Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2013, Article ID 276579, 12 pages
http://dx.doi.org/10.1155/2013/276579

Research Article

A New Approach to the Method of Lyapunov Functionals and Its Applications

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Received 18 July 2013; Revised 2 November 2013; Accepted 3 November 2013

Academic Editor: Xinyu Song

We show some results which can replace the graph theory used to construct global Lyapunov functions in some coupled systems of differential equations. We present an example of an epidemic model with stage structure and latency spreading in a heterogeneous host population and obtain a more general threshold for the extinction and persistence of a disease. Using some results obtained by mathematical induction and suitable Lyapunov functionals, we prove the global stability of the endemic equilibrium. For some coupled systems of differential equations, by a similar approach to the discussion of the epidemic model, the conditions of threshold property or global stability can be established without the assumption that the relative matrix is irreducible.

1. Introduction

Graph theory has developed into a substantial body of knowledge. A graph theoretic approach developed in [1, 2] is used to resolve a long-standing open problem on the uniqueness and global stability of the endemic equilibrium of a class of multigroup models in mathematical epidemiology. Using results from graph theory, a systematic approach developed in [3] allows one to construct global Lyapunov functions for large-scale coupled systems from building blocks of individual vertex systems. The graph-theoretical approach has been applied to various classes of coupled systems in engineering, ecology, and epidemiology (see, e.g., [1–13]). However, as in [14], while using the same Lyapunov function [3], sometimes graph theory can be replaced by positive operator theory. Furthermore, it seems that all authors use the graph theory under the assumption that the relative matrix is irreducible (see, e.g., [1–13]).

Motivated by the above discussion, in this paper, we show some results which can replace graph theory used to construct global Lyapunov functions in some coupled systems of differential equations (see, e.g., [1–13]) and a more general threshold without the assumption that the relative matrix is irreducible. For some coupled systems of differential equations (see, e.g., [1–13]), by a similar approach to the one discussed in this paper, the conditions of threshold property or global stability can be established without the assumption that the relative matrix is irreducible.

Various epidemics continue to pose a public health threat to humans. One of the most important subjects in this study of epidemic models (see, e.g., [1–3, 7–13]) is to obtain a threshold that determines the persistence or extinction of a disease. In the real world, some epidemics, such as malaria, dengue, fever, gonorrhea, and bacterial infections, may have a different ability to transmit the infections in different ages. For example, measles and varicella always occur in juveniles, while it is reasonable to consider the disease transmission in adult population such as typhus and diphtheria. A heterogeneous host population can be divided into several homogeneous groups according to models of transmission, contact patterns, or geographic distributions. Since the time it takes from the moment of new infection to the moment of becoming infectious may differ from individual to individual, it is indeed a random variable. We present an example of an epidemic model with stage structure and latency spreading in a heterogeneous host population and show a general threshold to improved existing results.

Some results which can replace the graph theory used to construct global Lyapunov functions in some coupled systems of differential equations are shown in the next sections. In Section 3, a new approach to the method of Lyapunov functionals is applied to an epidemic model to obtain a more general threshold for the extinction and persistence of a disease.

2. Some Results Obtained by Mathematical Induction

In this section, we show some results which can replace graph theory used to construct global Lyapunov functions in some coupled systems of differential equations.

Assumption 1. There exist $v_k > 0$, such that

$$\sum_{j=1}^{n} \overline{\beta}_{kj} v_k - \sum_{j=1}^{n} \overline{\beta}_{jk} v_j = 0, \quad (1)$$

where $\overline{\beta}_{kj} \geq 0$, $k, j = 1, 2, \ldots, n$, $n \geq 2$.

Theorem 2. *Under Assumption 1, the following results hold:*

$$\sum_{k,j=1}^{n} v_k \overline{\beta}_{kj} \left[F_k(x_k) - F_j(x_j) \right] = 0, \quad (2)$$

where $F_k(x_k)$, $k = 1, 2, \ldots, n$, are arbitrary functions. In particular, if $F_k(x_k) > 0$, $k = 1, 2, \ldots, n$, then

$$\sum_{k,j=1}^{n} v_k \overline{\beta}_{kj} \left[1 - \frac{F_k(x_k)}{F_j(x_j)} \right] \leq 0. \quad (3)$$

Proof. Note that

$$\begin{aligned} &\sum_{k,j=1}^{n} v_k \overline{\beta}_{kj} \left[F_k(x_k) - F_j(x_j) \right] \\ &\quad = \sum_{n \geq k \geq j \geq 1} \left(v_k \overline{\beta}_{kj} - v_j \overline{\beta}_{jk} \right) \left[F_k(x_k) - F_j(x_j) \right] \\ &\quad =: Z_n. \end{aligned} \quad (4)$$

Let

$$\Delta_L = \sum_{L \geq k \geq j \geq 2} \left(v_k \overline{\beta}_{kj} - v_j \overline{\beta}_{jk} \right) \left[F_k(x_k) - F_j(x_j) \right], \quad L = 2, 3, \ldots, n. \quad (5)$$

According to Assumption 1, it is easy to see that we only need to prove the following result:

$$\begin{aligned} Z_n = {} & \sum_{L=2}^{q+1} \left(v_L \overline{\beta}_{L1} - v_1 \overline{\beta}_{1L} + v_L \overline{\beta}_{Ln} - v_n \overline{\beta}_{nL} + v_L \overline{\beta}_{L,n-1} \right. \\ & \left. - v_{n-1} \overline{\beta}_{n-1,L} + \cdots + v_L \overline{\beta}_{L,q+2} - v_{q+2} \overline{\beta}_{q+2,L} \right) \\ & \times \left[F_L(x_L) - F_1(x_1) \right] + \Delta_{q+1} =: A_{q+1}, \\ & q = 1, 2, \ldots, n-2. \end{aligned} \quad (6)$$

In fact, if the above result holds, then we have

$$Z_n = \sum_{L=1}^{n} \left(v_2 \overline{\beta}_{2L} - v_L \overline{\beta}_{L2} \right) \left[F_2(x_2) - F_1(x_1) \right] + \Delta_2. \quad (7)$$

Using Assumption 1, we have $Z_n = \Delta_2 = 0$. First, we show that $Z_n = A_{n-1}$. We can rewrite Z_n as

$$\begin{aligned} Z_n = \sum_{L=2}^{n} & \left\{ \left(v_n \overline{\beta}_{nL} - v_L \overline{\beta}_{Ln} \right) \left[F_n(x_n) - F_L(x_L) \right] \right. \\ & \left. + \left(v_L \overline{\beta}_{L1} - v_1 \overline{\beta}_{1L} \right) \left[F_L(x_L) - F_1(x_1) \right] \right\} + \Delta_{n-1}. \end{aligned} \quad (8)$$

Then, using the fact that $ab + cd = a(b + d) + d(c - a)$, where a, b, c, and d are arbitrary numbers, we may obtain

$$\begin{aligned} Z_n &= \sum_{L=2}^{n} \left\{ \left(v_n \overline{\beta}_{nL} - v_L \overline{\beta}_{Ln} \right) \left[F_n(x_n) - F_1(x_1) \right] \right. \\ &\qquad + \left(v_L \overline{\beta}_{L1} - v_1 \overline{\beta}_{1L} + v_L \overline{\beta}_{Ln} - v_n \overline{\beta}_{nL} \right) \\ &\qquad \left. \times \left[F_L(x_L) - F_1(x_1) \right] \right\} + \Delta_{n-1} \\ &= \sum_{L=2}^{n} \left(v_n \overline{\beta}_{nL} - v_L \overline{\beta}_{Ln} \right) \left[F_n(x_n) - F_1(x_1) \right] \\ &\qquad + \left(v_n \overline{\beta}_{n1} - v_1 \overline{\beta}_{1n} \right) \left[F_n(x_n) - F_1(x_1) \right] \\ &\qquad + \sum_{L=2}^{n-1} \left(v_L \overline{\beta}_{L1} - v_1 \overline{\beta}_{1L} + v_L \overline{\beta}_{Ln} - v_n \overline{\beta}_{nL} \right) \\ &\qquad \times \left[F_L(x_L) - F_1(x_1) \right] + \Delta_{n-1} \\ &= \sum_{L=1}^{n} \left(v_n \overline{\beta}_{nL} - v_L \overline{\beta}_{Ln} \right) \left[F_n(x_n) - F_1(x_1) \right] + \Delta_{n-1} \\ &\qquad + \sum_{L=2}^{n-1} \left(v_L \overline{\beta}_{L1} - v_1 \overline{\beta}_{1L} + v_L \overline{\beta}_{Ln} - v_n \overline{\beta}_{nL} \right) \\ &\qquad \times \left[F_L(x_L) - F_1(x_1) \right]. \end{aligned} \quad (9)$$

