v|^2dx + c\int_{\mathbb{R}^n}\varphi\left(\frac{|x|^2}{k^2}\right)|u|^pdx\\ &\quad + \frac{c}{k}\left(\|u\|_{2p-2}^{2p-2} + \|\nabla u\|^2 + 1\right)\end{aligned}$$

$$+ c\int_{\mathbb{R}^n}\varphi\left(\frac{|x|^2}{k^2}\right)\left(|\psi_2|^2+|\psi_3|^2\right)dx$$
$$+ c\int_{\mathbb{R}^n}\varphi\left(\frac{|x|^2}{k^2}\right)$$
$$\times\left(|\nabla z(\theta_t\omega)|^2+|\Delta z(\theta_t\omega)|^2+|\Delta z(\theta_t\omega)|^p\right)dx. \tag{77}$$

Then by (68)–(71) and (77), we get

$$\frac{d}{dt}\int_{\mathbb{R}^n}\varphi\left(\frac{|x|^2}{k^2}\right)|\nabla v|^2dx$$
$$+\lambda\int_{\mathbb{R}^n}\varphi\left(\frac{|x|^2}{k^2}\right)|\nabla v|^2dx+\int_{\mathbb{R}^n}\varphi\left(\frac{|x|^2}{k^2}\right)|\Delta v|^2dx$$
$$\le\frac{c}{k}\left(\|v_t\|^2+\|\nabla v\|^2+\|v\|^2+\|u\|_{2p-2}^{2p-2}+\|\nabla u\|^2+1\right)$$
$$+4\beta\int_{\mathbb{R}^n}\varphi\left(\frac{|x|^2}{k^2}\right)|\nabla v|^2dx$$
$$+c\int_{\mathbb{R}^n}\varphi\left(\frac{|x|^2}{k^2}\right)|u|^pdx$$
$$+c\int_{\mathbb{R}^n}\varphi\left(\frac{|x|^2}{k^2}\right)\left(|\nabla z(\theta_t\omega)|^2\right.$$
$$\left.+|\Delta z(\theta_t\omega)|^2+|\Delta z(\theta_t\omega)|^p\right)dx$$
$$+c\int_{\mathbb{R}^n}\varphi\left(\frac{|x|^2}{k^2}\right)\left(|g|^2+|\psi_2|^2+|\psi_3|^2\right)dx. \tag{78}$$

In particular,

$$\frac{d}{dt}\int_{\mathbb{R}^n}\varphi\left(\frac{|x|^2}{k^2}\right)|\nabla v(t)|^2dx$$
$$\le 4\beta\int_{\mathbb{R}^n}\varphi\left(\frac{|x|^2}{k^2}\right)|\nabla v(t)|^2dx$$
$$+c\int_{\mathbb{R}^n}\varphi\left(\frac{|x|^2}{k^2}\right)|u(t)|^pdx$$
$$+\frac{c}{k}\left(\|v_t(t)\|^2+\|\nabla v(t)\|^2+\|v(t)\|^2\right.$$
$$+\|v(t)\|_{2p-2}^{2p-2}+\|z(\theta_t\omega)\|_{2p-2}^{2p-2}$$
$$\left.+\|\nabla z(\theta_t\omega)\|^2+1\right)$$
$$+c\int_{\mathbb{R}^n}\varphi\left(\frac{|x|^2}{k^2}\right)\left(|\nabla z(\theta_t\omega)|^2+|\Delta z(\theta_t\omega)|^2\right.$$
$$\left.+|\Delta z(\theta_t\omega)|^p\right)dx$$
$$+c\int_{\mathbb{R}^n}\varphi\left(\frac{|x|^2}{k^2}\right)\left(|g|^2+|\psi_2|^2+|\psi_3|^2\right)dx. \tag{79}$$

Let $s\in(t,t+1)$ and integrate the aforementioned inequality from s to $t+1$:

$$\int_{\mathbb{R}^n}\varphi\left(\frac{|x|^2}{k^2}\right)|\nabla v(t+1,\omega,v_0(\omega))|^2dx$$
$$\le\int_{\mathbb{R}^n}\varphi\left(\frac{|x|^2}{k^2}\right)|\nabla v(s,\omega,v_0(\omega))|^2dx$$
$$+4\beta\int_t^{t+1}\int_{\mathbb{R}^n}\varphi\left(\frac{|x|^2}{k^2}\right)|\nabla v(\tau,\omega,v_0(\omega))|^2dx\,d\tau$$
$$+c\int_t^{t+1}\int_{\mathbb{R}^n}\varphi\left(\frac{|x|^2}{k^2}\right)|u(\tau,\omega,u_0(\omega))|^pdx\,d\tau$$
$$+\frac{c}{k}\left(\int_t^{t+1}\|v_\tau(\tau,\omega,v_0(\omega))\|^2d\tau\right.$$
$$+\int_t^{t+1}\|\nabla v(\tau,\omega,v_0(\omega))\|^2d\tau$$
$$+\int_t^{t+1}\|v(\tau,\omega,v_0(\omega))\|^2d\tau$$
$$+\int_t^{t+1}\|v(\tau,\omega,v_0(\omega))\|_{2p-2}^{2p-2}d\tau$$
$$+\int_t^{t+1}\|z(\theta_\tau\omega)\|_{2p-2}^{2p-2}d\tau$$
$$\left.+\int_t^{t+1}\|\nabla z(\theta_\tau\omega)\|^2d\tau+1\right)$$
$$+c\int_t^{t+1}\int_{\mathbb{R}^n}\varphi\left(\frac{|x|^2}{k^2}\right)$$
$$\times\left(|\nabla z(\theta_\tau\omega)|^2+|\Delta z(\theta_\tau\omega)|^2\right.$$
$$\left.+|\Delta z(\theta_\tau\omega)|^p\right)dx\,d\tau$$
$$+c\int_{\mathbb{R}^n}\varphi\left(\frac{|x|^2}{k^2}\right)\left(|g|^2+|\psi_2|^2+|\psi_3|^2\right)dx. \tag{80}$$

Integrating the aforementioned inequality with respect to s over $(t,t+1)$, and replacing ω by $\theta_{-t-1}\omega$, we obtain

$$\int_{\mathbb{R}^n}\varphi\left(\frac{|x|^2}{k^2}\right)|\nabla v(t+1,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))|^2dx$$
$$\le c_3\int_t^{t+1}\int_{\mathbb{R}^n}\varphi\left(\frac{|x|^2}{k^2}\right)$$
$$\times|\nabla v(\tau,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))|^2dx\,d\tau$$

$$
\begin{aligned}
&+ c_3 \int_t^{t+1} \int_{\mathbb{R}^n} \varphi\left(\frac{|x|^2}{k^2}\right) \\
&\qquad \times \left|u\left(\tau, \theta_{-t-1}\omega, u_0\left(\theta_{-t-1}\omega\right)\right)\right|^p dx\, d\tau \\
&+ \frac{c_3}{k}\left(\int_t^{t+1} \left\|v_\tau\left(\tau, \theta_{-t-1}\omega, v_0\left(\theta_{-t-1}\omega\right)\right)\right\|^2 d\tau\right. \\
&\qquad + \int_t^{t+1} \left\|\nabla v\left(\tau, \theta_{-t-1}\omega, v_0\left(\theta_{-t-1}\omega\right)\right)\right\|^2 d\tau \\
&\qquad + \int_t^{t+1} \left\|v\left(\tau, \theta_{-t-1}\omega, v_0\left(\theta_{-t-1}\omega\right)\right)\right\|^2 d\tau \\
&\qquad + \int_t^{t+1} \left\|v\left(\tau, \theta_{-t-1}\omega, v_0\left(\theta_{-t-1}\omega\right)\right)\right\|_{2p-2}^{2p-2} d\tau \\
&\qquad + \int_t^{t+1} \left\|z\left(\theta_{\tau-t-1}\omega\right)\right\|_{2p-2}^{2p-2} d\tau \\
&\qquad \left. + \int_t^{t+1} \left\|\nabla z\left(\theta_{\tau-t-1}\omega\right)\right\|^2 d\tau + 1\right) \\
&+ c_3 \int_t^{t+1} \int_{\mathbb{R}^n} \varphi\left(\frac{|x|^2}{k^2}\right) \\
&\qquad \times \left(\left|\nabla z\left(\theta_{\tau-t-1}\omega\right)\right|^2 + \left|\Delta z\left(\theta_{\tau-t-1}\omega\right)\right|^2\right. \\
&\qquad \left. + \left|\Delta z\left(\theta_{\tau-t-1}\omega\right)\right|^p\right) dx\, d\tau \\
&+ c_3 \int_{\mathbb{R}^n} \varphi\left(\frac{|x|^2}{k^2}\right)\left(|g|^2 + \left|\psi_2\right|^2 + \left|\psi_3\right|^2\right) dx.
\end{aligned} \tag{81}
$$

In the sequel, we estimate each term in the right-hand side of (81) to show that they are arbitrary small when k and t are large enough.

To estimate the first term in the right-hand side of (81), we cite the result (4.46) in [1]; that is,

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt} \int_{\mathbb{R}^n} \varphi\left(\frac{|x|^2}{k^2}\right) |v|^2 dx \\
&\qquad + \frac{1}{2}\alpha_1 \int_{\mathbb{R}^n} \varphi\left(\frac{|x|^2}{k^2}\right) |u|^p dx + \int_{\mathbb{R}^n} \varphi\left(\frac{|x|^2}{k^2}\right) |\nabla v|^2 dx \\
&\leq \frac{c}{k}\left(\|\nabla v\|^2 + \|v\|^2\right) \\
&\qquad + \int_{\mathbb{R}^n} \varphi\left(\frac{|x|^2}{k^2}\right)\left(\frac{1}{\lambda}|g|^2 + \left|\psi_1\right| + \frac{1}{2}\left|\psi_2\right|^2\right) dx \\
&\qquad + c \int_{\mathbb{R}^n} \varphi\left(\frac{|x|^2}{k^2}\right)\left(\left|\Delta z\left(\theta_t\omega\right)\right|^2 + \left|z\left(\theta_t\omega\right)\right|^2\right. \\
&\qquad \left. + \left|z\left(\theta_t\omega\right)\right|^p\right) dx.
\end{aligned} \tag{82}
$$

