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Research Article

On Uniform Exponential Stability and Exact Admissibility of Discrete Semigroups

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We prove that a discrete semigroup $\mathbb{T} = \{T(n) : n \in \mathbb{Z}_+\}$ of bounded linear operators acting on a complex Banach space X is uniformly exponentially stable if and only if, for each $x \in AP_0(\mathbb{Z}_+, X)$, the sequence $n \mapsto \sum_{k=0}^{n} T(n-k)x(k) : \mathbb{Z}_+ \to X$ belongs to $AP_0(\mathbb{Z}_+, X)$. Similar results for periodic discrete evolution families are also stated.

1. Introduction

The solutions of the autonomous discrete systems $x_{n+1} = Ax_n$ or $y_{n+1} = Ay_n + h_n$ lead to the idea of discrete semigroups. There are a lot of spectral criteria which characterize different types of stability (or other types of asymptotic behavior) of the solutions of above systems. For further results on asymptotic behavior of semigroups, we refer to [1].

New difficulties appear in the study of the nonautonomous systems, especially because the part of the solution generated by the forced term (h_n), that is, $\sum_{k=\nu}^{n} U(n,k)h_k$, is not a convolution in the classical sense. These difficulties may be passed by using the so-called evolution semigroups.

The evolution semigroups were exhaustively studied in [2]. Having in mind the well-known results stated in the continuous case, see for example [2, 3], we can say that this method is a very efficient one. See also [4, 5] for recent developments concerning the semigroups of evolution acting on almost periodic function spaces.

Recently, the discrete version of [6] was obtained in [7].

In this note, we study the asymptotic behavior of the discrete semigroups in terms of exact admissibility of the space of almost periodic sequences.

In this regard, we develop the theory of discrete evolution semigroups on a special space of bounded sequences. Results of this type in the continuous case may be found in [8] and the references therein. However, by contrast with the continuous case, we did not find in the existent literature papers written in the spirit of the present one referring to the discrete evolution semigroups. These results could be new and useful for people whose area of research is restricted to difference equations.

2. Definitions and Preliminary Results

Let X be a complex Banach space and $\mathscr{B}(X)$ the Banach algebra of all linear and bounded operators acting on X. The norms in X and in $\mathscr{B}(X)$ will be denoted by $\|\cdot\|$. Let $\mathbb{Z}_+$ be the set of all nonnegative integer numbers. A sequence $x : \mathbb{Z}_+ \to X$ is said to be almost periodic if for any $\epsilon > 0$ there exists an integer $l_\epsilon > 0$ such that any discrete interval of length l_ϵ contains an integer τ, such that

$$\left\| x_{n+\tau} - x_n \right\| \le \epsilon, \quad \forall n \in \mathbb{Z}_+. \tag{1}$$

The integer number τ is called ϵ-translation number of (x_n). The set of all almost periodic sequences will be denoted by $AP(\mathbb{Z}_+, X)$. For further details about almost periodic functions, we refer to the books [9, 10]. The set $l^\infty(\mathbb{Z}_+, X)$ of all bounded sequences becomes a Banach space when it is endowed with the "sup" norm denoted by $\|\cdot\|_\infty$. Clearly, $AP(\mathbb{Z}_+, X)$ is a subset of $l^\infty(\mathbb{Z}_+, X)$. Let $P_q^0(\mathbb{Z}_+, X)$ be the space of all q-periodic ($q \ge 2$ is an integer number) sequences

x with $x(0) = 0$. Denote by $\mathscr{A}_0(\mathbb{Z}_+, X)$ the set of all sequences $\{x(n)\}_{n\geq 0}$ for which there exists $n_x \in \mathbb{Z}_+$ with $n_x > 0$ and $y_x \in P_q^0(\mathbb{Z}_+, X)$ such that

$$x(n) = \begin{cases} 0, & \forall 0 \leq n < n_x, \\ y_x(n), & \text{if } n \geq n_x. \end{cases} \tag{2}$$

Let $AP_0(\mathbb{Z}_+, X) := \overline{\operatorname{span}}\{\mathscr{A}_0(\mathbb{Z}_+, X)\}$. Here the closeness is considered in the space $l^\infty(\mathbb{Z}_+, X)$.