Using Assumption 1, we have $Z_n = A_{n-1}$. Next, we show that $A_{q+1} = A_q$, $q = 1, 2, \ldots, n-2$. We can rewrite A_{q+1} as

$$\begin{aligned} A_{q+1} = & \sum_{L=2}^{q+1} \left(v_L \overline{\beta}_{L1} - v_1 \overline{\beta}_{1L} + v_L \overline{\beta}_{Ln} - v_n \overline{\beta}_{nL} + v_L \overline{\beta}_{L,n-1} \right. \\ & \left. - v_{n-1} \overline{\beta}_{n-1,L} + \cdots + v_L \overline{\beta}_{L,q+2} - v_{q+2} \overline{\beta}_{q+2,L} \right) \\ & \times \left[F_L (x_L) - F_1 (x_1) \right] + \Delta_q \\ & + \sum_{j=2}^{q+1} \left(v_{q+1} \overline{\beta}_{q+1,j} - v_j \overline{\beta}_{j,q+1} \right) \left[F_{q+1} (x_{q+1}) - F_j (x_j) \right]. \end{aligned} \tag{10}$$

By a similar argument as for the discussion

$$\begin{aligned} & \sum_{L=2}^{n} \left\{ \left(v_n \overline{\beta}_{nL} - v_L \overline{\beta}_{Ln} \right) \left[F_n (x_n) - F_L (x_L) \right] \right. \\ & \quad \left. + \left(v_L \overline{\beta}_{L1} - v_1 \overline{\beta}_{1L} \right) \left[F_L (x_L) - F_1 (x_1) \right] \right\} \\ & = \sum_{L=1}^{n} \left(v_n \overline{\beta}_{nL} - v_L \overline{\beta}_{Ln} \right) \left[F_n (x_n) - F_1 (x_1) \right] \\ & \quad + \sum_{L=2}^{n-1} \left(v_L \overline{\beta}_{L1} - v_1 \overline{\beta}_{1L} + v_L \overline{\beta}_{Ln} - v_n \overline{\beta}_{nL} \right) \\ & \quad \times \left[F_L (x_L) - F_1 (x_1) \right], \end{aligned} \tag{11}$$

in the proof of $Z_n = A_{n-1}$, we can obtain

$$\begin{aligned} & \sum_{L=2}^{q+1} \left(v_L \overline{\beta}_{L1} - v_1 \overline{\beta}_{1L} \right) \left[F_L (x_L) - F_1 (x_1) \right] \\ & \quad + \sum_{j=2}^{q+1} \left(v_{q+1} \overline{\beta}_{q+1,j} - v_j \overline{\beta}_{j,q+1} \right) \left[F_{q+1} (x_{q+1}) - F_j (x_j) \right] \\ & = \sum_{L=1}^{q+1} \left(v_{q+1} \overline{\beta}_{q+1,L} - v_L \overline{\beta}_{L,q+1} \right) \left[F_{q+1} (x_{q+1}) - F_1 (x_1) \right] \\ & \quad + \sum_{L=2}^{q} \left(v_L \overline{\beta}_{L1} - v_1 \overline{\beta}_{1L} + v_L \overline{\beta}_{L,q+1} - v_{q+1} \overline{\beta}_{q+1,L} \right) \\ & \quad \times \left[F_L (x_L) - F_1 (x_1) \right]. \end{aligned} \tag{12}$$

Substituting (12) into (10), we have

$$\begin{aligned} A_{q+1} = & \sum_{L=2}^{q+1} \left(v_L \overline{\beta}_{Ln} - v_n \overline{\beta}_{nL} + v_L \overline{\beta}_{L,n-1} - v_{n-1} \overline{\beta}_{n-1,L} \right. \\ & \left. + \cdots + v_L \overline{\beta}_{L,q+2} - v_{q+2} \overline{\beta}_{q+2,L} \right) \\ & \times \left[F_L (x_L) - F_1 (x_1) \right] + \Delta_q \\ & + \sum_{L=1}^{q+1} \left(v_{q+1} \overline{\beta}_{q+1,L} - v_L \overline{\beta}_{L,q+1} \right) \left[F_{q+1} (x_{q+1}) - F_1 (x_1) \right] \\ & + \sum_{L=2}^{q} \left(v_L \overline{\beta}_{L1} - v_1 \overline{\beta}_{1L} + v_L \overline{\beta}_{L,q+1} - v_{q+1} \overline{\beta}_{q+1,L} \right) \\ & \times \left[F_L (x_L) - F_1 (x_1) \right] \\ = & \sum_{L=1}^{n} \left(v_{q+1} \overline{\beta}_{q+1,L} - v_L \overline{\beta}_{L,q+1} \right) \left[F_{q+1} (x_{q+1}) - F_1 (x_1) \right] \\ & + A_q. \end{aligned} \tag{13}$$

By Assumption 1, we can get $A_{q+1} = A_q$. Using $1 - (F_k(x_k)/F_j(x_j)) + \ln(F_k(x_k)/F_j(x_j)) \le 0$ and the result above, we can obtain that $\sum_{k,j=1}^{n} v_k \overline{\beta}_{kj} (1 - (F_k(x_k)/F_j(x_j))) \le 0$. This completes the proof. □

Remark 3. Some results in [1–13] from graph theory can be obtained by Theorem 2. For example, $\sum_{k,j=1}^{n} v_k \overline{\beta}_{kj} \ln((E_k E_j^*)/(E_k^* E_j)) = 0$ in [11] and $\sum_{k,j=1}^{n} v_k \overline{\beta}_{kj} [n + 2 - (S_k^*/S_k) - ((S_k I_j y_{k,1}^*)/(S_k^* I_j^* y_{k,1})) - \sum_{i=2}^{n} ((y_{k,i-1} y_{k,i}^*)/(y_{k,i-1}^* y_{k,i})) - ((y_{k,n} I_k^*)/(y_{k,n}^* I_k)) \le 0]$ in [10].

The following result is one result of Kirchhoff's Three Theorem (a result in graph theory). In the following, using mathematical induction, we show the result.

Theorem 4. *If $(\overline{\beta}_{kj})_{n \times n}$, $n \ge 2$, is irreducible, then Assumption 1 holds.*

Proof. Obviously, the result holds for $n = 2$. We assume $n \ge 3$ and can rewrite (1) as

$$\begin{bmatrix} \sum_{l \neq 1} \overline{\beta}_{1l} & -\overline{\beta}_{21} & \cdots & -\overline{\beta}_{n1} \\ -\overline{\beta}_{12} & \sum_{l \neq 2} \overline{\beta}_{2l} & \cdots & -\overline{\beta}_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ -\overline{\beta}_{1n} & -\overline{\beta}_{2n} & \cdots & \sum_{l \neq n} \overline{\beta}_{nl} \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{14}$$

Note that (14) is equivalent to the following system:

$$\begin{aligned} & \begin{bmatrix} \sum_{l \neq 1} \overline{\beta}_{1l} & -\overline{\beta}_{21} & \cdots & -\overline{\beta}_{n1} \\ 0 & \sum_{l \neq 2} \overline{\beta}_{2l} - \overline{\beta}_{21} \dfrac{\overline{\beta}_{12}}{\sum_{l \neq 1} \overline{\beta}_{1l}} & \cdots & -\overline{\beta}_{n2} - \overline{\beta}_{n1} \dfrac{\overline{\beta}_{12}}{\sum_{l \neq 1} \overline{\beta}_{1l}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & -\overline{\beta}_{2n} - \overline{\beta}_{21} \dfrac{\overline{\beta}_{1n}}{\sum_{l \neq 1} \overline{\beta}_{1l}} & \cdots & \sum_{l \neq n} \overline{\beta}_{nl} - \overline{\beta}_{n1} \dfrac{\overline{\beta}_{1n}}{\sum_{l \neq 1} \overline{\beta}_{1l}} \end{bmatrix} \\ & \quad \times \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \end{aligned} \tag{15}$$

Consider the following system:

$$\begin{bmatrix} \sum_{l\neq 2}\overline{\beta}_{2l}-\overline{\beta}_{21}\frac{\overline{\beta}_{12}}{\sum_{l\neq 1}\overline{\beta}_{1l}} & -\overline{\beta}_{32}-\overline{\beta}_{31}\frac{\overline{\beta}_{12}}{\sum_{l\neq 1}\overline{\beta}_{1l}} & \cdots & -\overline{\beta}_{n2}-\overline{\beta}_{n1}\frac{\overline{\beta}_{12}}{\sum_{l\neq 1}\overline{\beta}_{1l}} \\ -\overline{\beta}_{23}-\overline{\beta}_{21}\frac{\overline{\beta}_{13}}{\sum_{l\neq 1}\overline{\beta}_{1l}} & \sum_{l\neq 3}\overline{\beta}_{3l}-\overline{\beta}_{31}\frac{\overline{\beta}_{13}}{\sum_{l\neq 1}\overline{\beta}_{1l}} & \cdots & -\overline{\beta}_{n3}-\overline{\beta}_{n1}\frac{\overline{\beta}_{13}}{\sum_{l\neq 1}\overline{\beta}_{1l}} \\ \vdots & \vdots & \ddots & \vdots \\ -\overline{\beta}_{2n}-\overline{\beta}_{21}\frac{\overline{\beta}_{1n}}{\sum_{l\neq 1}\overline{\beta}_{1l}} & -\overline{\beta}_{3n}-\overline{\beta}_{31}\frac{\overline{\beta}_{1n}}{\sum_{l\neq 1}\overline{\beta}_{1l}} & \cdots & \sum_{l\neq n}\overline{\beta}_{nl}-\overline{\beta}_{n1}\frac{\overline{\beta}_{1n}}{\sum_{l\neq 1}\overline{\beta}_{1l}} \end{bmatrix} \times \begin{pmatrix} v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \quad (16)$$