Integrating the aforementioned inequality over $(t, t+1)$ and replacing ω by $\theta_{-t-1}\omega$ we get

$$
\begin{aligned}
&\int_t^{t+1} \int_{\mathbb{R}^n} \varphi\left(\frac{|x|^2}{k^2}\right) \left|\nabla v\left(s, \theta_{-t-1}\omega, v_0\left(\theta_{-t-1}\omega\right)\right)\right|^2 dx\, ds \\
&\quad \leq \frac{1}{2} \int_{\mathbb{R}^n} \varphi\left(\frac{|x|^2}{k^2}\right) \left|v\left(t, \theta_{-t-1}\omega, v_0\left(\theta_{-t-1}\omega\right)\right)\right|^2 dx \\
&\qquad + \frac{c}{k}\left(\int_t^{t+1} \left\|\nabla v\left(s, \theta_{-t-1}\omega, v_0\left(\theta_{-t-1}\omega\right)\right)\right\|^2 ds\right. \\
&\qquad \left. + \int_t^{t+1} \left\|v\left(s, \theta_{-t-1}\omega, v_0\left(\theta_{-t-1}\omega\right)\right)\right\|^2 ds\right) \\
&\qquad + \int_{\mathbb{R}^n} \varphi\left(\frac{|x|^2}{k^2}\right)\left(\frac{1}{\lambda}|g|^2 + \left|\psi_1\right| + \frac{1}{2}\left|\psi_2\right|^2\right) dx \\
&\qquad + c \int_t^{t+1} \int_{\mathbb{R}^n} \varphi\left(\frac{|x|^2}{k^2}\right) \\
&\qquad\quad \times \left(\left|\Delta z\left(\theta_{s-t-1}\omega\right)\right|^2 + \left|z\left(\theta_{s-t-1}\omega\right)\right|^2\right. \\
&\qquad\quad \left. + \left|z\left(\theta_{s-t-1}\omega\right)\right|^p\right) dx\, ds.
\end{aligned} \tag{83}
$$

To estimate the first term of the right-hand side of (83), we need a result in [1]. Substituting $\theta_{-t-1}\omega$ for ω of (4.49) in [1], we get

$$
\begin{aligned}
&\int_{\mathbb{R}^n} \varphi\left(\frac{|x|^2}{k^2}\right) \left|v\left(t, \theta_{-t-1}\omega, v_0\left(\theta_{-t-1}\omega\right)\right)\right|^2 dx \\
&\quad \leq e^{\lambda(T-t)} \int_{\mathbb{R}^n} \varphi\left(\frac{|x|^2}{k^2}\right) \left|v\left(T, \theta_{-t-1}\omega, v_0\left(\theta_{-t-1}\omega\right)\right)\right|^2 dx \\
&\qquad + \frac{c}{k} \int_T^t e^{\lambda(s-t)} \left(\left\|v\left(s, \theta_{-t-1}\omega, v_0\left(\theta_{-t-1}\omega\right)\right)\right\|^2\right. \\
&\qquad\quad \left. + \left\|\nabla v\left(s, \theta_{-t-1}\omega, v_0\left(\theta_{-t-1}\omega\right)\right)\right\|^2\right) ds \\
&\qquad + \int_T^t e^{\lambda(s-t)} \int_{\mathbb{R}^n} \varphi\left(\frac{|x|^2}{k^2}\right) \\
&\qquad\quad \times \left(\frac{2}{\lambda}|g|^2 + 2\left|\psi_1\right| + \left|\psi_2\right|^2\right) dx\, ds \\
&\qquad + c \int_T^t e^{\lambda(s-t)} \int_{\mathbb{R}^n} \varphi\left(\frac{|x|^2}{k^2}\right) \\
&\qquad\quad \times \left(\left|\Delta z\left(\theta_{s-t-1}\omega\right)\right|^2 + \left|z\left(\theta_{s-t-1}\omega\right)\right|^2\right. \\
&\qquad\quad \left. + \left|z\left(\theta_{s-t-1}\omega\right)\right|^p\right) dx\, ds \\
&\quad \leq c e^{\lambda(T-t-1)} \int_{\mathbb{R}^n} \varphi\left(\frac{|x|^2}{k^2}\right) \\
&\qquad\quad \times \left|v\left(T, \theta_{-t-1}\omega, v_0\left(\theta_{-t-1}\omega\right)\right)\right|^2 dx
\end{aligned}
$$

$$+\frac{c}{k}\int_T^{t+1} e^{\lambda(s-t-1)}\left(\left\|v\left(s,\theta_{-t-1}\omega,v_0\left(\theta_{-t-1}\omega\right)\right)\right\|^2 + \left\|\nabla v\left(s,\theta_{-t-1}\omega,v_0\left(\theta_{-t-1}\omega\right)\right)\right\|^2\right)ds$$
$$+c\int_T^{t+1} e^{\lambda(s-t-1)}\int_{\mathbb{R}^n}\varphi\left(\frac{|x|^2}{k^2}\right)\times\left(\frac{2}{\lambda}|g|^2+2\left|\psi_1\right|+\left|\psi_2\right|^2\right)dx\,ds$$
$$+c\int_T^{t+1} e^{\lambda(s-t-1)}\int_{\mathbb{R}^n}\varphi\left(\frac{|x|^2}{k^2}\right)\times\left(\left|\Delta z\left(\theta_{s-t-1}\omega\right)\right|^2+\left|z\left(\theta_{s-t-1}\omega\right)\right|^2 + \left|z\left(\theta_{s-t-1}\omega\right)\right|^p\right)dx\,ds, \tag{84}$$

where $T = T_{\widehat{D}}(\omega)$ is an absorbing time in Lemma 14 and $t \geq T$. It is a direct result of [1] that there exist $T_1 = T^1_{\widehat{D}}(\omega,\varepsilon) \geq T$, $R_1 = R_1(\omega,\varepsilon)$ such that, for all $t \geq T_1$, $k \geq R_1$ the right-hand side of the aforementioned inequality is less than or equal to $\varepsilon/20c_3$, so we get

$$\frac{1}{2}\int_{\mathbb{R}^n}\varphi\left(\frac{|x|^2}{k^2}\right)\left|v\left(t,\theta_{-t-1}\omega,v_0\left(\theta_{-t-1}\omega\right)\right)\right|^2dx \leq \frac{\varepsilon}{40c_3}. \tag{85}$$

By Lemma 15 we find that there exists $T_2 = T^2_{\widehat{D}}(\omega) > 0$, for all $t \geq T_2$,

$$\frac{c}{k}\left(\int_t^{t+1}\left\|\nabla v\left(s,\theta_{-t-1}\omega,v_0\left(\theta_{-t-1}\omega\right)\right)\right\|^2ds + \int_t^{t+1}\left\|v\left(s,\theta_{-t-1}\omega,v_0\left(\theta_{-t-1}\omega\right)\right)\right\|^2ds\right) \leq \frac{c}{k}\left(1+r(\omega)\right). \tag{86}$$

Choosing $R_2 = R_2(\omega,\varepsilon) \geq 0$ such that $(c/k)(1+r(\omega)) \leq \varepsilon/40c_3$ for $k \geq R_2$, we deduce that

$$\frac{c}{k}\left(\int_t^{t+1}\left\|\nabla v\left(s,\theta_{-t-1}\omega,v_0\left(\theta_{-t-1}\omega\right)\right)\right\|^2ds + \int_t^{t+1}\left\|v\left(s,\theta_{-t-1}\omega,v_0\left(\theta_{-t-1}\omega\right)\right)\right\|^2ds\right) \leq \frac{\varepsilon}{40c_3}, \quad \forall t \geq T_2,\ \forall k \geq R_2. \tag{87}$$

For the third term of the right-hand side of (83), we have

$$\int_{\mathbb{R}^n}\varphi\left(\frac{|x|^2}{k^2}\right)\left(\frac{1}{\lambda}|g|^2+\left|\psi_1\right|+\frac{1}{2}\left|\psi_2\right|^2\right)dx \leq \int_{|x|\geq k}\left(\frac{1}{\lambda}|g|^2+\left|\psi_1\right|+\frac{1}{2}\left|\psi_2\right|^2\right)dx, \tag{88}$$

since $\psi_1 \in L^1(\mathbb{R}^n)$, $\psi_2 \in L^2(\mathbb{R}^n)$, and $g \in L^2(\mathbb{R}^n)$ there is a constant $R_3 = R_3(\varepsilon)$ such that, for all $k \geq R_3$,

$$\int_{\mathbb{R}^n}\varphi\left(\frac{|x|^2}{k^2}\right)\left(\frac{1}{\lambda}|g|^2+\left|\psi_1\right|+\frac{1}{2}\left|\psi_2\right|^2\right)dx \leq \frac{\varepsilon}{40c_3}. \tag{89}$$

Note that $z(\theta_t\omega) = \sum_{j=1}^m h_j z_j(\theta_t\omega_j)$ and $h_j \in H^2(\mathbb{R}^n)\bigcap W^{2,p}(\mathbb{R}^n)\bigcap L^\infty(\mathbb{R}^n)$ for $j = 1,2,\ldots,m$. Hence there exists $R_4 = R_4(\omega,\varepsilon)$ such that for all $k \geq R_4$ and $j = 1,2,\ldots,m$,

$$\int_{|x|\geq k}\left(\left|h_j(x)\right|^2+\left|h_j(x)\right|^p+\left|\Delta h_j(x)\right|^2\right)dx \leq \frac{\lambda e^{-(\lambda/2)}\varepsilon}{160cc_3m^p r(\omega)}, \tag{90}$$

where c is the constant in (83), and thus we have the following estimate:

$$\begin{aligned}
&c\int_t^{t+1}\int_{\mathbb{R}^n}\varphi\left(\frac{|x|^2}{k^2}\right)\times\left(\left|\Delta z\left(\theta_{s-t-1}\omega\right)\right|^2+\left|z\left(\theta_{s-t-1}\omega\right)\right|^2+\left|z\left(\theta_{s-t-1}\omega\right)\right|^p\right)dx\,ds\\
&\leq c\int_t^{t+1}\int_{|x|\geq k}\left(\left|\Delta z\left(\theta_{s-t-1}\omega\right)\right|^2+\left|z\left(\theta_{s-t-1}\omega\right)\right|^2+\left|z\left(\theta_{s-t-1}\omega\right)\right|^p\right)dx\,ds\\
&\leq cm^p\int_t^{t+1}\sum_{j=1}^m\int_{|x|\geq k}\left(\left|\Delta h_j\right|^2\left|z_j\left(\theta_{s-t-1}\omega_j\right)\right|^2+\left|h_j\right|^2\left|z_j\left(\theta_{s-t-1}\omega_j\right)\right|^2+\left|h_j\right|^p\left|z_j\left(\theta_{s-t-1}\omega_j\right)\right|^p\right)dx\,ds\\
&= cm^p\int_t^{t+1}\left[\sum_{j=1}^m\left|z_j\left(\theta_{s-t-1}\omega_j\right)\right|^2\int_{|x|\geq k}\left|\Delta h_j\right|^2dx+\sum_{j=1}^m\left|z_j\left(\theta_{s-t-1}\omega_j\right)\right|^2\int_{|x|\geq k}\left|h_j\right|^2dx+\sum_{j=1}^m\left|z_j\left(\theta_{s-t-1}\omega_j\right)\right|^p\int_{|x|\geq k}\left|h_j\right|^pdx\right]ds\\
&\leq \frac{\lambda e^{-(\lambda/2)}\varepsilon}{160cc_3m^p r(\omega)}cm^p\int_t^{t+1}\sum_{j=1}^m\left(2\left|z_j\left(\theta_{s-t-1}\omega_j\right)\right|^2+\left|z_j\left(\theta_{s-t-1}\omega_j\right)\right|^p\right)ds
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{\lambda e^{-(\lambda/2)}\varepsilon}{80c_3 r(\omega)} \int_t^{t+1} \sum_{j=1}^{m} \left(\left|z_j\left(\theta_{s-t-1}\omega_j\right)\right|^2 + \left|z_j\left(\theta_{s-t-1}\omega_j\right)\right|^p\right) ds \\
&\leq \frac{\lambda e^{-(\lambda/2)}\varepsilon}{80c_3 r(\omega)} \int_t^{t+1} p\left(\theta_{s-t-1}\omega\right) ds \\
&= \frac{\lambda e^{-(\lambda/2)}\varepsilon}{80c_3 r(\omega)} \int_{-1}^{0} p\left(\theta_s\omega\right) ds \\
&\leq \frac{\lambda e^{-(\lambda/2)}\varepsilon}{80c_3 r(\omega)} r(\omega) \int_{-1}^{0} e^{-(\lambda/2)s} ds \leq \frac{\varepsilon}{40c_3}.
\end{aligned} \tag{91}$$