For a bounded linear operator L, acting on X, we denote by $\sigma(L)$ the spectrum of L and by $\rho(L)$ its resolvent set. Recall that a subset $\mathbb{T} = \{T(n)\}_{n\in\mathbb{Z}_+}$ of $\mathscr{B}(X)$ is called discrete semigroup if it satisfies the following conditions:

(i) $T(0) = I$, where I is the identity operator on X.

(ii) $T(n+m) = T(n)T(m)$, for all $n, m \in \mathbb{Z}_+$.

A discrete semigroup $\mathbb{T}$ is said to be uniformly exponentially stable if there exist $N, \nu > 0$ such that

$$\|T(n)\| \leq Ne^{-\nu n} \quad \forall n \in \mathbb{Z}_+. \tag{3}$$

The spectral radius of $T(1)$ denoted by $r(T(1))$ is defined as

$$r(T(1)) := \sup\{|\lambda| : \lambda \in \sigma(T(1))\}. \tag{4}$$

It is well known that, see for example [11, page 42],

$$r(T(1)) = \lim_{n\to\infty} \left\|(T(1))^n\right\|^{1/n}. \tag{5}$$

As a consequence of (5), a discrete semigroup $\{T(n)\}_{n\in\mathbb{Z}_+}$ is uniformly exponentially stable if and only if $r(T(1)) < 1$.

Having in mind the continuous case, the "infinitesimal generator" of the discrete semigroup denoted by G is defined by $G := T(1) - I$. For discrete semigroups, the Taylor formula of order one is

$$T(n)x - x = \sum_{k=0}^{n-1} T(k)Gx, \quad \forall n \in \mathbb{Z}_+,\ n \geq 1,\ \forall x \in X. \tag{6}$$

A discrete semigroup $\mathbb{T}$ is said to be $AP_0(\mathbb{Z}_+, X)$ exact admissible, if for every $x \in AP_0(\mathbb{Z}_+, X)$ the sequence $(\sum_{k=0}^{n} T(n-k)h(k))_{n\in\mathbb{Z}_+}$ belongs with $AP_0(\mathbb{Z}_+, X)$.

The evolution semigroup $\mathbb{S} = \{S(n), n \in \mathbb{Z}_+\}$ associated with $\mathbb{T}$ on $AP_0(\mathbb{Z}_+, X)$ is defined by

$$(S(r)x)(n) = \begin{cases} T(r)x(n-r), & \forall n \geq r, \\ 0, & 0 \leq n \leq r. \end{cases} \tag{7}$$

3. Results

The following lemma shows that the associated evolution semigroup $\{S(n)\}_{n\in\mathbb{Z}_+}$ acts on $AP_0(\mathbb{Z}_+, X)$.

Lemma 1. *Let $x \in AP_0(\mathbb{Z}_+, X)$ and $\mathbb{T} = \{T(j)\}_{j\in\mathbb{Z}_+}$ be a discrete semigroup of bounded linear operators on X. The sequence $S(r)x$, given by*

$$(S(r)x)(n) = \begin{cases} T(r)x(n-r), & \forall n \geq r \\ 0, & 0 \leq n \leq r, \end{cases} \tag{8}$$

belongs to $AP_0(\mathbb{Z}_+, X)$.

Proof. First we show that $S(r)x \in \mathscr{A}_0(\mathbb{Z}_+, X)$ for any $x \in \mathscr{A}_0(\mathbb{Z}_+, X)$. Since $x \in \mathscr{A}_0(\mathbb{Z}_+, X)$ there exist $n_x \in \mathbb{Z}_+$ with $n_x > 0$, and $(y_x(n)) \in P_q^0(\mathbb{Z}_+, X)$, such that

$$x(n) = \begin{cases} 0, & \text{if } 0 \leq n < n_x \\ y_x(n), & \text{if } n \geq n_x. \end{cases} \tag{9}$$