Let

$$\begin{bmatrix} -a_{11} & -a_{21} & \cdots & -a_{n-1,1} \\ -a_{12} & -a_{22} & \cdots & -a_{n-1,2} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{1,n-1} & -a_{2,n-1} & \cdots & -a_{n-1,n-1} \end{bmatrix} = \begin{bmatrix} \sum_{l\neq 2}\overline{\beta}_{2l}-\overline{\beta}_{21}\frac{\overline{\beta}_{12}}{\sum_{l\neq 1}\overline{\beta}_{1l}} & -\overline{\beta}_{32}-\overline{\beta}_{31}\frac{\overline{\beta}_{12}}{\sum_{l\neq 1}\overline{\beta}_{1l}} & \cdots & -\overline{\beta}_{n2}-\overline{\beta}_{n1}\frac{\overline{\beta}_{12}}{\sum_{l\neq 1}\overline{\beta}_{1l}} \\ -\overline{\beta}_{23}-\overline{\beta}_{21}\frac{\overline{\beta}_{13}}{\sum_{l\neq 1}\overline{\beta}_{1l}} & \sum_{l\neq 3}\overline{\beta}_{3l}-\overline{\beta}_{31}\frac{\overline{\beta}_{13}}{\sum_{l\neq 1}\overline{\beta}_{1l}} & \cdots & -\overline{\beta}_{n3}-\overline{\beta}_{n1}\frac{\overline{\beta}_{13}}{\sum_{l\neq 1}\overline{\beta}_{1l}} \\ \vdots & \vdots & \ddots & \vdots \\ -\overline{\beta}_{2n}-\overline{\beta}_{21}\frac{\overline{\beta}_{1n}}{\sum_{l\neq 1}\overline{\beta}_{1l}} & -\overline{\beta}_{3n}-\overline{\beta}_{31}\frac{\overline{\beta}_{1n}}{\sum_{l\neq 1}\overline{\beta}_{1l}} & \cdots & \sum_{l\neq n}\overline{\beta}_{nl}-\overline{\beta}_{n1}\frac{\overline{\beta}_{1n}}{\sum_{l\neq 1}\overline{\beta}_{1l}} \end{bmatrix}. \quad (17)$$

We can rewrite (16) as

$$\begin{bmatrix} \sum_{l\neq 1}a_{1l} & -a_{21} & \cdots & -a_{n-1,1} \\ -a_{12} & \sum_{l\neq 2}a_{2l} & \cdots & -a_{n-1,2} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{1,n-1} & -a_{2,n-1} & \cdots & \sum_{l\neq n-1}a_{n-1,l} \end{bmatrix} \begin{pmatrix} v_2 \\ v_3 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \quad (18)$$

From (17), it is easy to see that $(a_{kj})_{(n-1)\times(n-1)}$, $n \geq 3$, is also irreducible. Note that $\sum_{l\neq 1}\overline{\beta}_{1l}\neq 0$. According to the discussion above, we can deduce that if the result (if $(\overline{\beta}_{kj})_{n\times n}$, $n \geq 2$, is irreducible, then Assumption 1 holds) holds for $n = L$, $L \geq 2$, then it holds also for $n = L + 1$. The proof is complete. □

3. An Example of an Epidemic Model with Stage Structure and Latency Spreading in a Heterogeneous Host Population

In this section, we present an epidemic model with stage structure and latency spreading in a heterogeneous host population and obtain a more general threshold for the extinction and persistence of a disease. Using the results obtained by mathematical induction and suitable Lyapunov functionals, we prove the global stability of the endemic equilibrium. For some coupled systems of differential equations, by a similar approach to the discussion of the epidemic model, the conditions of threshold property or global stability can be established without the assumption that the relative matrix is irreducible.

We formulate an epidemic model with latency spreading in a heterogeneous host population. Let $S_k^{(1)}$, $S_k^{(2)}$, I_k, and R_k denote the immature susceptible, mature susceptible, infectious, and recovered population in the kth group, respectively. The disease incidence in the kth group can be calculated as

$$\sum_{i=1}^{2}\sum_{j=1}^{n}\beta_{kj}^{(i)}S_k^{(i)}I_j, \tag{19}$$

where the sum takes into account cross-infections from all groups and $\beta_{kj}^{(i)}$ is the transmission coefficient between compartments $S_k^{(i)}$ and I_j. Let $d_k^{(1)}$ and $d_k^{(2)}$ represent death rates of $S_k^{(1)}$ and $S_k^{(2)}$ populations, respectively. Let τ be the latent period of the population. $e^{-d_k^{(i)}\tau}\beta_{kj}^{(i)}S_k^{(i)}(t-\tau)I_j(t-\tau)$ represents the individuals surviving in the latent period τ and becoming infective at time t. Let $p_k^{(i)}(\tau) : [0,h] \to \mathbb{R}_+$ be integrable function with $\int_0^h p_k^{(i)}(\tau)d\tau = 1$. We assume that τ is distributed according to $p_k^{(i)}(\tau)$ over the interval $[0,h]$, where h is the upper bound of the latent period. Then, we obtain the following model for a disease with latency:

$$\begin{aligned}
\dot{S}_k^{(1)} &= b_k - d_k^{(1)}S_k^{(1)} - \sum_{j=1}^{n}\beta_{kj}^{(1)}S_k^{(1)}I_j - a_kS_k^{(1)},\\
\dot{S}_k^{(2)} &= a_kS_k^{(1)} - \sum_{j=1}^{n}\beta_{kj}^{(2)}S_k^{(2)}I_j - d_k^{(2)}S_k^{(2)},\\
\dot{I}_k &= \sum_{i=1}^{2}\sum_{j=1}^{n}\int_0^h p_k^{(i)}(\tau)e^{-d_k^{(i)}\tau}\beta_{kj}^{(i)}S_k^{(i)}(t-\tau)I_j(t-\tau)d\tau\\
&\quad - (d_k+\mu_k+\gamma_k)I_k,\\
\dot{R}_k &= \gamma_kI_k - d_kR_k, \quad k = 1,2,\ldots,n,
\end{aligned} \tag{20}$$

where $b_k^{(1)}$ denotes influx of individuals into the immature susceptible class in the kth group. a_k is the conversion rate from immature individual to mature individual in group k. d_k, μ_k, and γ_k are the natural death rate, the disease-related death rate, and the recovery rate in the kth group, respectively. All parameter values are assumed to be nonnegative and b_k, h, a_k, $d_k^{(i)}$, $d_k > 0$.

Let $f_k^{(i)}(\tau) = p_k^{(i)}(\tau)e^{-d_k^{(i)}\tau}$, $c_k^{(i)} = \int_0^h p_k^{(i)}(\tau)e^{-d_k^{(i)}\tau}d\tau$, $\varphi_k(S_k^{(1)}) = b_k - d_k^{(1)}S_k^{(1)}$, and $m_k = d_k + \mu_k + \gamma_k$. Since the variables R_k do not appear in the remaining three equations of (20), we can consider the following reduced system:

$$\begin{aligned}
\dot{S}_k^{(1)} &= \varphi_k\left(S_k^{(1)}\right) - \sum_{j=1}^{n}\beta_{kj}^{(1)}S_k^{(1)}I_j - a_kS_k^{(1)},\\
\dot{S}_k^{(2)} &= a_kS_k^{(1)} - \sum_{j=1}^{n}\beta_{kj}^{(2)}S_k^{(2)}I_j - d_k^{(2)}S_k^{(2)},\\
\dot{I}_k &= \sum_{i=1}^{2}\sum_{j=1}^{n}\int_0^h f_k^{(i)}(\tau)\beta_{kj}^{(i)}S_k^{(i)}(t-\tau)I_j(t-\tau)d\tau - m_kI_k,\\
&\qquad k = 1,2,\ldots,n.
\end{aligned} \tag{21}$$

The initial conditions for system (21) take the form

$$\begin{gathered}
S_k^{(1)}(\theta) = \phi_k^{(1)}(\theta) \ge 0, \qquad S_k^{(2)}(\theta) = \phi_k^{(2)}(\theta) \ge 0,\\
I_k(\theta) = \phi_k(\theta) \ge 0, \qquad \phi_k^{(i)}(0) > 0, \qquad \phi_k(0) > 0,\\
i = 1,2, \quad k = 1,2,\ldots,n, \quad \theta \in [-h,0],
\end{gathered} \tag{22}$$

where $(\phi_1^{(1)}(\theta), \phi_1^{(2)}(\theta), \ldots, \phi_n(\theta)) \in C([-h,0], \mathbb{R}_+^{3n})$, the Banach space of continuous functions mapping the interval $[-h,0]$ into $\mathbb{R}_+^{3n}$.

