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Research Article

A Note on the Tail Behavior of Randomly Weighted Sums with Convolution-Equivalently Distributed Random Variables

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We investigate the tailed asymptotic behavior of the randomly weighted sums with increments with convolution-equivalent distributions. Our obtained result can be directly applied to a discrete-time insurance risk model with insurance and financial risks and derive the asymptotics for the finite-time probability of the above risk model.

1. Introduction and Main Result

Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed (i.i.d.) real-valued random variables with common distribution F, and let $\{Y_n, n \geq 1\}$ be another sequence of i.i.d. nonnegative r.v.s with common distribution G and right endpoint $\widehat{x}_G = \sup\{x : \mathbb{P}(Y_1 \leq x) < 1\}$. Assume that $\{X_n, n \geq 1\}$ are independent of $\{Y_n, n \geq 1\}$. In this paper, we are interested in the randomly weighted sum

$$S_n^Y = \sum_{i=1}^{n} X_i \prod_{j=1}^{i} Y_j, \quad n \geq 1. \tag{1}$$

This is because the study for the tail probability $\mathbb{P}(S_n^Y > x)$ can be directly applied to risk theory. Consider a discrete-time insurance risk model. Within period i, $i \geq 1$, the net insurance loss is denoted by a real-valued (r.v.) X_i. The insurer makes both risk-free and risky investments, leading to an overall stochastic discounted factor Y_i from time i to time $i-1$. In the terminology of Norberg [1], the sequences $\{X_n, n \geq 1\}$ and $\{Y_n, n \geq 1\}$ are called the insurance and financial risks, respectively. Then, the randomly weighted sum S_n^Y in (1) represents the stochastic discounted value of aggregate net losses up to time n, $n \geq 1$. As usual, the probability of ruin by time n can be defined by

$$\Psi(x, n) = \mathbb{P}\left(\max_{1 \leq m \leq n} \sum_{i=1}^{m} X_i \prod_{j=1}^{i} Y_j > x\right), \quad n \geq 1, \tag{2}$$

where $x \geq 0$ is interpreted as the initial capital reserve of an insurance company. Clearly, for each $n \geq 1$,

$$\mathbb{P}\left(S_n^Y > x\right) \leq \Psi(x, n) \leq \mathbb{P}\left(\sum_{i=1}^{n} X_i^+ \prod_{j=1}^{i} Y_j > x\right), \tag{3}$$

where $X_i^+ = X_i \mathbf{1}_{\{X_i \geq 0\}}$ denotes the positive part of X_i, $i \geq 1$. If we can establish an asymptotic formula for $\mathbb{P}(S_n^Y > x)$ while doing so does not require $F(0-) > 0$, then the same asymptotic formula should hold for the right-hand side of (3) as well. In this way the ruin probability $\Psi(x, n)$ has the same asymptotic behavior as that of the tail probability $\mathbb{P}(S_n^Y > x)$ as x tends to infinity.

There has been a vast amount of literature studying the asymptotic behavior of the tail probability of the randomly weighted sum S_n^Y. Many works have considered the heavy-tailed case; that is, the distribution F of X belongs to some classes of heavy-tailed distributions, even under some dependence structures. For example, one can refer to Tang and Tsitsiashvili [2, 3], Wang and Tang [4], Zhang et al. [5], Shen et al. [6], Chen and Yuen [7], Gao and Wang [8], and Yi et al. [9] among others for some details in this direction, where the distribution F is heavily heavy tailed; as for some lightly heavy-tailed distribution F, some related results were obtained by Tang and Tsitsiashvili [3, 10], Chen and Su [11], Hashorva et al. [12], Yang et al. [13], Yang and Hashorva [14], and Yang and Wang [15] among others. We pointed out that Tang and Tsitsiashvili [3] achieved some interesting results on the asymptotics for the tail probability $\mathbb{P}(S_n^Y > x)$ in some cases where F belongs to the intersection between the subexponential distribution class and the rapidly varying distribution class.