Let $n_{S(r)x} := r + n_x$ and set $y_{S(r)x}(\cdot) = T(r)y_x(\cdot - r)$. Clearly $y_{S(r)x}$ is q-periodic sequence. It remains to show that

$$(S(r)x)(n) = \begin{cases} 0, & \text{if } 0 \leq n < n_{S(r)x} \\ y_{S(r)x}(n), & \text{if } n \geq n_{S(r)x}. \end{cases} \tag{10}$$

If $n \leq n_{S(r)x} = r + n_x$, then $n - r < n_x$ and $x(n-r) = 0$, so

$$(S(r)x)(n) = T(r)x(n-r) = 0. \tag{11}$$

If $n \geq n_{S(r)x} = r + n_x$, then $n - r \geq n_x$ and $x(n-r) = y_x(n-r)$; hence

$$\begin{aligned} (S(r)x)(n) &= T(r)x(n-r) \\ &= T(r)y_x(n-r) \\ &= y_{S(r)x}(n). \end{aligned} \tag{12}$$

Thus $S(r)x \in \mathscr{A}_0(\mathbb{Z}_+, X)$. Now, from linearity it follows that $S(r)z$ belongs to $\operatorname{span}\{\mathscr{A}_0(\mathbb{Z}_+, X)\}$ whenever $z \in \operatorname{span}\{\mathscr{A}_0(\mathbb{Z}_+, X)\}$. Let now $\epsilon > 0$, $x \in AP_0(\mathbb{Z}_+, X)$, and let $z \in \operatorname{span}\{\mathscr{A}_0(\mathbb{Z}_+, X)\}$, such that $\|x - z\|_{l^\infty(\mathbb{Z}_+, X)} < \epsilon$. Clearly $S(r)z$ belongs to $\operatorname{span}\{\mathscr{A}_0(\mathbb{Z}_+, X)\}$, and

$$\begin{aligned} \|S(r)z - S(r)x\|_{l^\infty(\mathbb{Z}_+, X)} &= \sup_{n\geq r} \|T(r)[z(n-r) - x(n-r)]\| \\ &\leq Me^{\nu r} \sup_{n\geq r} \|z(n-r) - x(n-r)\| \\ &\leq Me^{\nu r}\epsilon, \end{aligned} \tag{13}$$

that is, $S(r)x$ is in $AP_0(\mathbb{Z}_+, X)$. This completes the proof. □

Lemma 2. *Let $\mathbb{T} = \{T(n)\}_{n\in\mathbb{Z}_+}$ be a discrete semigroup of bounded linear operators on X, and let $\mathbb{S} = \{S(n), n \in \mathbb{Z}_+\}$ be the evolution semigroup associated with $\mathbb{T}$ on $AP_0(\mathbb{Z}_+, X)$, having $G_{\mathbb{S}}$ as generator. Let $x, z \in AP_0(\mathbb{Z}_+, X)$. The following two statements are equivalent:*

(i) $G_{\mathbb{S}}x = -z$,

(ii) $x(n) = \sum_{k=0}^{n} T(n-k)z(k)$, *for all $n \in \mathbb{Z}_+$.*

Proof. (i) ⇒ (ii): Using the Taylor formula (6), one has

$$S(n)x - x = \sum_{m=0}^{n-1} S(m)G_{\mathbb{S}}x = -\sum_{m=0}^{n-1} S(m)z. \tag{14}$$

Then, for every $n \in \mathbb{Z}_+$, one has

$$\begin{aligned} x(n) &= (S(n)x)(n) + \sum_{m=0}^{n-1} (S(m)z)(n) \\ &= T(n)x(0) + \sum_{m=0}^{n-1} T(m)z(n-m) \\ &= \sum_{k=0}^{n} T(n-k)z(k). \end{aligned} \tag{15}$$

(ii) $\Rightarrow$ (i): For each $n \in \mathbb{Z}_+$, one has

$$\begin{aligned} (G_{\mathbb{S}}x)(n) &= (S(1)-I)x(n) \\ &= T(1)x(n-1) - x(n) \\ &= T(1)\sum_{k=0}^{n-1} T(n-1-k)z(k) - x(n) \\ &= \sum_{k=0}^{n-1} T(n-k)z(k) - \sum_{k=0}^{n} T(n-k)z(k) \\ &= -z(n). \end{aligned} \tag{16}$$

This completes the proof. □

See also [12], for a variant of this lemma in other space.