We see that system (21) exits a disease-free equilibrium $P_0 = (S_{10}^{(1)}, S_{10}^{(2)}, \ldots, S_{n0}^{(1)}, S_{n0}^{(2)}, 0, 0, \ldots, 0)$, where

$$\varphi_k\left(S_{k0}^{(1)}\right) = d_k^{(2)}S_{k0}^{(2)}, \quad a_kS_{k0}^{(1)} = d_k^{(2)}S_{k0}^{(2)}, \quad k = 1,2,\ldots,n. \tag{23}$$

Let

$$A_{kj} = \frac{\sum_{i=1}^{2} c_k^{(i)}\beta_{kj}^{(i)}S_{k0}^{(i)}}{m_k}, \quad k,j = 1,2,\ldots,n, \tag{24}$$

and $\mathbf{Q} = (A_{kj})_{n\times n}$. We assume that

(H1) there exist $(w_1, w_2, \ldots, w_n) > 0$ $(w_k > 0,\ k = 1,2,\ldots,n)$, such that

$$(w_1, w_2, \ldots, w_n)\,\mathbf{Q} < (w_1, w_2, \ldots, w_n); \tag{25}$$

(H2) there exist $(w_1, w_2, \ldots, w_n) > 0$, such that

$$(w_1, w_2, \ldots, w_n)\,\mathbf{Q} = (w_1, w_2, \ldots, w_n); \tag{26}$$

(H3) there exist $(w_1, w_2, \ldots, w_n) > 0$, such that

$$(w_1, w_2, \ldots, w_n)\,\mathbf{Q} > (w_1, w_2, \ldots, w_n). \tag{27}$$

Let

(i) $R_0 < 1$ if and only if (H1) holds;

(ii) $R_0 = 1$ if and only if (H2) holds;

(iii) $R_0 > 1$ if and only if (H3) holds.

Remark 5. By an approach as the one in [1–3, 7–13], we define r_0. Let $r_0 = \rho(\mathbf{Q})$, where ρ denotes the spectral radius. If a matrix is irreducible, then, for the eigenvalue of maximum, the associated eigenvector is positive. Note that the authors in [1–3, 7–13] discussed some coupled systems of differential equations under a definition with an approach as the one of definition of r_0 and the assumption that the relevant matrix is irreducible. In fact, if $r_0 < 1$ and the relevant matrix is irreducible, then $R_0 < 1$; if $r_0 = 1$ and the relevant matrix is irreducible, then $R_0 = 1$; if $r_0 > 1$ and the relevant matrix is irreducible, then $R_0 > 1$. However, the reverse is not true. For example,

$$\left(A_{kj}\right)_{2\times 2} = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}. \tag{28}$$

Furthermore, let

$$\left(A_{kj}\right)_{4\times 4} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}. \tag{29}$$

Obviously, $R_0 > 1$ holds also for arbitrary $(w_1, w_2, w_3, w_4) > 0$. $(A_{kj})_{4\times 4}$ conforms to the conditions of Theorem 8 but is not in accord with the conditions of Corollary 10. The definition of r_0 and the assumption that the relative matrix is irreducible in the results of Corollaries 9 and 10 are analogous with relative definition and assumption in [1–3, 7–13].

The equilibria of (21) are the same as those of the associated ODE system:

$$\begin{aligned} \dot{S}_k^{(1)} &= \varphi_k\left(S_k^{(1)}\right) - S_k^{(1)} \sum_{j=1}^{n} \beta_{kj}^{(1)} I_j - a_k S_k^{(1)}, \\ \dot{S}_k^{(2)} &= a_k S_k^{(1)} - S_k^{(2)} \sum_{j=1}^{n} \beta_{kj}^{(2)} I_j - d_k^{(2)} S_k^{(2)}, \\ \dot{I}_k &= \sum_{i=1}^{2} c_k^{(i)} S_k^{(i)} \sum_{j=1}^{n} \beta_{kj}^{(i)} I_j - m_k I_k, \quad k = 1, 2, \ldots, n. \end{aligned} \tag{30}$$

Let

$$N_k = S_k^{(1)} + S_k^{(2)} + I_k, \qquad \underline{d}_k = \min\left\{d_k^{(1)}, d_k^{(2)}, d_k\right\}, \quad k = 1, 2, \ldots, n. \tag{31}$$

From (30) and $c_k^{(i)} \leq 1$, we have

$$\dot{N}_k \leq \varphi_k\left(S_k^{(1)}\right) + d_k^{(1)} S_k^{(1)} - \underline{d}_k N_k. \tag{32}$$

We derive from (32) that the region

$$\begin{aligned} \Gamma = \Bigg\{ & \left(S_1^{(1)}, S_1^{(2)}, \ldots, S_n^{(1)}, S_n^{(2)}, I_1, \ldots, I_n\right) \\ & \in \mathbb{R}_+^{3n} : S_k^{(1)} \leq S_{k0}^{(1)}, S_k^{(2)} \leq S_{k0}^{(2)}, S_k^{(1)} + S_k^{(2)} + I_k \\ & \leq \frac{\varphi_k(0) + d_k^{(1)} S_{k0}^{(1)}}{\underline{d}_k}, k = 1, 2, \ldots, n \Bigg\} \end{aligned} \tag{33}$$

is a forward invariant compact absorbing set with respect to (30). Let Γ° denote the interior of Γ.

Note that Γ is positively invariant with respect to (21). In fact, let

$$\begin{aligned} \dot{E}_k = & \sum_{i=1}^{2} \sum_{j=1}^{n} \beta_{kj}^{(i)} S_k^{(i)} I_j \\ & - \sum_{i=1}^{2} \sum_{j=1}^{n} \int_0^h f_k^{(i)}(\tau)\, \beta_{kj}^{(i)} S_k^{(i)} (t-\tau)\, I_j (t-\tau)\, d\tau - d_k E_k, \\ & k = 1, 2, \ldots, n. \end{aligned} \tag{34}$$

Then, we have

$$\dot{N}_k + \dot{E}_k \leq \varphi_k\left(S_k^{(1)}\right) + d_k^{(1)} S_k^{(1)} - \underline{d}_k\left(N_k + E_k\right). \tag{35}$$

Furthermore, using the fact that $N_k \leq N_k + E_k$, we may obtain that Γ is positively invariant with respect to (21).

Lemma 6. *If $R_0 > 1$, then P_0 of system (30) is unstable in Γ and there exists a positive equilibrium P^* in Γ°.*

Proof. Let $\mathbf{I} = (I_1, I_2, \ldots, I_n)$ and $M = \sum_{k=1}^{n}((\omega_k I_k)/(m_k))$. Thus

$$\begin{aligned} \dot{M} &= \sum_{k=1}^{n} \omega_k \left[\frac{\sum_{i=1}^{2} c_k^{(i)} \sum_{j=1}^{n} \beta_{kj}^{(i)} S_k^{(i)} I_j}{m_k} - I_k \right] \\ &\leq \left(\omega_1, \omega_2, \ldots, \omega_n\right)\left(\mathbf{Q}\mathbf{I}^T - \mathbf{I}^T\right). \end{aligned} \tag{36}$$

If $R_0 > 1$, by continuity, we obtain $\dot{M} > 0$, in a neighborhood of P_0 in Γ°. This implies that P_0 is unstable. Using the uniform persistence result from [15] and by a similar argument to that in the proof in [1], we can show that if $R_0 > 1$, the instability of P_0 implies the uniform persistence of system (30). This, together with the uniform boundedness of solutions of (30) in Γ°, implies (30) has at least a positive equilibrium P^* in Γ°. The proof is completed. □

Let

$$P^* = \left(S_1^{(1)*}, S_1^{(2)*}, \ldots, S_n^{(1)*}, S_n^{(2)*}, I_1^*, I_2^*, \ldots, I_n^*\right); \tag{37}$$

then the components of P^* satisfy

$$\varphi_k\left(S_k^{(1)*}\right) = \sum_{i=1}^{2} S_k^{(i)*} \sum_{j=1}^{n} \beta_{kj}^{(i)} I_j^* + d_k^{(2)} S_k^{(2)*}, \tag{38}$$

$$\sum_{i=1}^{2} c_k^{(i)} S_k^{(i)*} \sum_{j=1}^{n} \beta_{kj}^{(i)} I_j^* = m_k I_k^*, \tag{39}$$

$$a_k S_k^{(1)*} = d_k^{(2)} S_k^{(2)*} + S_k^{(2)*} \sum_{j=1}^{n} \beta_{kj}^{(2)} I_j^*, \quad k = 1, 2, \ldots, n. \tag{40}$$

Next, we will study the global stability of equilibria of system (21).