In this paper, we aim to consider the light-tailed case, more exactly, to investigate the asymptotic behavior of the tail probability of the randomly weighted sums with increments with convolution-equivalent distributions.

Hereafter, all the limit relationships hold for x tending to infinity. For two positive functions $a(x)$ and $b(x)$, we write $a(x) \sim b(x)$ if $\lim a(x)/b(x) = 1$; write $a(x) = o(b(x))$ if $\lim a(x)/b(x) = 0$; and write $a(x) = O(b(x))$ if $\limsup a(x)/b(x) < \infty$.

Firstly we introduce some definitions on some classes of convolution-equivalent distributions. A distribution V on $[0,\infty)$ belongs to the class of convolution-equivalent distributions, denoted by $\mathcal{S}(\gamma)$, $\gamma \geq 0$, if for any $y \in \mathbb{R}$,

$$\lim \frac{\overline{V}(x-y)}{\overline{V}(x)} = e^{\gamma y}, \tag{4}$$

$$\lim \frac{\overline{V^{*2}}(x)}{\overline{V}(x)} = 2\int_0^\infty e^{\gamma u} V(du) < \infty, \tag{5}$$

where V^{*2} denotes the convolution of V with itself. More generally, a distribution V on $\mathbb{R}$ belongs to the class $\mathcal{S}(\gamma)$, $\gamma \geq 0$, if and only if its right-hand distribution $V^+(x) = V(x)\mathbf{1}_{\{x\geq 0\}}$ belongs to this class; see Corollary 2.1 of Pakes [16]. The class $\mathcal{S} := \mathcal{S}(0)$ is called the class of subexponential distributions. A distribution V on $\mathbb{R}$ belongs to the class $\mathcal{L}(\gamma)$, $\gamma \geq 0$ if only relation (4) holds. In the case $\gamma = 0$, we say that $\mathcal{L} := \mathcal{L}(0)$ is the class of long-tailed distributions. Similarly, a positive function $f(\cdot)$ is said to be long tailed if $\lim f(x-y)/f(x) = 1$ for any $y \in \mathbb{R}$. Clearly, if a distribution $V \in \mathcal{L}$, then its tail probability $\overline{V}(x)$ is long tailed. Closely related is the class $\mathcal{A}$, which was introduced by Konstantinides et al. [17]. A distribution V on $\mathbb{R}$ belongs to the class $\mathcal{A}$ if V is subexponential, and, for some $y > 1$,

$$\limsup \frac{\overline{V}(xy)}{\overline{V}(x)} < 1. \tag{6}$$

Clearly, all distributions in the classes $\mathcal{A}$, $\mathcal{S}$, and $\mathcal{L}$ are heavy tailed. A distribution V of r.v. ξ is said to be heavy tailed if $\mathbb{E}e^{s\xi} = \infty$ for any $s > 0$; otherwise it is said to be light tailed.

For each $n \geq 1$, denote the distribution of $X_n \prod_{j=1}^n Y_j$ by H_n, by convention, $H = H_1$. Now we state our main result as follows.

Theorem 1. *If $F \in \mathcal{S}(\gamma)$ for some $\gamma > 0$, $\hat{x}_G = \infty$, and, for all $u > 0$,*

$$\overline{G}(ux) = o\left(\overline{H}(x)\right), \tag{7}$$

then, for each $n \geq 1$,

$$\mathbb{P}\left(S_n^Y > x\right) \sim \overline{H_n}(x). \tag{8}$$

Remark 2. We remark that Tang [18] considered a similar result for $0 < \hat{x}_G < \infty$, whereas Theorem 1 deals with the case with $\hat{x}_G = \infty$ for a complement.

Remark 3. In Theorem 1, relation (7) is a mild condition. According to Corollary 1.1 of Tang [19], relation (7) can be further implied by either

(a) $\overline{G}(vx) = o(\overline{G}(x))$ for some $v > 1$ or

(b) $\overline{G}(vx) = o(\overline{F}(x))$ for some $v > 0$.