Let

$$\begin{aligned}
T_1^* &= T_1^*\left(\widehat{D}, \omega, \varepsilon\right) = \max\left\{T_1, T_2\right\}, \\
R_1^* &= R_1^*(\omega, \varepsilon) = \max\left\{R_1, R_2, R_3, R_4\right\}.
\end{aligned} \tag{92}$$

From (83), (85), (87), (89), and (91), $\forall t \geq T_1^*$, $k \geq R_1^*$, we can estimate the first term of the right-hand side of (81) as

$$c_3 \int_t^{t+1} \int_{\mathbb{R}^n} \varphi\left(\frac{|x|^2}{k^2}\right) \times \left|\nabla v\left(s, \theta_{-t-1}\omega, v_0\left(\theta_{-t-1}\omega\right)\right)\right|^2 dx\, ds \leq \frac{\varepsilon}{10}. \tag{93}$$

Next, from (82) and a similar process as the proof of (93), one can also get the result for the second term of right-hand side of (81); that is, there exist $R_2^* = R_2^*(\omega, \varepsilon) > 0$ and $T_2^* = T_2^*(\widehat{D}, \omega, \varepsilon) > 0$ such that, for all $t \geq T_2^*$, $k \geq R_2^*$,

$$c_3 \int_t^{t+1} \int_{\mathbb{R}^n} \varphi\left(\frac{|x|^2}{k^2}\right) \times \left|u\left(s, \theta_{-t-1}\omega, v_0\left(\theta_{-t-1}\omega\right)\right)\right|^p dx\, ds \leq \frac{\varepsilon}{10}. \tag{94}$$

From Lemmas 15 and 16 we see that there exist $T_3^* = T_3^*(\widehat{D}, \omega) > 0$ and $R_3^* = R_3^*(\omega, \varepsilon) > 0$ such that $\forall t \geq T_3^*$, $\forall k \geq R_3^*$,

$$\begin{aligned}
&\frac{c_3}{k} \int_t^{t+1} \left\|\nabla v\left(s, \theta_{-t-1}\omega, v_0\theta_{-t-1}\omega\right)\right\|^2 ds \leq \frac{c}{k}\left(1 + r(\omega)\right) \leq \frac{\varepsilon}{10}, \\
&\frac{c_3}{k} \int_t^{t+1} \left\|v\left(s, \theta_{-t-1}\omega, v_0\theta_{-t-1}\omega\right)\right\|^2 ds \leq \frac{c}{k}\left(1 + r(\omega)\right) \leq \frac{\varepsilon}{10}, \\
&\frac{c_3}{k} \int_t^{t+1} \left\|v_s\left(s, \theta_{-t-1}\omega, v_0\theta_{-t-1}\omega\right)\right\|^2 ds \leq \frac{c}{k}\left(1 + r(\omega)\right) \leq \frac{\varepsilon}{10}, \\
&\frac{c_3}{k} \int_t^{t+1} \left\|v\left(s, \theta_{-t-1}\omega, v_0\theta_{-t-1}\omega\right)\right\|_{2p-2}^{2p-2} ds \leq \frac{c}{k}\left(1 + r(\omega)\right) \leq \frac{\varepsilon}{10}.
\end{aligned} \tag{95}$$

For the seventh and eighth terms of the right-hand side of (81), we have

$$\begin{aligned}
&\int_t^{t+1} \left(\left\|z\left(\theta_{s-t-1}\omega\right)\right\|_{2p-2}^{2p-2} + \left\|\nabla z\left(\theta_{s-t-1}\omega\right)\right\|^2\right) ds \\
&\leq c_0 \int_t^{t+1} p\left(\theta_{s-t-1}\omega\right) ds = c_0 \int_{-1}^{0} p\left(\theta_s\omega\right) ds \\
&\leq c_0 r(\omega) \int_{-1}^{0} e^{-(\lambda/2)s} ds \leq c r(\omega).
\end{aligned} \tag{96}$$

So there exists $R_4^* = R_4^*(\omega, \varepsilon) > 0$ such that, for all $k \geq R_4^*$ and all $t \geq 0$, we have

$$\frac{c_3}{k} \int_t^{t+1} \left(\left\|z\left(\theta_{s-t-1}\omega\right)\right\|_{2p-2}^{2p-2} + \left\|\nabla z\left(\theta_{s-t-1}\omega\right)\right\|^2\right) ds \leq \frac{c}{k} r(\omega) \leq \frac{\varepsilon}{10}. \tag{97}$$

Similar to the proof of (91), one can show that there exists $R_5^* = R_5^*(\omega, \varepsilon) > 0$ such that, for all $k \geq R_5^*$ and all $t \geq 0$, we have

$$c_3 \int_t^{t+1} \int_{\mathbb{R}^n} \varphi\left(\frac{|x|^2}{k^2}\right) \left(\left|\nabla z\left(\theta_{s-t-1}\omega\right)\right|^2 + \left|\Delta z\left(\theta_{s-t-1}\omega\right)\right|^2 + \left|\Delta z\left(\theta_{s-t-1}\omega\right)\right|^p\right) dx\, ds \leq \frac{\varepsilon}{10}. \tag{98}$$

Since $\psi_2 \in L^2(\mathbb{R}^n)$, $\psi_3 \in L^2(\mathbb{R}^n)$, and $g \in L^2(\mathbb{R}^n)$, we can easily show that there exists $R_6^* = R_6^*(\varepsilon) > 0$ such that the last term of the right-hand side of (81) is bounded by

$$\begin{aligned}
&c_3 \int_{\mathbb{R}^n} \varphi\left(\frac{|x|^2}{k^2}\right) \left(|g|^2 + \left|\psi_2\right|^2 + \left|\psi_3\right|^2\right) dx \\
&\leq c_3 \int_{|x| \geq k} \left(|g|^2 + \left|\psi_2\right|^2 + \left|\psi_3\right|^2\right) dx \leq \frac{\varepsilon}{10},
\end{aligned} \tag{99}$$

when $k \geq R_6^*$.

Finally, let

$$\begin{aligned}
R^* &= R^*(\omega, \varepsilon) = \max\left\{R_1^*, R_2^*, \ldots, R_6^*\right\}, \\
T^* &= T_{\widehat{D}}^*(\omega, \varepsilon) = \max\left\{T_1^*, T_2^*, T_3^*\right\}.
\end{aligned} \tag{100}$$

From (81), (93)–(95), (97)–(99) we get

$$\int_{\mathbb{R}^n} \varphi\left(\frac{|x|^2}{k^2}\right) \left|\nabla v\left(t+1, \theta_{-t-1}\omega, v_0\left(\theta_{-t-1}\omega\right)\right)\right|^2 dx \leq \varepsilon, \quad t \geq T^*, \; k \geq R^*. \tag{101}$$

Therefore, $\forall t \geq T^*$,

$$\int_{|x|\geq\sqrt{2}R^*} \left|\nabla v\left(t+1,\theta_{-t-1}\omega, v_0\left(\theta_{-t-1}\omega\right)\right)\right|^2 dx \leq \int_{\mathbb{R}^n} \varphi\left(\frac{|x|^2}{R^{*2}}\right) \left|\nabla v\left(t+1,\theta_{-t-1}\omega, v_0\left(\theta_{-t-1}\omega\right)\right)\right|^2 dx \leq \varepsilon. \tag{102}$$

The proof is complete. □

Lemma 18. *Assume that $g \in L^2(\mathbb{R}^n)$ and (3)–(6) hold. Let $\widehat{D} = \{D(\omega)\}_{\omega\in\Omega} \in \mathscr{D}_2$ and $u_0(\omega) \in D(\omega)$. Then, for $\mathbb{P}$-a.e. $\omega \in \Omega$ and for every $\varepsilon > 0$, there exist $T^* = T^*_{\widehat{D}}(\omega, \varepsilon) > 0$ and $R^* = R^*(\omega, \varepsilon)$, such that the solution $u(t, \omega, u_0(\omega))$ of problem* (1)-(2) *satisfies that, $\forall t \geq T^*_{\widehat{D}}(\omega, \varepsilon)$,*

$$\int_{|x|\geq R^*} \left(\left|u\left(t,\theta_{-t}\omega, u_0\left(\theta_{-t}\omega\right)\right)\right|^2 + \left|\nabla u\left(t,\theta_{-t}\omega, u_0\left(\theta_{-t}\omega\right)\right)\right|^2\right) dx \leq \varepsilon. \tag{103}$$

Proof. Considering $\forall\varepsilon > 0$, we choose $R_5 = R_5(\omega, \varepsilon)$ large enough such that

$$\int_{|x|\geq R_5} \left(\left|h_j\right|^2 + \left|\nabla h_j\right|^2\right) dx \leq \frac{\varepsilon}{2m^2 r(\omega)}, \quad j = 1, 2, \dots, m, \tag{104}$$

and set $R^{*\prime} = \max\{R^*, R_5\}$. Then by Lemma 17 and (34), one can easily show that

$$\int_{|x|\geq R^{*\prime}} \left(\left|u\left(t,\theta_{-t}\omega, u_0\left(\theta_{-t}\omega\right)\right)\right|^2 + \left|\nabla u\left(t,\theta_{-t}\omega, u_0\left(\theta_{-t}\omega\right)\right)\right|^2\right) dx \leq 3\varepsilon, \tag{105}$$

with $\forall t \geq T^*_{\widehat{D}}(\omega, \varepsilon)$, where R^*, $T^*_{\widehat{D}}(\omega, \varepsilon)$ are the constants in Lemma 17. The proof is complete. □

4.3. Asymptotic Compactness in Bounded Balls. In what follows, we prove the asymptotic compactness in any bounded ball, which together with Lemma 18 and Theorem 10 is a necessary condition for verifying the $(L^2(\mathbb{R}^n), H^1(\mathbb{R}^n))$-asymptotic compactness. For this purpose, we set $\psi = 1 - \varphi$, where φ is the function described in Lemma 17.