Theorem 7. *If $c_k^{(1)} \geq c_k^{(2)}$, $k = 1, 2, \ldots, n$, and $R_0 \leq 1$, then P_0 of system* (21) *is globally asymptotically stable in* Γ.

Proof. Let $\mathbf{S^0} = (S_{10}^{(1)}, S_{10}^{(2)}, \ldots, S_{n0}^{(1)}, S_{n0}^{(2)})$. Consider a Lyapunov functional $L_1 + L_2$, where

$$L_1 = \sum_{k=1}^{n} \frac{\omega_k}{m_k} \left[\sum_{i=1}^{2} c_k^{(i)} \left(S_k^{(i)} - S_{k0}^{(i)} - S_{k0}^{(i)} \ln \frac{S_k^{(i)}}{S_{k0}^{(i)}} \right) + I_k \right],$$
$$L_2 = \sum_{k=1}^{n} \frac{\omega_k}{m_k} \left[\sum_{i=1}^{2} \sum_{j=1}^{n} \beta_{kj}^{(i)} \int_0^h f_k^{(i)}(\tau) \int_{t-\tau}^{t} S_k^{(i)}(x) I_j(x)\, dx\, d\tau \right]. \tag{41}$$

Differentiating L_1 along the solution of system (21), we obtain

$$\begin{aligned}
\dot{L}_1 = \sum_{k=1}^{n} \frac{\omega_k}{m_k} \Bigg\{ & c_k^{(1)} \left[\varphi_k\left(S_k^{(1)}\right) - \sum_{j=1}^{n} \beta_{kj}^{(1)} S_k^{(1)} I_j - a_k S_k^{(1)} \right] \\
& + c_k^{(2)} \left[a_k S_k^{(1)} - \sum_{j=1}^{n} \beta_{kj}^{(2)} S_k^{(2)} I_j - d_k^{(2)} S_k^{(2)} \right] \\
& + \sum_{i=1}^{2} \sum_{j=1}^{n} \int_0^h f_k^{(i)}(\tau) \beta_{kj}^{(i)} S_{kj}^{(i)}(t-\tau) \\
& \times I_j(t-\tau)\, d\tau - m_k I_k \\
& - \frac{c_k^{(1)} S_{k0}^{(1)}}{S_k^{(1)}} \left[\varphi_k\left(S_k^{(1)}\right) - \sum_{j=1}^{n} \beta_{kj}^{(1)} S_k^{(1)} I_j - a_k S_k^{(1)} \right] \\
& - \frac{c_k^{(2)} S_{k0}^{(2)}}{S_k^{(2)}} \left[a_k S_k^{(1)} - \sum_{j=1}^{n} \beta_{kj}^{(2)} S_k^{(2)} I_j - d_k^{(2)} S_k^{(2)} \right] \Bigg\}.
\end{aligned} \tag{42}$$

Differentiating L_2 along the solution of system (21), we obtain

$$\begin{aligned}
L_2 = \sum_{k=1}^{n} \frac{\omega_k}{m_k} \Bigg[& \sum_{i=1}^{2} \sum_{j=1}^{n} c_k^{(i)} \beta_{kj}^{(i)} S_k^{(i)} I_j \\
& - \sum_{i=1}^{2} \sum_{j=1}^{n} \beta_{kj}^{(i)} \int_0^h f_k^{(i)}(\tau) S_k^{(i)}(t-\tau) I_j(t-\tau)\, d\tau \Bigg].
\end{aligned} \tag{43}$$

Therefore

$$\begin{aligned}
\dot{L} = \sum_{k=1}^{n} \frac{\omega_k}{m_k} \Bigg\{ & \varphi_k\left(S_k^{(1)}\right) \left(c_k^{(1)} - \frac{c_k^{(1)} S_{k0}^{(1)}}{S_k^{(1)}} \right) + d_k^{(2)} S_{k0}^{(2)} \\
& \times \left(c_k^{(2)} - \frac{c_k^{(2)} S_k^{(2)}}{S_{k0}^{(2)}} \right) + a_k S_{k0}^{(1)} \\
& \times \left[c_k^{(1)} - \frac{c_k^{(1)} S_k^{(1)}}{S_{k0}^{(1)}} + \frac{c_k^{(2)} S_k^{(1)}}{S_{k0}^{(1)}} - \frac{c_k^{(2)} S_k^{(1)} S_{k0}^{(2)}}{S_{k0}^{(1)} S_k^{(2)}} \right] \\
& + \sum_{i=1}^{2} \sum_{j=1}^{n} S_{k0}^{(i)} c_k^{(i)} \beta_{kj}^{(i)} I_j - m_k I_k \Bigg\} \\
= \sum_{k=1}^{n} \frac{\omega_k}{m_k} \Bigg\{ & \varphi_k\left(S_k^{(1)}\right) \left(c_k^{(1)} - \frac{c_k^{(1)} S_{k0}^{(1)}}{S_k^{(1)}} \right) \\
& + \sum_{i=1}^{2} \sum_{j=1}^{n} S_{k0}^{(i)} c_k^{(i)} \beta_{kj}^{(i)} I_j - m_k I_k \\
& + d_k^{(2)} S_{k0}^{(2)} \Bigg[c_k^{(1)} + c_k^{(2)} - \frac{c_k^{(1)} S_k^{(1)}}{S_{k0}^{(1)}} + \frac{c_k^{(2)} S_k^{(1)}}{S_{k0}^{(1)}} \\
& - \frac{c_k^{(2)} S_k^{(1)} S_{k0}^{(2)}}{S_{k0}^{(1)} S_k^{(2)}} - \frac{c_k^{(2)} S_k^{(2)}}{S_{k0}^{(2)}} \Bigg] \Bigg\}.
\end{aligned} \tag{44}$$

From (23), we know that

$$\varphi_k\left(S_{k0}^{(1)}\right) \left(c_k^{(1)} - \frac{c_k^{(1)} S_{k0}^{(1)}}{S_k^{(1)}} \right) = d_k^{(2)} S_{k0}^{(2)} \left(c_k^{(1)} - \frac{c_k^{(1)} S_{k0}^{(1)}}{S_k^{(1)}} \right), \quad k = 1, 2, \ldots, n. \tag{45}$$

By (45), we obtain

$$\begin{aligned}
\dot{L} = \sum_{k=1}^{n} \frac{\omega_k}{m_k} \Bigg\{ & \left[\varphi_k\left(S_k^{(1)}\right) - \varphi_k\left(S_{k0}^{(1)}\right) \right] \left(c_k^{(1)} - \frac{c_k^{(1)} S_{k0}^{(1)}}{S_k^{(1)}} \right) \\
& + \sum_{i=1}^{2} \sum_{j=1}^{n} S_{k0}^{(i)} c_k^{(i)} \beta_{kj}^{(i)} I_j - m_k I_k \\
& + d_k^{(2)} S_{k0}^{(2)} \Bigg[2c_k^{(1)} + c_k^{(2)} - \frac{c_k^{(1)} S_k^{(1)}}{S_{k0}^{(1)}} - \frac{c_k^{(1)} S_{k0}^{(1)}}{S_k^{(1)}} \\
& + \frac{c_k^{(2)} S_k^{(1)}}{S_{k0}^{(1)}} - \frac{c_k^{(2)} S_k^{(1)} S_{k0}^{(2)}}{S_{k0}^{(1)} S_k^{(2)}} - \frac{c_k^{(2)} S_k^{(2)}}{S_{k0}^{(2)}} \Bigg] \Bigg\}.
\end{aligned} \tag{46}$$

We can rewrite the equation as

$$\dot{L}=\sum_{k=1}^{n}\frac{\omega_k}{m_k}\left\{\left[\varphi_k\left(S_k^{(1)}\right)-\varphi_k\left(S_{k0}^{(1)}\right)\right]\left(c_k^{(1)}-\frac{c_k^{(1)}S_{k0}^{(1)}}{S_k^{(1)}}\right)+\sum_{i=1}^{2}\sum_{j=1}^{n}S_{k0}^{(i)}c_k^{(i)}\beta_{kj}^{(i)}I_j-m_kI_k+d_k^{(2)}S_{k0}^{(2)}\left[2c_k^{(1)}+c_k^{(2)}-\left(c_k^{(1)}-c_k^{(2)}\right)\times\left(\frac{S_k^{(1)}}{S_{k0}^{(1)}}+\frac{S_{k0}^{(1)}}{S_k^{(1)}}\right)-c_k^{(2)}\left(\frac{S_{k0}^{(1)}}{S_k^{(1)}}+\frac{S_k^{(1)}S_{k0}^{(2)}}{S_{k0}^{(1)}S_k^{(2)}}+\frac{S_k^{(2)}}{S_{k0}^{(2)}}\right)\right]\right\}. \tag{47}$$