2. Proof of the Main Result

We start this section by a series of lemmas. The first two lemmas are due to Lemma 3.2 and Theorem 2.1 of Tang [20].

Lemma 4. *For two distributions G and H with $\overline{G}(x) > 0$ and $\overline{H}(x) > 0$ for all $x \geq 0$, relation (7) holds for each $u > 0$, if and only if there is a nonnegative function $a(\cdot)$ such that*

$$a(x) \nearrow \infty, \qquad \frac{a(x)}{x} \searrow 0, \qquad \overline{G}(a(x)) = o\left(\overline{H}(x)\right). \tag{9}$$

Lemma 5. *Consider the product XY. The distribution H of the product belongs to the class $\mathcal{A}$ if and only if $F \in \mathcal{A}$ and relation (7) holds for all $u > 0$.*

Tang [19] obtained an interesting result to show that a light-tailed random variable can be transferred into a heavy-tailed one through multiplier.

Lemma 6. *Consider the product XY with $F \in \mathcal{S}(\gamma)$ for some $\gamma > 0$ and $\hat{x}_G = \infty$. If relation (7) holds for all $u > 0$, then $H \in \mathcal{S}$.*

The last lemma can be found in, for example, Theorem 3.14 of Foss et al. [21].

Lemma 7. *Let a reference distribution V on $\mathbb{R}$ belong to the class $\mathcal{S}$. Assume that distributions $V_1, \ldots, V_n$ on $\mathbb{R}$ satisfy that, for each $i = 1, \ldots, n$, the function $\overline{V} + \overline{V_i}$ is long tailed and $\overline{V_i}(x) = O(\overline{V}(x))$. Then, it holds that*

$$\overline{V_1 * \cdots * V_n}(x) = \overline{V_1}(x) + \cdots + \overline{V_n}(x) + o\left(\overline{V}(x)\right). \tag{10}$$

Proof of Theorem 1. Now we begin to prove the main result of Theorem 1.

For each $n \geq 1$, write

$$T_n := \sum_{i=1}^n X_i \prod_{j=i}^n Y_j \stackrel{d}{=} S_n^Y, \tag{11}$$

where $\overset{d}{=}$ stands for equality in distribution. Since $F \in \mathcal{S}(\gamma) \subset \mathcal{L}(\gamma)$, $\gamma > 0$, its tail distribution $\overline{F}$ is rapidly varying in the sense that

$$\lim \frac{\overline{F}(xy)}{\overline{F}(x)} = 0, \quad \forall y > 1. \tag{12}$$

By Lemma 6, we get that $H \in \mathcal{S}$. Further, by Lemma 4, there exists a nonnegative function $a(\cdot)$ such that (9) holds. Thus, by (9) and (12), for any $y > 1$,

$$\begin{aligned}
&\limsup \frac{\overline{H^+}(xy)}{\overline{H^+}(x)} \\
&\quad = \limsup \frac{\left(\int_0^{a(x)} + \int_{a(x)}^{\infty}\right)\overline{F}(xy/u)\,G(du)}{\overline{H^+}(x)} \\
&\quad \le \limsup \frac{\int_0^{a(x)} \overline{F}(xy/u)\,G(du)}{\int_0^{a(x)} \overline{F}(x/u)\,G(du)} + \limsup \frac{\overline{G}(a(x))}{\overline{H^+}(x)} \\
&\quad \le \limsup \sup_{0 \le u \le a(x)} \frac{\overline{F}(xy/u)}{\overline{F}(x/u)} \\
&\quad = \limsup \frac{\overline{F}(xy)}{\overline{F}(x)} = 0 < 1,
\end{aligned} \tag{13}$$

which, together with $H \in \mathcal{S}$, implies that $H \in \mathcal{A}$.