For fixed $k \geq 1$, define

$$\widetilde{v}\left(t,\omega, v_0(\omega)\right) = \psi\left(\frac{|x|^2}{k^2}\right) v\left(t,\omega, v_0(\omega)\right). \tag{106}$$

Then $\widetilde{v}(t, \omega, v_0(\omega)) \in H^1_0(Q_{2k})$ and

$$\left\|\widetilde{v}\left(t,\omega, v_0(\omega)\right)\right\|_{H^1_0(Q_{2k})} \leq c\left\|v\left(t,\omega, v_0(\omega)\right)\right\|_{H^1(\mathbb{R}^n)}, \tag{107}$$

where c is a positive constant, independent of t, ω, and k. Then we have

$$\widetilde{v}_t(t) = \psi\left(\frac{|x|^2}{k^2}\right) v_t(t), \qquad \Delta\widetilde{v} = (\Delta\psi) v + 2\nabla\psi\cdot\nabla v + \psi\Delta v. \tag{108}$$

Multiplying (30) by $\psi(|x|^2/k^2)$, then we can easily show that

$$\widetilde{v}_t + \lambda\widetilde{v} - \Delta\widetilde{v} = \psi f(x, u) + \psi g + \psi\Delta z\left(\theta_t\omega\right) - \nu\Delta\psi - 2\nabla\psi\cdot\nabla v. \tag{109}$$

Consider the following eigenvalue problem:

$$-\Delta\widetilde{v} = \lambda\widetilde{v}, \quad \text{in } Q_{2\kappa}, \qquad \widetilde{v}|_{\partial Q_{2\kappa}} = 0, \tag{110}$$

and then problem (110) has a family of eigenfunctions $\{e_j\}_{j=1}^{\infty}$ with corresponding eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$ such that $\{e_j\}_{j=1}^{\infty}$ forms an orthogonal basis in both $L^2(Q_{2k})$ and $H^1_0(Q_{2k})$ and

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \longrightarrow \infty \quad \text{as } j \longrightarrow \infty. \tag{111}$$

Given n, let $X_n = \operatorname{span}\{e_1, \dots, e_n\}$ and $P_n : L^2(Q_{2k}) \to X_n$ be the projection operator. For any $\widetilde{v} \in H^1_0(Q_{2k})$, we write

$$\widetilde{v} = P_m\widetilde{v} + \left(I - P_m\right)\widetilde{v} =: \widetilde{v}_1 + \widetilde{v}_2. \tag{112}$$

In order to prove the asymptotic compactness we need the following lemma, which can be found in [19].

Lemma 19 (see [19]). *Let $r > 0$, $\lambda > 0$, $\tau \in \mathbb{R}$, and for $s > \tau$*

$$y'(s) + \lambda y(s) \leq h(s), \tag{113}$$

where the functions y, y', h are assumed to be locally integrable and y, h nonnegative on the interval $t < s < t + r$, for some $t \geq \tau$. Then, for any $\delta \in (0, r)$,

$$y(t+r) \leq e^{-\lambda(r-\delta)}\frac{1}{\delta}\int_t^{t+r} y(s)\,ds + e^{-\lambda(t+r)}\int_t^{t+r} e^{\lambda s}h(s)\,ds. \tag{114}$$

In particular, let $\delta = (r/2)$, then

$$y(t+r) \leq e^{-\lambda(r/2)}\frac{2}{r}\int_t^{t+r} y(s)\,ds + e^{-\lambda(t+r)}\int_t^{t+r} e^{\lambda s}h(s)\,ds. \tag{115}$$

Lemma 20. *Assume that $g \in L^2(\mathbb{R}^n)$ and (3)–(6) hold. Let $\widehat{D} = \{D(\omega)\}_{\omega\in\Omega} \in \mathscr{D}_2$ and $u_0(\omega) \in D(\omega)$. Then, for $\mathbb{P}$-a.e. $\omega \in \Omega$ and for every $\varepsilon > 0$, there exist $M(\omega, \varepsilon) > 0$, $\eta(\omega, \varepsilon) > 0$, and $T_{\widehat{D}}(\omega) > 0$ such that the solution $v(t, \omega, v_0(\omega))$ of* (30)-(31) *satisfies that, $\forall t \geq T_{\widehat{D}}(\omega)$,*

$$\begin{aligned}&\int_{t-\eta}^{t}\int_{Q_{2k}(v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\geq M)} \left|v\left(s,\theta_{-t}\omega, v_0\left(\theta_{-t}\omega\right)\right)\right|^{p-1} \times \left|v\left(s,\theta_{-t}\omega, v_0\left(\theta_{-t}\omega\right)\right) - M\right|^{p-1} dx\,ds \leq \varepsilon,\\ &\int_{t-\eta}^{t}\int_{Q_{2k}(v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\leq -M)} \left|v\left(s,\theta_{-t}\omega, v_0\left(\theta_{-t}\omega\right)\right)\right|^{p-1} \times \left|v\left(s,\theta_{-t}\omega, v_0\left(\theta_{-t}\omega\right)\right) + M\right|^{p-1} dx\,ds \leq \varepsilon.\end{aligned} \tag{116}$$

Proof. If $p = 2$, one can easily show the results by (36). So in the following we assume $p > 2$. Let

$$(v - M)_+ = \begin{cases} v - M, & x \in Q_{2k}^1, \\ 0, & \text{elsewhere}, \end{cases}$$
$$(v + M)_- = \begin{cases} v + M, & x \in Q_{2k}^2, \\ 0, & \text{elsewhere}, \end{cases} \quad (117)$$

where $Q_{2k}^1 = Q_{2k}(v \geq M) = Q_{2k} \bigcap \{x : v(t, \omega, v_0(\omega)) \geq M\}$ and $Q_{2k}^2 = Q_{2k}(v \leq -M) = Q_{2k} \bigcap \{x : v(t, \omega, v_0(\omega)) \leq -M\}$.

Multiplying (30) with $|(v - M)_+|^{p-1}$ and integrating over $\mathbb{R}^n$, we get

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\|(v - M)_+\|_p^p + \lambda \int_{Q_{2k}^1} v(v - M)^{p-1} dx \\ &\quad + (p - 1) \int_{Q_{2k}^1} |\nabla (v - M)|^2 (v - M)^{p-2} dx \\ &= \int_{Q_{2k}^1} f(x, u)(v - M)^{p-1} dx \\ &\quad + \int_{Q_{2k}^1} g(v - M)^{p-1} dx \\ &\quad + \int_{Q_{2k}^1} \Delta z(\theta_t\omega)(v - M)^{p-1} dx. \end{aligned} \quad (118)$$

From (41), when $x \in Q_{2k}^1$, we have

$$\begin{aligned} &f(x, v + z(\theta_t\omega))(v - M)^{p-1} \\ &= f(x, v + z(\theta_t\omega)) v \frac{(v - M)^{p-1}}{v} \\ &\leq -\frac{\alpha_1}{2^p}|v|^p \frac{(v - M)^{p-1}}{v} \\ &\quad + c\left(|z(\theta_t\omega)|^p + |z(\theta_t\omega)|^2\right)\frac{(v - M)^{p-1}}{v} \\ &\quad + \left(\psi_1 + \frac{1}{2}\psi_2^2\right)\frac{(v - M)^{p-1}}{v} \\ &\leq -\frac{\alpha_1}{2^p}|v|^{p-1}(v - M)^{p-1} \\ &\quad + \frac{c}{M}\left(|z(\theta_t\omega)|^p + |z(\theta_t\omega)|^2\right)(v - M)^{p-1} \\ &\quad + \frac{1}{M}\left(\psi_1 + \frac{1}{2}\psi_2^2\right)(v - M)^{p-1} \\ &\leq -\frac{\alpha_1}{2^p}|v|^{p-1}(v - M)^{p-1} \\ &\quad + c\left(|z(\theta_t\omega)|^p + |z(\theta_t\omega)|^2\right)(v - M)^{p-1} \\ &\quad + c\left(\psi_1 + \frac{1}{2}\psi_2^2\right)(v - M)^{p-1}, \end{aligned} \quad (119)$$

where the constant c in the right hand side of (119) is independent of M when we assume, without loss of generality, that $M \geq 1$.

From (118)-(119), we get

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\|(v - M)_+\|_p^p + c_4 \int_{Q_{2k}^1} |v|^{p-1}(v - M)^{p-1} dx \\ &\leq c_5 \int_{Q_{2k}^1} \left(|\psi_1| + \frac{1}{2}|\psi_2|^2 + |g|\right)(v - M)^{p-1} dx \\ &\quad + c_5 \int_{Q_{2k}^1} \left(|z(\theta_t\omega)|^p + |z(\theta_t\omega)|^2\right. \\ &\qquad \left. + |\Delta z(\theta_t\omega)|\right)(v - M)^{p-1} dx. \end{aligned} \quad (120)$$

Applying Cauchy's inequality, we obtain

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\|(v - M)_+\|_p^p + \frac{c_4}{2} \int_{Q_{2k}^1} |v|^{p-1}(v - M)^{p-1} dx \\ &\quad + \frac{c_4}{2} \int_{Q_{2k}^1} (v - M)^{2p-2} dx \\ &\leq \frac{c_4}{2} \int_{Q_{2k}^1} (v - M)^{2p-2} dx \\ &\quad + c \int_{Q_{2k}^1} \left(|\psi_1|^2 + |\psi_2|^4 + g^2\right) dx \\ &\quad + c \int_{Q_{2k}^1} \left(|z(\theta_t\omega)|^{2p} + |z(\theta_t\omega)|^4 + |\Delta z(\theta_t\omega)|^2\right) dx, \end{aligned} \quad (121)$$

that is,

$$\begin{aligned} &\frac{2}{p}\frac{d}{dt}\|(v - M)_+\|_p^p + c_4 \int_{Q_{2k}^1} |v|^{p-1}(v - M)^{p-1} dx \\ &\leq c \int_{Q_{2k}^1} \left(|\psi_1|^2 + |\psi_2|^4 + g^2\right) dx \\ &\quad + c \int_{Q_{2k}^1} \left(|z(\theta_t\omega)|^{2p} + |z(\theta_t\omega)|^4 + |\Delta z(\theta_t\omega)|^2\right) dx, \end{aligned} \quad (122)$$

$$\begin{aligned} &\frac{2}{p}\frac{d}{dt}\|(v - M)_+\|_p^p + c_4 M^{p-2} \int_{Q_{2k}^1} (v - M)^p dx \\ &\leq c \int_{Q_{2k}^1} \left(|\psi_1|^2 + |\psi_2|^4 + g^2\right) dx \\ &\quad + c \int_{Q_{2k}^1} \left(|z(\theta_t\omega)|^{2p} + |z(\theta_t\omega)|^4 + |\Delta z(\theta_t\omega)|^2\right) dx. \end{aligned} \quad (123)$$

By (123) and Lemma 19 with $r = 1$, we get

$$\begin{aligned}\|(v-M)_+(t)\|_p^p &\le 2e^{-\mu/2}\int_{t-1}^{t}\|(v-M)_+(\tau)\|_p^p d\tau \\ &+ ce^{-\mu t}\int_{t-1}^{t} e^{\mu\tau}\left(\|z(\theta_\tau\omega)\|_{2p}^{2p} + \|z(\theta_\tau\omega)\|_4^4 \right.\\ &\left.+\|\Delta z(\theta_\tau\omega)\|^2\right)d\tau \\ &+ ce^{-\mu t}\int_{t-1}^{t} e^{\mu\tau}\left(\|\psi_1\|^2 + \|\psi_2\|_4^4 + \|g\|^2\right)d\tau,\end{aligned} \tag{124}$$

where $\mu = c_4 M^{p-2}p/2$.