By the fact that φ_k is strictly decreasing function and the arithmetic-geometric mean, we have

$$\dot{L}\le\sum_{k=1}^{n}\frac{\omega_k}{m_k}\left\{\sum_{i=1}^{2}\sum_{j=1}^{n}S_{k0}^{(i)}c_k^{(i)}\beta_{kj}^{(i)}I_j-m_kI_k+d_k^{(2)}S_{k0}^{(2)}\times\left[2c_k^{(1)}+c_k^{(2)}-2\left(c_k^{(1)}-c_k^{(2)}\right)-3c_k^{(2)}\right]\right\} \tag{48}$$

$$=\sum_{k=1}^{n}\frac{\omega_k}{m_k}\left(\sum_{i=1}^{2}\sum_{j=1}^{n}S_{k0}^{(i)}c_k^{(i)}\beta_{kj}^{(i)}I_j-m_kI_k\right)=:U,$$

where equality holds if and only if

$$S_k^{(1)}=S_{k0}^{(1)},\quad S_k^{(2)}=S_{k0}^{(2)},\quad k=1,2,\ldots,n. \tag{49}$$

Thus

$$U=\sum_{k=1}^{n}\omega_k\left[\frac{\sum_{i=1}^{2}c_k^{(i)}\sum_{j=1}^{n}\beta_{kj}^{(i)}S_{k0}^{(i)}I_j}{m_k}-I_k\right]=\left(\omega_1,\omega_2,\ldots,\omega_n\right)\left(\mathbf{Q}\mathbf{I}^T-\mathbf{I}^T\right). \tag{50}$$

If $R_0<1$, then $\dot{L}=0$ if and only if $\mathbf{I}^T=\mathbf{0}$. If $R_0=1$, then $\dot{L}=0$ implies $U=0$. If $R_0=1$ and $\dot{U}=0$, then (49) holds. If (49) holds, then, from the first two equations of (21), we may obtain $\sum_{j=1}^{n}\beta_{kj}^{(1)}S_k^{(1)}I_j\equiv 0$, $\sum_{j=1}^{n}\beta_{kj}^{(2)}S_k^{(2)}I_j\equiv 0$, $k=1,2\ldots,n$. Therefore, if $R_0=1$ and $\dot{U}=0$, then we have $I_k=0$. Therefore, $\dot{L}=0$ if and only if $\mathbf{I}=\mathbf{0}$ and $\mathbf{S}=\mathbf{S^0}$. Hence the largest invariant subset of the set where $\dot{L}=0$ is the singleton $\{P_0\}$. By LaSalle's Invariance Principle, P_0 is globally attractive. Using the same proof as the one for Corollary 5.3.1 in [16], we can show that P_0 is locally stable. Hence, the disease-free equilibrium P_0 is globally asymptotically stable in Γ for $R_0\le 1$. This completes the proof. □

Theorem 8. *Under Assumption 1, P^* of system (21) is globally asymptotically stable in Γ, if $c_k^{(1)}\ge c_k^{(2)}$, $k=1,2,\ldots,n$, and $R_0>1$.*

Proof. Set $\overline{\beta}_{kj}=\sum_{i=1}^{2}c_k^{(i)}\beta_{kj}^{(i)}S_k^{(i)*}I_j^*$, $1\le k,j\le n$. Consider a Lyapunov functional $V=V_1+V_2$, where

$$V_1=\sum_{k=1}^{n}v_k\left[\sum_{i=1}^{2}c_k^{(i)}\left(S_k^{(i)}-S_k^{(i)*}-S_k^{(i)*}\ln\frac{S_k^{(i)}}{S_k^{(i)*}}\right)+I_k-I_k^*-I_k^*\ln\frac{I_k}{I_k^*}\right],$$

$$V_2=\sum_{k=1}^{n}v_k\sum_{i=1}^{2}\sum_{j=1}^{n}\beta_{kj}^{(i)}\times\int_0^h f_k^{(i)}(\tau)\int_{t-\tau}^{t}\left(S_k^{(i)}(x)I_j(x)-S_k^{(i)*}I_j^*-S_k^{(i)*}I_j^*\times\ln\frac{S_k^{(i)}(x)I_j(x)}{S_k^{(i)*}I_j^*}\right)dx\,d\tau. \tag{51}$$

Differentiating V_1 along the solution of system (21), we obtain

$$\begin{aligned}\dot{V}_1=\sum_{k=1}^{n}v_k\Bigg\{&c_k^{(1)}\left[\varphi_k\left(S_k^{(1)}\right)-\sum_{j=1}^{n}\beta_{kj}^{(1)}S_k^{(1)}I_j-a_kS_k^{(1)}\right]+c_k^{(2)}\left[a_kS_k^{(1)}-\sum_{j=1}^{n}\beta_{kj}^{(2)}S_k^{(2)}I_j-d_k^{(2)}S_k^{(2)}\right]\\&+\sum_{i=1}^{2}\sum_{j=1}^{n}\int_0^h f_k^{(i)}(\tau)\beta_{kj}^{(i)}S_k^{(i)}(t-\tau)I_j(t-\tau)\,d\tau-m_kI_k\\&-\frac{c_k^{(1)}S_k^{(1)*}}{S_k^{(1)}}\left[\varphi_k\left(S_k^{(1)}\right)-\sum_{j=1}^{n}\beta_{kj}^{(1)}S_k^{(1)}I_j-a_kS_k^{(1)}\right]-\frac{c_k^{(2)}S_k^{(2)*}}{S_k^{(2)}}\left[a_kS_k^{(1)}-\sum_{j=1}^{n}\beta_{kj}^{(2)}S_k^{(2)}I_j-d_k^{(2)}S_k^{(2)}\right]\\&-\frac{I_k^*}{I_k}\left[\sum_{i=1}^{2}\sum_{j=1}^{n}\int_0^h f_k^{(i)}(\tau)\beta_{kj}^{(i)}S_k^{(i)}(t-\tau)\times I_j(t-\tau)\,d\tau-m_kI_k\right]\Bigg\}.\end{aligned} \tag{52}$$

Differentiating V_2 along the solution of system (21), we obtain

$$\dot{V}_2 = \sum_{k=1}^{n} v_k \sum_{i=1}^{2}\sum_{j=1}^{n}\beta_{kj}^{(i)} \times \left\{ c_k^{(i)} S_k^{(i)} I_j - \int_0^h f_k^{(i)}(\tau)\left[S_k^{(i)}(t-\tau) I_j(t-\tau) + S_k^{(i)*} I_j^* \times \ln \frac{S_k^{(i)} I_j}{S_k^{(i)}(t-\tau) I_j(t-\tau)} \right] d\tau \right\}. \quad (53)$$

Therefore

$$\dot{V} = \sum_{k=1}^{n} v_k \left\{ \varphi_k\left(S_k^{(1)}\right)\left(c_k^{(1)} - \frac{c_k^{(1)} S_k^{(1)*}}{S_k^{(1)}}\right) + d_k^{(2)} S_k^{(2)*}\left(c_k^{(2)} - \frac{c_k^{(2)} S_k^{(2)}}{S_k^{(2)*}}\right) + \sum_{i=1}^{2} c_k^{(i)} \sum_{j=1}^{n}\beta_{kj}^{(i)} S_k^{(i)*} I_j + m_k I_k^* - m_k I_k + a_k S_k^{(1)*} \times \left[c_k^{(1)} - \frac{c_k^{(1)} S_k^{(1)}}{S_k^{(1)*}} + \frac{c_k^{(2)} S_k^{(1)}}{S_k^{(1)*}} - \frac{c_k^{(2)} S_k^{(1)} S_k^{(2)*}}{S_k^{(1)*} S_k^{(2)}} \right] - \frac{I_k^*}{I_k}\left[\sum_{i=1}^{2}\sum_{j=1}^{n} \int_0^h f_k^{(i)}(\tau)\beta_{kj}^{(i)} S_k^{(i)} \times (t-\tau) I_j(t-\tau)\, d\tau \right] - \Delta \right\}, \quad (54)$$

where

$$\Delta = \sum_{i=1}^{2}\sum_{j=1}^{n}\beta_{kj}^{(i)} \int_0^h f_k^{(i)}(\tau) S_k^{(i)*} I_j^* \ln \frac{S_k^{(i)} I_j}{S_k^{(i)}(t-\tau) I_j(t-\tau)} d\tau. \quad (55)$$

From (38), we know that

$$\varphi_k\left(S_k^{(1)*}\right)\left(c_k^{(1)} - \frac{c_k^{(1)} S_k^{(1)*}}{S_k^{(1)}}\right) = \left(\sum_{i=1}^{2}\sum_{j=1}^{n}\beta_{kj}^{(i)} S_k^{(i)*} I_j^* + d_k^{(2)} S_k^{(2)*} \right)\left(c_k^{(1)} - \frac{c_k^{(1)} S_k^{(1)*}}{S_k^{(1)}}\right). \quad (56)$$