We proceed to prove relation (8) by induction on n. Trivially, the distribution $H_1 = H$ of T_1 or S_1 belongs to the class $\mathcal{A}$, and relation (8) holds for $n = 1$. Assume that $H_n \in \mathcal{A}$ and (8) holds for n. We aim to prove that $H_{n+1} \in \mathcal{A}$ and (8) holds for $n + 1$, which, by (11), is equivalent to

$$\mathbb{P}(T_{n+1} > x) \sim \overline{H_{n+1}}(x). \tag{14}$$

First of all, according to Lemma 2.17 of Foss et al. [21] and $H_n \in \mathcal{A} \subset \mathcal{L}$, we have, that for any $s > 0$,

$$\lim \overline{H_n}(x)\,e^{sx} = \infty, \tag{15}$$

which, together with $F \in \mathcal{S}(\gamma)$, $\gamma > 0$, implies that

$$\overline{F}(x) = o\left(\overline{H_n}(x)\right). \tag{16}$$

By (16) and $H_n \in \mathcal{A} \subset \mathcal{L}$, we have that for any $y \in \mathbb{R}$,

$$\begin{aligned}
&\lim \frac{\overline{H_n}(x-y) + \overline{F}(x-y)}{\overline{H_n}(x) + \overline{F}(x)} \\
&\quad = \lim \left(\left(\frac{\overline{H_n}(x-y)}{\overline{H_n}(x)}\right) + \left(\frac{\overline{F}(x-y)}{\overline{H_n}(x-y)}\right)\right. \\
&\qquad \left. \cdot \left(\frac{\overline{H_n}(x-y)}{\overline{H_n}(x)}\right)\right) \times \left(1 + \left(\frac{\overline{F}(x)}{\overline{H_n}(x)}\right)\right)^{-1} \\
&\quad = 1,
\end{aligned} \tag{17}$$

which shows that the function $\overline{H_n}(x) + \overline{F}(x)$ is long tailed. Since T_n and X_{n+1} are independent of each other, thus, by $H_n \in \mathcal{A} \subset \mathcal{S}$, we can apply Lemma 7 to derive from the induction assumption and (16) that

$$\begin{aligned}
\mathbb{P}(T_n + X_{n+1} > x) &= \overline{H_n}(x) + \overline{F}(x) + o\left(\overline{H_n}(x)\right) \\
&= (1 + o(1))\,\overline{H_n}(x).
\end{aligned} \tag{18}$$

For the above-mentioned nonnegative function $a(\cdot)$, from (9) and (18), we obtain that

$$\begin{aligned}
&\mathbb{P}(T_{n+1} > x) \\
&\quad = \mathbb{P}\left((T_n + X_{n+1})\,Y_{n+1} > x\right) \\
&\quad = \left(\int_0^{a(x)} + \int_{a(x)}^{\infty}\right) \mathbb{P}\left(T_n + X_{n+1} > \frac{x}{u}\right) G(du) \\
&\quad = (1 + o(1)) \int_0^{a(x)} \overline{H_n}\left(\frac{x}{u}\right) G(du) + o\left(\overline{H}(x)\right) \\
&\quad = (1 + o(1))\,\overline{H_{n+1}}(x),
\end{aligned} \tag{19}$$

where the last step used the fact that $\overline{H}(x) = O(\overline{H_{n+1}}(x))$, because, for any $x \ge 0$,

$$\begin{aligned}
\overline{H_{n+1}}(x) &\ge \mathbb{P}\left(X_{n+1} \prod_{j=1}^{n+1} Y_j > x, Y_1 > 1, \ldots, Y_n > 1\right) \\
&\ge \overline{H}(x)\left(\overline{G}(1)\right)^n,
\end{aligned} \tag{20}$$

and $\overline{G}(1) > 0$ by $\hat{x}_G = \infty$. Relation (19) means that (14) holds.

Finally, by (9) and (20), we have that

$$\overline{G}(a(x)) = o\left(\overline{H_{n+1}}(x)\right), \tag{21}$$

from which and $H_n \in \mathcal{A}$, Lemma 5 gives that $H_{n+1} \in \mathcal{A}$.

This completes the proof of Theorem 1. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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