Set $s \in [t-1, t]$; we first substitute s for t in the aforementioned inequality and then we replace ω by $\theta_{-t}\omega$ to get

$$\begin{aligned}&\|(v(s,\theta_{-t}\omega, v_0(\theta_{-t}\omega)) - M)_+\|_p^p \\ &\le 2e^{-\mu/2}\int_{s-1}^{s}\|(v(\tau,\theta_{-t}\omega, v_0(\theta_{-t}\omega)) - M)_+\|_p^p d\tau \\ &+ ce^{-\mu s}\int_{s-1}^{s} e^{\mu\tau}\left(\|z(\theta_{\tau-t}\omega)\|_{2p}^{2p} + \|z(\theta_{\tau-t}\omega)\|_4^4\right.\\ &\left.+\|\Delta z(\theta_{\tau-t}\omega)\|^2\right)d\tau \\ &+ ce^{-\mu s}\int_{s-1}^{s} e^{\mu\tau}d\tau\left(\|\psi_1\|^2 + \|\psi_2\|_4^4 + \|g\|^2\right) \\ &\le 2e^{-\mu/2}\int_{t-2}^{t}\|(v(\tau,\theta_{-t}\omega, v_0(\theta_{-t}\omega)) - M)_+\|_p^p d\tau \\ &+ ce^{-\mu s}\int_{s-1}^{s} e^{\mu\tau}\left(\|z(\theta_{\tau-t}\omega)\|_{2p}^{2p} + \|z(\theta_{\tau-t}\omega)\|_4^4\right.\\ &\left.+\|\Delta z(\theta_{\tau-t}\omega)\|^2\right)d\tau \\ &+ ce^{-\mu s}\int_{s-1}^{s} e^{\mu\tau}d\tau\left(\|\psi_1\|^2 + \|\psi_2\|_4^4 + \|g\|^2\right).\end{aligned} \tag{125}$$

For the first term of the right-hand side of (125), we use Lemma 15:

$$\begin{aligned}&\int_{t-2}^{t}\|(v(\tau,\theta_{-t}\omega, v_0(\theta_{-t}\omega)) - M)_+\|_p^p d\tau \\ &\le \int_{t-2}^{t}\|v(\tau,\theta_{-t}\omega, v_0(\theta_{-t}\omega))\|_p^p d\tau \\ &\le c(1+r(\omega)), \quad \forall t \ge T_{\widehat{D}}(\omega).\end{aligned} \tag{126}$$

Then there exists $M_1 = M_1(\omega,\varepsilon) > 0$ such that, for all $t \ge T_{\widehat{D}}(\omega)$ and all $M \ge M_1$,

$$\begin{aligned}&2e^{-\mu/2}\int_{t-2}^{t}\|(v(\tau,\theta_{-t}\omega, v_0(\theta_{-t}\omega)) - M)_+\|_p^p d\tau \\ &\le ce^{-\mu/2}(1+r(\omega)) \le \frac{\varepsilon}{3}.\end{aligned} \tag{127}$$

For the second term of the right-hand side of (125), $\forall s \in [t-1, t]$, by (26) we have

$$\begin{aligned}&ce^{-\mu s}\int_{s-1}^{s} e^{\mu\tau}\left(\|z(\theta_{\tau-t}\omega)\|_{2p}^{2p} + \|z(\theta_{\tau-t}\omega)\|_4^4\right.\\ &\left.+\|\Delta z(\theta_{\tau-t}\omega)\|^2\right)d\tau \\ &\le c\int_{s-1}^{s} e^{\mu(\tau-s)}p(\theta_{\tau-t}\omega)d\tau \\ &= c\int_{-1}^{0} e^{\mu\tau}p(\theta_{\tau-t+s}\omega)d\tau \\ &\le cr(\omega)\int_{-1}^{0} e^{\mu\tau}e^{(\lambda/2)|\tau-t+s|}d\tau \\ &\le cr(\omega)\int_{-1}^{0} e^{\mu\tau}e^{\lambda}d\tau \\ &\le cr(\omega)\int_{-1}^{0} e^{\mu\tau}d\tau \le \frac{cr(\omega)}{\mu}.\end{aligned} \tag{128}$$

This implies that there exists $M_2 = M_2(\omega,\varepsilon) > 0$ such that $\forall M \ge M_2$ we have

$$\begin{aligned}&ce^{-\mu s}\int_{s-1}^{s} e^{\mu\tau}\left(\|z(\theta_{\tau-t}\omega)\|_{2p}^{2p} + \|z(\theta_{\tau-t}\omega)\|_4^4\right.\\ &\left.+\|\Delta z(\theta_{\tau-t}\omega)\|^2\right)d\tau \le \frac{\varepsilon}{3}.\end{aligned} \tag{129}$$

For the last term of the right-hand side of (125), we can easily see that there exists $M_3 = M_3(\varepsilon) > 0$ such that, for all $M \ge M_3$,

$$\begin{aligned}&ce^{-\mu s}\int_{s-1}^{s} e^{\mu\tau}d\tau\left(\|\psi_1\|^2 + \|\psi_2\|_4^4 + \|g\|^2\right) \\ &= c\int_{-1}^{0} e^{\mu\tau}d\tau\left(\|\psi_1\|^2 + \|\psi_2\|_4^4 + \|g\|^2\right) \le \frac{c}{\mu} \le \frac{\varepsilon}{3}.\end{aligned} \tag{130}$$

Letting $N^* = N^*(\omega,\varepsilon) = \max\{M_1, M_2, M_3\}$, then combining with (125), (127) and (129)–(130), we can show that, for all $t \ge T_{\widehat{D}}(\omega)$ and all $M \ge N^*(\omega,\varepsilon)$,

$$\|(v(s,\theta_{-t}\omega, v_0(\theta_{-t}\omega)) - M)_+\|_p^p \le \varepsilon, \quad \forall s \in [t-1, t]. \tag{131}$$

Next, we set

$$\eta = \min\left\{\frac{2}{\lambda}\ln\left(\frac{\lambda\varepsilon}{6c_8 r(\omega)} + 1\right), \frac{\varepsilon}{3c_7\left(\|\psi_1\|^2 + \|\psi_2\|_4^4 + \|g\|^2\right)}, 1\right\}, \tag{132}$$

where $c_7 = c/c_4$, $c_8 = c_0c_7$, and c is the constants in (122). Integrating (122) over $(t-\eta, t)$, we get

$$\begin{aligned}\int_{t-\eta}^{t}\int_{Q_{2k}^1}|v(s)|^{p-1}(v(s)-M)^{p-1}dx\,ds &\le c_6\|(v(t-\eta)-M)_+\|_p^p \\ &+ c_7\int_{t-\eta}^{t}\int_{Q_{2k}^1}\left(|z(\theta_s\omega)|^{2p}+|z(\theta_s\omega)|^4 +|\Delta z(\theta_s\omega)|^2\right)dx\,ds \\ &+ c_7\int_{t-\eta}^{t}\int_{Q_{2k}^1}\left(|\psi_1|^2+|\psi_2|^4+g^2\right)dx\,ds.\end{aligned} \tag{133}$$

Replacing ω by $\theta_{-t}\omega$ and setting $Q_{2k}^* = Q_{2k}(v(s,\theta_{-t}\omega, v_0(\theta_{-t}\omega)) \ge M)$, we obtain

$$\begin{aligned}\int_{t-\eta}^{t}\int_{Q_{2k}^*}&|v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))|^{p-1} \times (v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))-M)^{p-1}dx\,ds \\ &\le c_7\int_{t-\eta}^{t}\int_{Q_{2k}^*}\left(|z(\theta_{s-t}\omega)|^{2p}+|z(\theta_{s-t}\omega)|^4 +|\Delta z(\theta_{s-t}\omega)|^2\right)dx\,ds \\ &+ c_6\|(v(t-\eta,\theta_{-t}\omega,v_0(\theta_{-t}\omega))-M)_+\|_p^p \\ &+ c_7\eta\left(\|\psi_1\|^2+\|\psi_2\|_4^4+\|g\|^2\right).\end{aligned} \tag{134}$$

For fixed $\omega \in \Omega$, by (131) and (132), $\forall t \ge T_{\widehat{D}}(\omega)$, $\forall M \ge M_1^*(\omega,\varepsilon) = N^*(\omega, \varepsilon/3c_6)$,

$$c_6\|(v(t-\eta,\theta_{-t}\omega,v_0(\theta_{-t}\omega))-M)_+\|_p^p \le \frac{\varepsilon}{3}, \tag{135}$$

where $N^*(\omega,\varepsilon)$ is the constant in (131).