It follows from (39), (40), and (56) that

$$\dot{V} = \sum_{k=1}^{n} v_k \left\{ \left[\varphi_k\left(S_k^{(1)}\right) - \varphi_k\left(S_k^{(1)*}\right)\right]\left(c_k^{(1)} - \frac{c_k^{(1)} S_k^{(1)*}}{S_k^{(1)}}\right) + d_k^{(2)} S_k^{(2)*}\left(c_k^{(2)} - \frac{c_k^{(2)} S_k^{(2)}}{S_k^{(2)*}}\right) + \sum_{i=1}^{2} c_k^{(i)}\beta_{kj}^{(i)} S_k^{(i)*} I_j^* \left(\frac{I_j}{I_j^*} - \frac{I_k}{I_k^*} + 1\right) + S_k^{(1)*}\sum_{j=1}^{n}\beta_{kj}^{(1)} I_j^* \left(c_k^{(1)} - \frac{c_k^{(1)} S_k^{(1)*}}{S_k^{(1)}}\right) + \left(d_k^{(2)} S_k^{(2)*} + S_k^{(2)*}\sum_{j=1}^{n}\beta_{kj}^{(2)} I_j^*\right) \times \left[2c_k^{(1)} - \frac{c_k^{(1)} S_k^{(1)*}}{S_k^{(1)}} - \frac{c_k^{(1)} S_k^{(1)}}{S_k^{(1)*}} + \frac{c_k^{(2)} S_k^{(1)}}{S_k^{(1)*}} - \frac{c_k^{(2)} S_k^{(1)} S_k^{(2)*}}{S_k^{(1)*} S_k^{(2)}} \right] - \frac{I_k^*}{I_k} \times \left[\sum_{i=1}^{2}\sum_{j=1}^{n}\int_0^h f_k^{(i)}(\tau)\beta_{kj}^{(i)} S_k^{(i)}(t-\tau) I_j(t-\tau)\, d\tau \right] - \Delta \right\}. \quad (57)$$

By Theorem 2 and the fact that φ_k is strictly decreasing function, we obtain

$$\dot{V} \le \sum_{k=1}^{n} v_k \left\{ d_k^{(2)} S_k^{(2)*}\left(c_k^{(2)} - \frac{c_k^{(2)} S_k^{(2)}}{S_k^{(2)*}}\right) + \sum_{i=1}^{2} c_k^{(i)}\beta_{kj}^{(i)} S_k^{(i)*} I_j^* + S_k^{(1)*}\sum_{j=1}^{n}\beta_{kj}^{(1)} I_j^* \left(c_k^{(1)} - \frac{c_k^{(1)} S_k^{(1)*}}{S_k^{(1)}}\right) + \left(d_k^{(2)} S_k^{(2)*} + S_k^{(2)*}\sum_{j=1}^{n}\beta_{kj}^{(2)} I_j^*\right) \times \left[2c_k^{(1)} - \frac{c_k^{(1)} S_k^{(1)*}}{S_k^{(1)}} - \frac{c_k^{(1)} S_k^{(1)}}{S_k^{(1)*}} + \frac{c_k^{(2)} S_k^{(1)}}{S_k^{(1)*}} - \frac{c_k^{(2)} S_k^{(1)} S_k^{(2)*}}{S_k^{(1)*} S_k^{(2)}} \right]$$

$$
\begin{aligned}
&-\frac{I_k^*}{I_k}\left[\sum_{i=1}^{2}\sum_{j=1}^{n}\int_0^h f_k^{(i)}(\tau)\,\beta_{kj}^{(i)}S_k^{(i)}\right.\\
&\qquad\left.\times(t-\tau)\,I_j(t-\tau)\,d\tau\right]-\Delta\Bigg\}\\
&=\sum_{k=1}^{n}v_k\Bigg\{d_k^{(2)}S_k^{(2)*}\Bigg(c_k^{(2)}+2c_k^{(1)}-\frac{c_k^{(1)}S_k^{(1)*}}{S_k^{(1)}}-\frac{c_k^{(1)}S_k^{(1)}}{S_k^{(1)*}}\\
&\qquad+\frac{c_k^{(2)}S_k^{(1)}}{S_k^{(1)*}}-\frac{c_k^{(2)}S_k^{(1)}S_k^{(2)*}}{S_k^{(1)*}S_k^{(2)}}-\frac{c_k^{(2)}S_k^{(2)}}{S_k^{(2)*}}\Bigg)\\
&\quad+S_k^{(2)*}\sum_{j=1}^{n}\beta_{kj}^{(2)}I_j^*\left[2c_k^{(1)}+c_k^{(2)}-\frac{c_k^{(1)}S_k^{(1)*}}{S_k^{(1)}}-\frac{c_k^{(1)}S_k^{(1)}}{S_k^{(1)*}}\right.\\
&\qquad\left.+\frac{c_k^{(2)}S_k^{(1)}}{S_k^{(1)*}}-\frac{c_k^{(2)}S_k^{(1)}S_k^{(2)*}}{S_k^{(1)*}S_k^{(2)}}\right]\\
&\quad+S_k^{(1)*}\sum_{j=1}^{n}\beta_{kj}^{(1)}I_j^*\left(2c_k^{(1)}-\frac{c_k^{(1)}S_k^{(1)*}}{S_k^{(1)}}\right)\\
&\quad-\frac{I_k^*}{I_k}\left[\sum_{i=1}^{2}\sum_{j=1}^{n}\int_0^h f_k^{(i)}(\tau)\,\beta_{kj}^{(i)}S_k^{(i)}\right.\\
&\qquad\left.\times(t-\tau)\,I_j(t-\tau)\,d\tau\right]-\Delta\Bigg\}\\
&=\sum_{k=1}^{n}v_k\Bigg\{d_k^{(2)}S_k^{(2)*}\Bigg[c_k^{(2)}+2c_k^{(1)}-\left(c_k^{(1)}-c_k^{(2)}\right)\\
&\qquad\times\left(\frac{S_k^{(1)*}}{S_k^{(1)}}+\frac{S_k^{(1)}}{S_k^{(1)*}}\right)\\
&\qquad-c_k^{(2)}\left(\frac{S_k^{(1)*}}{S_k^{(1)}}+\frac{S_k^{(1)}S_k^{(2)*}}{S_k^{(1)*}S_k^{(2)}}+\frac{S_k^{(2)}}{S_k^{(2)*}}\right)\Bigg]+S_k^{(2)*}\\
&\quad\times\sum_{j=1}^{n}\beta_{kj}^{(2)}I_j^*\Bigg[2c_k^{(1)}+c_k^{(2)}-\left(c_k^{(1)}-c_k^{(2)}\right)\\
&\qquad\times\left(\frac{S_k^{(1)*}}{S_k^{(1)}}+\frac{S_k^{(1)}}{S_k^{(1)*}}\right)\\
&\qquad-\int_0^h f_k^{(2)}(\tau)\\
&\qquad\times\Bigg(\frac{S_k^{(1)*}}{S_k^{(1)}}+\frac{S_k^{(1)}S_k^{(2)*}}{S_k^{(1)*}S_k^{(2)}}\\
&\qquad+\Big(\Big(S_k^{(2)}I_k^*S_k^{(2)}\\
&\qquad\times(t-\tau)\,I_j(t-\tau)\Big)\\
&\qquad\times\left(S_k^{(2)*}S_k^{(2)}I_kI_j^*\right)^{-1}\Big)\Bigg)d\tau\Bigg]\\
&\quad+S_k^{(1)*}\sum_{j=1}^{n}\beta_{kj}^{(1)}I_j^*\\
&\quad\times\Bigg[2c_k^{(1)}\\
&\qquad-\int_0^h f_k^{(1)}(\tau)\\
&\qquad\times\Bigg(\frac{S_k^{(1)*}}{S_k^{(1)}}\\
&\qquad+\Big(\Big(S_k^{(1)}I_k^*S_k^{(1)}(t-\tau)\,I_j(t-\tau)\Big)\\
&\qquad\times\left(S_k^{(1)*}S_k^{(1)}I_kI_j^*\right)^{-1}\Big)\Bigg)d\tau\Bigg]\\
&\quad-\Delta\Bigg\}=:B_1.
\end{aligned}
\tag{58}
$$