Using (132), the first term of the right-hand side of (134) is bounded by

$$\begin{aligned}c_7\int_{t-\eta}^{t}&\left(\|z(\theta_{\tau-t}\omega)\|_{2p}^{2p}+\|z(\theta_{\tau-t}\omega)\|_4^4 +\|\Delta z(\theta_{\tau-t}\omega)\|^2\right)d\tau \\ &\le c_8\int_{t-\eta}^{t}p(\theta_{\tau-t}\omega)\,d\tau = c_8\int_{-\eta}^{0}p(\theta_\tau\omega)\,d\tau \\ &\le c_8 r(\omega)\int_{-\eta}^{0}e^{-(\lambda/2)\tau}d\tau = \frac{2c_8}{\lambda}r(\omega)\left(e^{\lambda\eta/2}-1\right) \le \frac{\varepsilon}{3}.\end{aligned} \tag{136}$$

For the last term of right-hand side of (134), by using (132), we have

$$c_7\eta\left(\|\psi_1\|^2+\|\psi_2\|_4^4+\|g\|^2\right) \le \frac{\varepsilon}{3}. \tag{137}$$

By (134)–(137), $\forall M \ge M_1^*$, $\forall t \ge T_{\widehat{D}}(\omega)$, and $\forall u_0(\omega) \in D(\omega)$ with $v_0(\omega) = u_0(\omega) - z(\omega)$, we have

$$\begin{aligned}\int_{t-\eta}^{t}\int_{Q_{2k}^*}&|v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))|^{p-1} \\ &\times|v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))-M|^{p-1}dx\,ds \le \varepsilon.\end{aligned} \tag{138}$$

Similarly, multiplying (30) with $|(v+M)_-|^{p-1}$, we can prove that there exists $M_2^* = M_2^*(\omega,\varepsilon)$ such that $\forall M \ge M_2^*, \forall t \ge T_{\widehat{D}}(\omega)$, and $\forall u_0(\omega) \in D(\omega)$ with $v_0(\omega) = u_0(\omega) - z(\omega)$, over the region $Q_{2k}^{**} = Q_{2k}(v(s,\theta_{-t}\omega, v_0(\theta_{-t}\omega)) \le -M)$; we have

$$\begin{aligned}\int_{t-\eta}^{t}\int_{Q_{2k}^{**}}&|v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))|^{p-1} \\ &\times|v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))+M|^{p-1}dx\,ds \le \varepsilon.\end{aligned} \tag{139}$$

Let $M^* = \max\{M_1^*, M_2^*\}$; from (138) and (139) we can obtain our results. The proof is complete. □

Remark 21. The idea of the proof of the above lemma comes from [19] (this idea can be further traced back to Marion [16] and Robinson [18]). We see from (132) that the constant η in Lemma 20 is independent of t, which is different from the η in the Lemma 3.4 in [19]. This is crucial in the following estimates. As we know, the time t will vary to infinite when we consider the asymptotic behavior of an RDS. It means that if η is not a fixed constant with respect to t, the following estimate will be invalid. In other words, if the function g in (1) is dependent on t, our method will fail.

Lemma 22. *Assume that $g \in L^2(\mathbb{R}^n)$ and (3)–(6) hold. Let $\widehat{D} = \{D(\omega)\}_{\omega\in\Omega} \in \mathscr{D}_2$ and $u_0(\omega) \in D(\omega)$. Then, for $\mathbb{P}$-a.e. $\omega \in \Omega$ and for every $\varepsilon > 0$ and all $k \ge 1$, there exist $T_{\widehat{D}}(\omega) > 0$ and $N(k,\omega,\varepsilon) > 0$ such that the solution $v(t,\omega,v_0(\omega))$ of* (30)-(31) *satisfies that, $\forall t \ge T_{\widehat{D}}(\omega)$, $\forall m \ge N(k,\omega,\varepsilon)$,*

$$\|\widetilde{v}_2(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|_{H_0^1(Q_{2k})}^2 \le \varepsilon, \tag{140}$$

where $\widetilde{v}_2 = (I-P_m)\widetilde{v}$.

Proof. Multiplying (109) with $-\Delta\widetilde{v}_2$ and integrating over Q_{2k}, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\widetilde{v}_2\|_{H_0^1(Q_{2k})}^2 &+ \lambda\|\widetilde{v}_2\|_{H_0^1(Q_{2k})}^2 + \|\Delta\widetilde{v}_2\|_{L^2(Q_{2k})}^2 \\ &= -(\psi f(x,u),\Delta\widetilde{v}_2)_{L^2(Q_{2k})} - (\psi\Delta z(\theta_t\omega),\Delta\widetilde{v}_2)_{L^2(Q_{2k})} \\ &- (\psi g,\Delta\widetilde{v}_2)_{L^2(Q_{2k})} + (v\Delta\psi,\Delta\widetilde{v}_2)_{L^2(Q_{2k})} \\ &+ 2(\nabla\psi\cdot\nabla v,\Delta\widetilde{v}_2)_{L^2(Q_{2k})}.\end{aligned} \tag{141}$$

We now estimate each term in the right-hand side of the aforementioned equality. For the first term, by using Cauchy inequality and (4), we have

$$\begin{aligned}
&\left|\left(\psi f(x,u), \Delta\widetilde{v}_2\right)_{L^2(Q_{2k})}\right| \\
&\quad\leq \frac{1}{10}\|\Delta\widetilde{v}_2\|^2_{L^2(Q_{2k})} + c\int_{Q_{2k}}|f(x,u)|^2 dx \\
&\quad\leq \frac{1}{10}\|\Delta\widetilde{v}_2\|^2_{L^2(Q_{2k})} + c\int_{Q_{2k}}|u|^{2p-2}dx + c\int_{Q_{2k}}|\psi_2|^2 dx \\
&\quad\leq \frac{1}{10}\|\Delta\widetilde{v}_2\|^2_{L^2(Q_{2k})} + c\int_{Q_{2k}}|v|^{2p-2}dx \\
&\qquad + c\int_{Q_{2k}}|z(\theta_t\omega)|^{2p-2}dx + c \\
&\quad\leq \frac{1}{10}\|\Delta\widetilde{v}_2\|^2_{L^2(Q_{2k})} + c\int_{Q_{2k}(|v|\leq M)}|v|^{2p-2}dx \\
&\qquad + c\int_{Q_{2k}(|v|\geq M)}|v|^{2p-2}dx + c\int_{Q_{2k}}|z(\theta_t\omega)|^{2p-2}dx + c \\
&\quad\leq \frac{1}{10}\|\Delta\widetilde{v}_2\|^2_{L^2(Q_{2k})} + c\int_{Q_{2k}(|v|\geq M)}|v|^{2p-2}dx \\
&\qquad + c\|z(\theta_t\omega)\|^{2p-2}_{2p-2} + c\left(\|\psi_2\|^2, |Q_{2k}|, p, M\right),
\end{aligned} \tag{142}$$

where $|Q_{2k}|$ denotes the Lebesgue measure of Q_{2k}.

For the second to fifth term, we can estimate them as follows:

$$\begin{aligned}
&\left|\left(\psi\Delta z(\theta_t\omega), \Delta\widetilde{v}_2\right)_{L^2(Q_{2k})}\right| \\
&\quad\leq \frac{1}{10}\|\Delta\widetilde{v}_2\|^2_{L^2(Q_{2k})} + c\|\Delta z(\theta_t\omega)\|^2,
\end{aligned} \tag{143}$$

$$\begin{aligned}
&\left|\left(\psi g, \Delta\widetilde{v}_2\right)_{L^2(Q_{2k})}\right| \\
&\quad\leq \frac{1}{10}\|\Delta\widetilde{v}_2\|^2_{L^2(Q_{2k})} + c\|g\|^2,
\end{aligned} \tag{144}$$

$$\begin{aligned}
&\left|\left(v\Delta\psi, \Delta\widetilde{v}_2\right)_{L^2(Q_{2k})}\right| \\
&\quad\leq \frac{1}{10}\|\Delta\widetilde{v}_2\|^2_{L^2(Q_{2k})} + c\int_{Q_{2k}}|v|^2|\Delta\psi|^2 dx \\
&\quad\leq \frac{1}{10}\|\Delta\widetilde{v}_2\|^2_{L^2(Q_{2k})} + c\int_{\mathbb{R}^n}|v|^2|\Delta\psi|^2 dx \\
&\quad\leq \frac{1}{10}\|\Delta\widetilde{v}_2\|^2_{L^2(Q_{2k})} \\
&\qquad + c\int_{\mathbb{R}^n}|v|^2\left|\varphi''\left(\frac{|x|^2}{k^2}\right)\frac{4|x|^2}{k^4} + \varphi'\left(\frac{|x|^2}{k^2}\right)\frac{2n}{k^2}\right|^2 dx \\
&\quad\leq \frac{1}{10}\|\Delta\widetilde{v}_2\|^2_{L^2(Q_{2k})} + \frac{c}{k^4}\int_{k\leq|x|\leq\sqrt{2}k}|v|^2 dx \\
&\quad\leq \frac{1}{10}\|\Delta\widetilde{v}_2\|^2_{L^2(Q_{2k})} + c\|v\|^2,
\end{aligned} \tag{145}$$

$$\begin{aligned}
&2\left|\left(\nabla\psi\cdot\nabla v, \Delta\widetilde{v}_2\right)_{L^2(Q_{2k})}\right| \\
&\quad\leq \frac{1}{10}\|\Delta\widetilde{v}_2\|^2_{L^2(Q_{2k})} + c\int_{Q_{2k}}|\nabla\psi|^2|\nabla v|^2 dx \\
&\quad\leq \frac{1}{10}\|\Delta\widetilde{v}_2\|^2_{L^2(Q_{2k})} + \frac{c}{k^2}\|\nabla v\|^2 \\
&\quad\leq \frac{1}{10}\|\Delta\widetilde{v}_2\|^2_{L^2(Q_{2k})} + c\|\nabla v\|^2,
\end{aligned} \tag{146}$$

where the constant c in (145) and (146) is independent of k, since $k \geq 1$.

Combining (141)–(146) we get

$$\begin{aligned}
&\frac{d}{dt}\|\widetilde{v}_2\|^2_{H^1_0(Q_{2k})} + 2\lambda\|\widetilde{v}_2\|^2_{H^1_0(Q_{2k})} + \|\Delta\widetilde{v}_2\|^2_{L^2(Q_{2k})} \\
&\quad\leq c\int_{Q_{2k}(|v|\geq M)}|v|^{2p-2}dx + c\left(\|v\|^2 + \|\nabla v\|^2\right) \\
&\qquad + c\left(\|z(\theta_t\omega)\|^{2p-2}_{2p-2} + \|\Delta z(\theta_t\omega)\|^2\right) \\
&\qquad + c\left(\|\psi_2\|^2, \|g\|^2, |Q_{2k}|, p, M\right),
\end{aligned} \tag{147}$$

and this implies that

$$\begin{aligned}
&\frac{d}{dt}\|\widetilde{v}_2\|^2_{H^1_0(Q_{2k})} + \lambda_{m+1}\|\widetilde{v}_2\|^2_{H^1_0(Q_{2k})} \\
&\quad\leq c\int_{Q_{2k}(|v|\geq M)}|v|^{2p-2}dx + c\left(\|v\|^2 + \|\nabla v\|^2\right) \\
&\qquad + c\left(\|z(\theta_t\omega)\|^{2p-2}_{2p-2} + \|\Delta z(\theta_t\omega)\|^2\right) \\
&\qquad + c\left(\|\psi_2\|^2, \|g\|^2, |Q_{2k}|, p, M\right).
\end{aligned} \tag{148}$$

By using Lemma 19 with $r = 1$, we get

$$\begin{aligned}
&\|\widetilde{v}_2(t+1)\|^2_{H^1_0(Q_{2k})} \\
&\quad\leq 2e^{-\lambda_{m+1}/2}\int_t^{t+1}\|\widetilde{v}_2(s)\|^2_{H^1_0(Q_{2k})}ds \\
&\qquad + ce^{-\lambda_{m+1}(t+1)}\int_t^{t+1}e^{\lambda_{m+1}s}\int_{Q_{2k}(|v(s)|\geq M)}|v(s)|^{2p-2}dx\,ds \\
&\qquad + ce^{-\lambda_{m+1}(t+1)}\int_t^{t+1}e^{\lambda_{m+1}s} \\
&\qquad\qquad \times\left(\|z(\theta_s\omega)\|^{2p-2}_{2p-2} + \|\Delta z(\theta_s\omega)\|^2\right)ds \\
&\qquad + ce^{-\lambda_{m+1}(t+1)}\int_t^{t+1}e^{\lambda_{m+1}s}\left(\|v(s)\|^2 + \|\nabla v(s)\|^2\right)ds \\
&\qquad + c\left(\|\psi_2\|^2, \|g\|^2, |Q_{2k}|, p, M\right)e^{-\lambda_{m+1}(t+1)}\int_t^{t+1}e^{\lambda_{m+1}s}ds.
\end{aligned} \tag{149}$$

Substituting $\theta_{-t-1}\omega$ for ω, we get

$$\begin{aligned}
&\|\widetilde{v}_2(t+1,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|^2_{H_0^1(Q_{2k})} \\
&\quad\le 2e^{-\lambda_{m+1}/2}\int_t^{t+1}\|\widetilde{v}_2(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|^2_{H_0^1(Q_{2k})}ds \\
&\qquad+ce^{-\lambda_{m+1}(t+1)}\int_t^{t+1}e^{\lambda_{m+1}s} \\
&\qquad\qquad\times\int_{Q_{2k}(|v(*)|\ge M)}|v(*)|^{2p-2}dx\,ds \\
&\qquad+ce^{-\lambda_{m+1}(t+1)}\int_t^{t+1}e^{\lambda_{m+1}s}\left(\|z(\theta_{s-t-1}\omega)\|^{2p-2}_{2p-2}\right. \\
&\qquad\qquad\left.+\|\Delta z(\theta_{s-t-1}\omega)\|^2\right)ds \\
&\qquad+ce^{-\lambda_{m+1}(t+1)}\int_t^{t+1}e^{\lambda_{m+1}s}\left(\|v(*)\|^2+\|\nabla v(*)\|^2\right)ds \\
&\qquad+c\left(\|\psi_2\|^2,\|g\|^2,|Q_{2k}|,p,M\right)e^{-\lambda_{m+1}(t+1)}\int_t^{t+1}e^{\lambda_{m+1}s}ds,
\end{aligned}\tag{150}$$

for simplicity, hereafter, we write $v(*)=v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))$.

In the sequel, we estimate each term in the right-hand side of (150). For the first term, we use Lemma 15 and (107) to obtain that, for $t\ge T_{\widehat{D}}(\omega)$,

$$\begin{aligned}
&2e^{-\lambda_{m+1}/2}\int_t^{t+1}\|\widetilde{v}_2(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|^2_{H_0^1(Q_{2k})}ds \\
&\quad\le 2e^{-\lambda_{m+1}/2}\int_t^{t+1}\|\widetilde{v}(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|^2_{H_0^1(Q_{2k})}ds \\
&\quad\le ce^{-\lambda_{m+1}/2}\int_t^{t+1}\|v(*)\|^2_{H^1(\mathbb{R}^n)}ds \\
&\quad\le ce^{-\lambda_{m+1}/2}(1+r(\omega)).
\end{aligned}\tag{151}$$

Thus, there exists $N_1=N_1(\omega,\varepsilon)$, for all $m\ge N_1$ and all $t\ge T_{\widehat{D}}(\omega)$, we get

$$2e^{-\lambda_{m+1}/2}\int_t^{t+1}\|\widetilde{v}_2(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|^2_{H_0^1(Q_{2k})}ds\le\frac{\varepsilon}{5}.\tag{152}$$

For the second term in the right-hand side of (150), we can estimate it as follows:

$$\begin{aligned}
&\int_{Q_{2k}(|v(*)|\ge M)}|v(*)|^{2p-2}dx \\
&\quad=\int_{Q_{2k}(v(*)\ge M)}|v(*)|^{2p-2}dx \\
&\qquad+\int_{Q_{2k}(v(*)\le -M)}|v(*)|^{2p-2}dx \\
&\quad\le 2^{p-2}\left[\int_{Q_{2k}(v(*)\ge M)}|v(*)|^{p-1}\right. \\
&\qquad\qquad\times\left(|v(*)-M|^{p-1}+M^{p-1}\right)dx \\
&\qquad\qquad+\int_{Q_{2k}(v(*)\le -M)}|v(*)|^{p-1} \\
&\qquad\qquad\left.\times\left(|v(*)+M|^{p-1}+M^{p-1}\right)dx\right] \\
&\quad\le 2^{p-2}\left[\int_{Q_{2k}(v(*)\ge M)}|v(*)|^{p-1}|v(*)-M|^{p-1}dx\right. \\
&\qquad\qquad\left.+\int_{Q_{2k}(v(*)\le -M)}|v(*)|^{p-1}|v(*)+M|^{p-1}dx\right] \\
&\qquad+2^{p-2}M^{p-2}\int_{Q_{2k}(|v(*)|\ge M)}|v(*)|^{p}dx,
\end{aligned}\tag{153}$$

$$\begin{aligned}
&ce^{-\lambda_{m+1}(t+1)}\int_t^{t+1}e^{\lambda_{m+1}s}\int_{Q_{2k}(|v(*)|\ge M)}|v(*)|^{2p-2}dx\,ds \\
&\quad=ce^{-\lambda_{m+1}(t+1)}\left(\int_{t+1-\eta}^{t+1}+\int_t^{t+1-\eta}\right)e^{\lambda_{m+1}s} \\
&\qquad\times\int_{Q_{2k}(|v(*)|\ge M)}|v(*)|^{2p-2}dx\,ds \\
&\quad\le ce^{-\lambda_{m+1}(t+1)}\int_{t+1-\eta}^{t+1}e^{\lambda_{m+1}s}2^{p-2} \\
&\qquad\times\left[\int_{Q_{2k}(v(*)\ge M)}|v(*)|^{p-1}|v(*)-M|^{p-1}dx\right. \\
&\qquad\qquad+\int_{Q_{2k}(v(*)\le -M)}|v(*)|^{p-1} \\
&\qquad\qquad\left.\times|v(*)+M|^{p-1}dx\right]ds \\
&\qquad+c2^{p-2}M^{p-2}e^{-\lambda_{m+1}(t+1)} \\
&\qquad\times\int_{t+1-\eta}^{t+1}e^{\lambda_{m+1}s}\int_{Q_{2k}(|v(*)|\ge M)}|v(*)|^{p}dx\,ds+ce^{-\lambda_{m+1}(t+1)} \\
&\qquad\times\int_t^{t+1-\eta}e^{\lambda_{m+1}s}\int_{Q_{2k}(|v(*)|\ge M)}|v(*)|^{2p-2}dx\,ds.
\end{aligned}\tag{154}$$

The first term in the right-hand side of (154) is bounded by

$$2^{p-2}c\int_{t+1-\eta}^{t+1}\left[\int_{Q_{2k}(v(*)\geq M)}|v(*)|^{p-1}\times|v(*)-M|^{p-1}dx+\int_{Q_{2k}(v(*)\leq -M)}|v(*)|^{p-1}|v(*)+M|^{p-1}dx\right]ds\leq\frac{\varepsilon}{15},\quad\forall t\geq T_{\widehat{D}}(\omega),\tag{155}$$

when we choose appropriate η and M by Lemma 20.

For the second term of the right-hand side of (154), we use Lemma 15, $\forall t\geq T_{\widehat{D}}(\omega)$,

$$\begin{aligned}&c2^{p-2}M^{p-2}e^{-\lambda_{m+1}(t+1)}\times\int_{t+1-\eta}^{t+1}e^{\lambda_{m+1}s}\int_{Q_{2k}(|v(*)|\geq M)}|v(*)|^{p}dx\,ds\\&\leq c2^{p-2}M^{p-2}e^{-\lambda_{m+1}(t+1)}\int_{t+1-\eta}^{t+1}e^{\lambda_{m+1}s}\|v(*)\|_p^p ds\\&\leq c2^{p-2}M^{p-2}e^{-\lambda_{m+1}(t+1)}(1+r(\omega))\int_{t+1-\eta}^{t+1}e^{\lambda_{m+1}s}ds\\&\leq c2^{p-2}M^{p-2}(1+r(\omega))\int_{t+1-\eta}^{t+1}e^{\lambda_{m+1}(s-t-1)}ds\\&=c2^{p-2}M^{p-2}(1+r(\omega))\int_{-\eta}^{0}e^{\lambda_{m+1}s}ds\\&\leq\frac{c2^{p-2}M^{p-2}(1+r(\omega))}{\lambda_{m+1}},\end{aligned}\tag{156}$$

where M is determined in (155), and then there exists $N_2^1(\omega,\varepsilon)>0$, such that $\forall m\geq N_2^1$, $\forall t\geq T_{\widehat{D}}(\omega)$, we get

$$c2^{p-2}M^{p-2}e^{-\lambda_{m+1}(t+1)}\times\int_{t+1-\eta}^{t+1}e^{\lambda_{m+1}s}\int_{Q_{2k}(|v(*)|\geq M)}|v(*)|^{p}dx\,ds\leq\frac{\varepsilon}{15}.\tag{157}$$

By Lemma 16, the last term of the right-hand side of (154) is bounded by

$$\begin{aligned}&ce^{-\lambda_{m+1}\eta}\int_{t}^{t+1-\eta}\int_{Q_{2k}(|v(*)|\geq M)}|v(*)|^{2p-2}dx\,ds\\&\leq ce^{-\lambda_{m+1}\eta}\int_{t}^{t+1}\|v(*)\|_{2p-2}^{2p-2}ds\\&\leq ce^{-\lambda_{m+1}\eta}(1+r(\omega)),\end{aligned}\tag{158}$$

for all $t\geq T_{\widehat{D}}(\omega)$. This implies that there exists $N_2^2=N_2^2(\omega,\varepsilon)>0$, such that, for all $m\geq N_2^2$, $t\geq T_{\widehat{D}}(\omega)$, we have

$$ce^{-\lambda_{m+1}(t+1)}\times\int_{t}^{t+1-\eta}e^{\lambda_{m+1}s}\int_{Q_{2k}(|v(*)|\geq M)}|v(*)|^{2p-2}dx\,ds\leq\frac{\varepsilon}{15}.\tag{159}$$