By the arithmetic-geometric mean, we easily see that

$$
\begin{aligned}
B_1&\le\sum_{k=1}^{n}v_k\Bigg\{d_k^{(2)}S_k^{(2)*}\left[c_k^{(2)}+2c_k^{(1)}-2\left(c_k^{(1)}-c_k^{(2)}\right)-3c_k^{(2)}\right]\\
&\quad+S_k^{(2)*}\sum_{j=1}^{n}\beta_{kj}^{(2)}I_j^*\Bigg[2c_k^{(1)}+c_k^{(2)}-2\left(c_k^{(1)}-c_k^{(2)}\right)\\
&\qquad-3\int_0^h f_k^{(2)}(\tau)\\
&\qquad\times\left(\frac{I_k^*S_k^{(2)}(t-\tau)\,I_j(t-\tau)}{S_k^{(2)}I_kI_j^*}\right)^{1/3}d\tau\Bigg]\\
&\quad+S_k^{(1)*}\sum_{j=1}^{n}\beta_{kj}^{(1)}I_j^*\\
&\quad\times\Bigg[2c_k^{(1)}\\
&\qquad-2\int_0^h f_k^{(1)}(\tau)\\
&\qquad\times\left(\frac{I_k^*S_k^{(1)}(t-\tau)\,I_j(t-\tau)}{S_k^{(1)}I_kI_j^*}\right)^{1/2}d\tau\Bigg]\\
&\quad-\Delta\Bigg\}\\
&=\sum_{k=1}^{n}v_k\Bigg\{3S_k^{(2)*}\sum_{j=1}^{n}\beta_{kj}^{(2)}I_j^*\\
&\quad\times\Bigg[c_k^{(2)}\\
&\qquad-\int_0^h f_k^{(2)}(\tau)
\end{aligned}
$$

$$\times\left(\frac{I_k^* S_k^{(2)}(t-\tau)I_j(t-\tau)}{S_k^{(2)} I_k I_j^*}\right)^{1/3} d\tau\Bigg]$$
$$+ 2S_k^{(1)*}$$
$$\times \sum_{j=1}^{n} \beta_{kj}^{(1)} I_j^* \Bigg[c_k^{(1)}$$
$$- \int_0^h f_k^{(1)}(\tau)$$
$$\times\left(\frac{I_k^* S_k^{(1)}(t-\tau) I_j(t-\tau)}{S_k^{(1)} I_k I_j^*}\right)^{1/2} d\tau \Bigg]$$
$$-\Delta\Bigg\} =: B_2. \tag{59}$$

Let $Y(x) = 1 - x + \ln x$. We can rewrite B_2 as

$$\begin{aligned}
&\sum_{k=1}^{n} \nu_k \Bigg\{ 3S_k^{(2)*} \sum_{j=1}^{n} \beta_{kj}^{(2)} I_j^* \\
&\quad\times \int_0^h f_k^{(2)}(\tau) Y \\
&\quad\times \left(\left[\frac{I_k^* S_k^{(2)}(t-\tau) I_j(t-\tau)}{S_k^{(2)} I_k I_j^*} \right]^{1/3} \right) d\tau \\
&\quad + 2S_k^{(1)*} \sum_{j=1}^{n} \beta_{kj}^{(1)} I_j^* \\
&\quad\times \int_0^h f_k^{(1)}(\tau) Y \\
&\quad\times \left(\left[\frac{I_k^* S_k^{(1)}(t-\tau) I_j(t-\tau)}{S_k^{(1)} I_k I_j^*} \right]^{1/2} \right) d\tau \\
&\quad - \sum_{i=1}^{2}\sum_{j=1}^{n} \beta_{kj}^{(i)} \\
&\quad\times \int_0^h f_k^{(i)}(\tau) S_k^{(i)*} I_j^* \\
&\quad\times \ln \frac{I_k^* S_k^{(i)}(t-\tau) I_j(t-\tau)}{I_k I_j^* S_k^{(i)}} d\tau - \Delta \Bigg\}.
\end{aligned} \tag{60}$$

By Theorem 2 and the fact that $Y(x) \le 0$, where equality holds if and only if $x = 1$, we obtain

$$\begin{aligned}
B_2 &\le \sum_{k=1}^{n} \nu_k \Bigg\{ -\sum_{i=1}^{2}\sum_{j=1}^{n} \beta_{kj}^{(i)} \int_0^h f_k^{(i)}(\tau) S_k^{(i)*} I_j^* \\
&\qquad \times \ln \frac{I_k^* S_k^{(i)}(t-\tau) I_j(t-\tau)}{I_k I_j^* S_k^{(i)}} d\tau - \Delta \Bigg\} \\
&= -\sum_{k=1}^{n} \nu_k \sum_{i=1}^{2}\sum_{j=1}^{n} c_k^{(i)} \beta_{kj}^{(i)} S_k^{(i)*} I_j^* \ln \frac{I_k^* I_j}{I_k I_j^*} \\
&= -\sum_{k,j=1}^{n} \nu_k \overline{\beta}_{kj} \ln \frac{I_k^* I_j}{I_k I_j^*} = 0.
\end{aligned} \tag{61}$$

From (58) and (59), we see that if $\dot{V} = 0$, then

$$S_k^{(i)} = S_k^{(i)*}, \quad i = 1,2, \ k = 1,2,\ldots,n. \tag{62}$$

If (62) holds, it follows from (21) that

$$\begin{aligned}
0 &= \varphi_k\left(S_k^{(1)*}\right) - \sum_{j=1}^{n} \beta_{kj}^{(1)} S_k^{(1)*} I_j - a_k S_k^{(1)*}, \\
0 &= a_k S_k^{(1)*} - \sum_{j=1}^{n} \beta_{kj}^{(2)} S_k^{(2)*} I_j - d_k^{(2)} S_k^{(2)*}, \quad k = 1,2,\ldots,n.
\end{aligned} \tag{63}$$

Then, we obtain that

$$\begin{aligned}
\sum_{j=1}^{n} \beta_{kj}^{(1)} S_k^{(1)*} I_j &= \varphi_k\left(S_k^{(1)*}\right) - a_k S_k^{(1)*}, \\
\sum_{j=1}^{n} \beta_{kj}^{(2)} S_k^{(2)*} I_j &= a_k S_k^{(1)*} - d_k^{(2)} S_k^{(2)*}, \quad k = 1,2,\ldots,n.
\end{aligned} \tag{64}$$

Thus,

$$\begin{aligned}
\dot{I}_k &= c_k^{(1)}\left(\varphi_k\left(S_k^{(1)*}\right) - a_k S_k^{(1)*}\right) \\
&\quad + c_k^{(2)}\left(a_k S_k^{(1)*} - d_k^{(2)} S_k^{(2)*}\right) - m_k I_k, \\
&\qquad k = 1,2,\ldots,n.
\end{aligned} \tag{65}$$

This implies that

$$\begin{aligned}
&\lim_{t\to+\infty} I_k \\
&\quad = \frac{c_k^{(1)}\left(\varphi_k\left(S_k^{(1)*}\right) - a_k S_k^{(1)*}\right) + c_k^{(2)}\left(a_k S_k^{(1)*} - d_k^{(2)} S_k^{(2)*}\right)}{m_k} \\
&\quad = I_k^*.
\end{aligned} \tag{66}$$

Hence, the largest invariant subset of the set where $\dot{V} = 0$ is the singleton $\{P^*\}$. By LaSalle's Invariance Principle, P^* is globally attractive. By a similar argument to that in the proof of Theorem 7, P^* is globally asymptotically stable in Γ for $R_0 > 1$. □

Following [17], we set matrices

$$\mathbf{F} := \left(\sum_{i=1}^{2} c_k^{(i)} \beta_{kj}^{(i)} S_{k0}^{(i)} \right)_{n\times n}, \qquad \mathbf{V} := \operatorname{diag}\left(m_1, m_2, \ldots, m_n\right). \tag{67}$$

The next generation matrix for system (16) is

$$\mathbf{Q} := \mathbf{F}\mathbf{V}^{-1} = \left(\frac{\sum_{i=1}^{2} c_k^{(i)} \beta_{kj}^{(i)} S_{k0}^{(i)}}{m_k} \right)_{n\times n}. \tag{68}$$

Let $r_0 = \rho(\mathbf{Q})$, where ρ denotes the spectral radius.

By Lemma 6 and Theorems 2, 4, 7, and 8, we can obtain the following results.

Corollary 9. *Assume that* $\mathbf{B} = [\sum_{i=1}^{2} \beta_{kj}^{(i)}]$ *is irreducible and* $c_k^{(1)} \geq c_k^{(2)}$, $k = 1, 2, \ldots, n$. *If* $r_0 \leq 1$, *then* P_0 *of system* (21) *is globally asymptotically stable in* Γ.

Corollary 10. *Assume that* $\mathbf{B} = [\sum_{i=1}^{2} \beta_{kj}^{(i)}]$ *is irreducible and* $c_k^{(1)} \geq c_k^{(2)}$, $k = 1, 2, \ldots, n$. *If* $r_0 > 1$, *then* P^* *of system* (21) *is globally asymptotically stable in* Γ.

Remark 11. Note that, for Theorems 7 and 8, we do not assume that $\mathbf{B} = [\sum_{i=1}^{2} \beta_{kj}^{(i)}]$ is irreducible. If Theorems 7 and 8 hold, then Corollaries 9 and 10 hold too, respectively. However, it is easy to show some examples which conform to the conditions of Theorem 7 or Theorem 8 but are not in accord with the conditions of Corollary 9 or Corollary 10. It seems that all authors use the the graph theory under the assumption that the relative matrix is irreducible which is analogous with the conditions of Corollaries 9 and 10 (see, e.g., [1–13]).

Acknowledgments

This work was supported by the National Basic Research Program of China (2010CB732501), and the National Natural Science Foundation of China (61273015).

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