Combining (154), (155), (157), and (159) and setting $N_2=N_2(\omega,\varepsilon)=\max\{N_2^1(\omega,\varepsilon),N_2^2(\omega,\varepsilon)\}$, then we have that, for all $m\geq N_2$, $t\geq T_{\widehat{D}}(\omega)$,

$$ce^{-\lambda_{m+1}(t+1)}\int_{t}^{t+1}e^{\lambda_{m+1}s}\times\int_{Q_{2k}(|v(*)|\geq M)}|v(*)|^{2p-2}dx\,ds\leq\frac{\varepsilon}{5}.\tag{160}$$

The third term of the right-hand side of (150) is bounded by

$$\begin{aligned}&ce^{-\lambda_{m+1}(t+1)}\int_{t}^{t+1}e^{\lambda_{m+1}s}p(\theta_{s-t-1}\omega)ds\\&=c\int_{t}^{t+1}e^{\lambda_{m+1}(s-t-1)}p(\theta_{s-t-1}\omega)ds\\&=c\int_{-1}^{0}e^{\lambda_{m+1}s}p(\theta_s\omega)ds\\&\leq cr(\omega)\int_{-1}^{0}e^{\lambda_{m+1}s}e^{-(\lambda/2)s}ds\leq\frac{cr(\omega)}{\lambda_{m+1}-\lambda/2},\end{aligned}\tag{161}$$

so there exists $N_3=N_3(\omega,\varepsilon)>0$, $\forall m\geq N_3$, $\forall t\geq 0$; we have

$$ce^{-\lambda_{m+1}(t+1)}\times\int_{t}^{t+1}e^{\lambda_{m+1}s}\left(\|z(\theta_{s-t-1}\omega)\|_{2p-2}^{2p-2}+\|\Delta z(\theta_{s-t-1}\omega)\|^{2}\right)ds\leq\frac{\varepsilon}{5}.\tag{162}$$

For the forth term of the right-hand side of (150), by using Lemma 15, we have

$$\begin{aligned}&ce^{-\lambda_{m+1}(t+1)}\int_{t}^{t+1}e^{\lambda_{m+1}s}\left(\|v(*)\|^{2}+\|\nabla v(*)\|^{2}\right)ds\\&\leq ce^{-\lambda_{m+1}(t+1)}\int_{t}^{t+1}e^{\lambda_{m+1}s}ds(1+r(\omega))\\&\leq c\int_{-1}^{0}e^{\lambda_{m+1}s}ds(1+r(\omega))\\&\leq\frac{c}{\lambda_{m+1}}(1+r(\omega)),\quad\forall t\geq T_{\widehat{D}}(\omega),\end{aligned}\tag{163}$$

and then there exists $N_4 = N_4(\omega, \varepsilon) > 0$, for all $m \geq N_4$ and all $t \geq T_{\widehat{D}}(\omega)$,

$$ce^{-\lambda_{m+1}(t+1)} \times \int_t^{t+1} e^{\lambda_{m+1}s} \left(\|v(*)\|^2 + \|\nabla v(*)\|^2\right) ds \leq \frac{\varepsilon}{5}. \tag{164}$$

For the last term of the right-hand side of (150), we have

$$c\left(\|\psi_2\|^2, \|g\|^2, |Q_{2k}|, p, M\right) e^{-\lambda_{m+1}(t+1)} \times \int_t^{t+1} e^{\lambda_{m+1}s} ds \leq \frac{c\left(\|\psi_2\|^2, \|g\|^2, |Q_{2k}|, p, M\right)}{\lambda_{m+1}}, \tag{165}$$

this implies that there exists $N_5 = N_5(k, \varepsilon)$ (M is determined in (155)), for all $m \geq N_5$ and all $t \geq 0$,

$$c\left(\|\psi_2\|^2, \|g\|^2, |Q_{2k}|, p, M\right) e^{-\lambda_{m+1}(t+1)} \times \int_t^{t+1} e^{\lambda_{m+1}s} ds \leq \frac{\varepsilon}{5}. \tag{166}$$

Finally, let $N(k, \omega, \varepsilon) = \max\{N_1, N_2, N_3, N_4, N_5\}$, and then from (150), (152), (160), (162), (164), and (166) we have that, for all $m \geq N(k, \omega, \varepsilon)$, $t \geq T_{\widehat{D}}(\omega)$,

$$\|\widetilde{v}_2(t+1, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))\|^2_{H_0^1(Q_{2k})} \leq \varepsilon. \tag{167}$$

The proof is complete. □

Lemma 23. *Assume that $g \in L^2(\mathbb{R}^n)$ and (3)–(6) hold, and ϕ is the RDS generated by problem (1)-(2); then, $\chi_{Q_k} \cdot \phi$ is $(L^2(\mathbb{R}^n), H^1(Q_k))$-asymptotically compact for all $k \geq 1$.*

Proof. We first prove that the sequence $\{\widetilde{v}(t_m, \theta_{-t_m}\omega, y_m)\}_{m=1}^{\infty}$ is precompact in $H_0^1(Q_{2k})$, for any $t_m \to \infty$, and any $x_m \in D(\theta_{-t_m}\omega)$ with $\widehat{D} = \{D(\omega)\}_{\omega\in\Omega} \in \mathscr{D}_2$, where the relationship between x_m and y_m is determined by (31), that is, $y_m = x_m - z(\theta_{-t_m}\omega)$. By Lemma 15 we see that, for all $t \geq T_{\widehat{D}}(\omega)$,

$$\|v(t, \theta_{-t}\omega, v_0(\theta_{-t}\omega))\|^2_{H^1(\mathbb{R}^n)} \leq c(1 + r(\omega)), \tag{168}$$

where $v_0(\theta_{-t}\omega) = u_0(\theta_{-t}\omega) - z(\theta_{-t}\omega)$ with $u_0(\omega) \in D(\omega)$. Thus there is $M_1 = M_1(\widehat{D}, \omega)$ such that, for all $m \geq M_1$, we have $t_m \geq T_{\widehat{D}}(\omega)$ and

$$\left\|v\left(t_m, \theta_{-t_m}\omega, y_m\right)\right\|^2_{H^1(\mathbb{R}^n)} \leq c(1 + r(\omega)), \quad \forall m \geq M_1. \tag{169}$$

By (107) and the aforementioned inequality we find that

$$\left\|\widetilde{v}\left(t_m, \theta_{-t_m}\omega, y_m\right)\right\|^2_{H_0^1(Q_{2k})} \leq c(1 + r(\omega)), \quad \forall m \geq M_1. \tag{170}$$

Given $\varepsilon > 0$, it follows from Lemma 22 that there are $T_{\widehat{D}}(\omega)$ and $N(k, \omega, \varepsilon)$ such that, for all $t \geq T_{\widehat{D}}(\omega)$,

$$\left\|(I - P_N)\,\widetilde{v}(t, \theta_{-t}\omega, v_0(\theta_{-t}\omega))\right\|^2_{H_0^1(Q_{2k})} \leq \varepsilon. \tag{171}$$

Taking $M_2 = M_2(\widehat{D}, \omega)$ large enough such that $t_m \geq T_{\widehat{D}}(\omega)$ for $m \geq M_2$, then we get from (171) that

$$\left\|(I - P_N)\,\widetilde{v}\left(t_m, \theta_{-t_m}\omega, y_m\right)\right\|^2_{H_0^1(Q_{2k})} \leq \varepsilon, \quad \forall m \geq M_2. \tag{172}$$

On the other hand, (170) shows that the sequence $\{P_N\widetilde{v}(t_m, \theta_{-t_m}\omega, y_m)\}$ is bounded in the finite-dimensional space $P_N H_0^1(Q_{2k})$ and hence is precompact in $P_N H_0^1(Q_{2k})$, which along with (172) implies the precompactness of $\{\widetilde{v}(t_m, \theta_{-t_m}\omega, y_m)\}$ in $H_0^1(Q_{2k})$.

Next we prove the $(L^2(\mathbb{R}^n), H^1(Q_k))$-asymptotic compactness for $\chi_{Q_k} \cdot \phi$. From the definition of $\widetilde{v}$, we see that $v = \widetilde{v}$ in Q_k, so $v(t_m, \theta_{-t_m}\omega, y_m)\}_{m=1}^{\infty}$ is precompact in $H^1(Q_k)$. By the relationship

$$u\left(t_m, \theta_{-t_m}\omega, x_m\right) = v\left(t_m, \theta_{-t_m}\omega, y_m\right) + z(\omega), \tag{173}$$

and the aforementioed assert, we obtain that $\{u(t_m, \theta_{-t_m}\omega, x_m)\}_{m=1}^{\infty}$ is precompact in $H^1(Q_k)$, for all $k \geq 1$. The proof is complete. □

5. Asymptotic Compactness and Random Attractors

In this section, we prove our main result, that is, the existence of an $(L^2(\mathbb{R}^n), H^1(\mathbb{R}^n))$-random attractor for the RDS ϕ associated with the initial value problem of SRDE (1)-(2). To this end, we should show the $(L^2(\mathbb{R}^n), H^1(\mathbb{R}^n))$-asymptotic compactness of ϕ. From Theorem 10 and Lemmas 18 and 23, we can immediately obtain the asymptotic compactness of ϕ.

Lemma 24. *Assume that $g \in L^2(\mathbb{R}^n)$ and (3)–(6) hold. Then the RDS ϕ generated by (1)-(2) is $(L^2(\mathbb{R}^n), H^1(\mathbb{R}^n))$ asymptotically compact.*

Now, we are in a position to present our main result.

Theorem 25. *Assume that $g \in L^2(\mathbb{R}^n)$ and (3)–(6) hold. Then the RDS ϕ generated by (1)-(2) has an $(L^2(\mathbb{R}^n), H^1(\mathbb{R}^n))$-random attractor $\widehat{A}_v$.*

Proof. The result can be obtained by Theorems 9, and 13, Lemmas 14, and 24 immediately. □

Remark 26. Our methods can be used to prove the existence of $(L^2(\mathbb{R}^n), H^1(\mathbb{R}^n))$-pullback attractors for the following non-autonomous reaction-diffusion equation on unbounded domains:

$$\frac{\partial u}{\partial t} + \lambda u - \Delta u = f(x, u) + g(x, t), \quad x \in \mathbb{R}^n, \tag{174}$$

with the initial condition

$$u(x, \tau) = u_\tau(x), \quad x \in \mathbb{R}^n, \tag{175}$$

for every $\tau \in \mathbb{R}$ and $t > \tau$ where the nonlinear term f satisfies (3)–(6) in this paper. g is a given function in $L^2_{\text{loc}}(\mathbb{R}, \mathbb{R}^n)$ with

$$\int_{-\infty}^{t} e^{\lambda s} \|g(s)\|^2 ds < \infty, \quad \forall t \in \mathbb{R}, \tag{176}$$

and the result is new in this case.

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