The next result is the main ingredient in the proof of Theorem 5 that follows.

Theorem 3 (see [7]). *Let $\mathbb{T} = \{T(n) : n \in \mathbb{Z}_+\}$ be a discrete semigroup on X, and let μ be a real number. If*

$$\sup_{n\geq 0}\left\|\sum_{k=0}^{n} e^{i\mu k}T(n-k)f(k)\right\| < \infty, \tag{17}$$

for every $f \in P_0^q(\mathbb{Z}_+, X)$, then $T(1)$ is power bounded (i.e., $\sup_{n\in\mathbb{Z}_+}\|A^n\| < \infty$) and $e^{i\mu} \in \rho(T(1))$.

As a corollary of this theorem, we state the following.

Corollary 4 (see [7]). *Let $\mathbb{T} = \{T(n) : n \in \mathbb{Z}_+\}$ be a discrete semigroup on X. If the condition* (17) *holds for every $\mu \in \mathbb{R}$ and every f in $P_0^q(\mathbb{Z}_+, X)$, then the semigroup $\mathbb{T}$ is uniformly exponentially stable.*

The result of this paper reads as follows.

Theorem 5. *Let $\mathbb{T} = \{T(n)\}_{n\in\mathbb{Z}_+}$ be a discrete semigroup on X. The following four statements are equivalent:*

(i) *$\mathbb{T}$ is uniformly exponentially stable.*

(ii) *The evolution semigroup $\mathbb{S}$ associated with $\mathbb{T}$ on $AP_0(\mathbb{Z}_+, X)$ is uniformly exponentially stable.*

(iii) *The semigroup $\mathbb{T}$ is $AP_0(\mathbb{Z}_+, X)$ exact admissible.*

(iv) $\sup_{n\in\mathbb{Z}_+}\|\sum_{k=0}^{n} T(n-k)z(k)\| = M_z < \infty$, *for all $z \in AP_0(\mathbb{Z}_+, X)$.*

Proof. (i) $\Rightarrow$ (ii): Let $\mathbb{T}$ be uniformly exponentially stable, and let N and ν be positive constants such that

$$\|T(n)\| \leq Ne^{-\nu n} \quad \forall n \in \mathbb{Z}_+. \tag{18}$$

Then for every f in $AP_0(\mathbb{Z}_+, X)$, one has

$$\|S(j)f\|_\infty = \sup_{n\geq j}\|T(j)f(n-j)\| \leq Ne^{-\nu j}\|f\|_\infty. \tag{19}$$

(ii) $\Rightarrow$ (iii): Since $\mathbb{S}$ is uniformly exponentially stable, $1 \in \rho(S(1))$, that is, $S(1) - I$ is invertible. Then for each z in $AP_0(\mathbb{Z}_+, X)$, there exists $u \in AP_0(\mathbb{Z}_+, X)$ such that $(S(1) - I)u = -z$.

On the other hand, by Lemma 2, $u(n) = \sum_{k=0}^{n} T(k)z(n-k)$, for every $n \in \mathbb{Z}_+$; hence $\mathbb{T}$ is $AP_0(\mathbb{Z}_+, X)$ exact admissible.

(iii) $\Rightarrow$ (iv) It is obvious.

(iv) $\Rightarrow$ (i) Obviously, if $z \in P_q^0(\mathbb{Z}_+, X)$ and μ is a real number, then $(e^{i\mu n}z(n))_{n\in\mathbb{Z}_+}$ belongs to $AP_0(\mathbb{Z}_+, X)$. Now, we can apply Corollary 4 to finish the proof. □

The following example is a concrete application of Theorem 5.

Example 6. Let X be a complex Banach space, and let A be a bounded linear operator acting on X. Consider the following two discrete Cauchy problems:

$$\begin{aligned} x_{j+1} &= Ax_j, \quad j \in \mathbb{Z}_+, \\ x_0 &= b, \end{aligned} \tag{20}$$

$$\begin{aligned} y_{j+1} &= Ay_j + f_{j+1}, \quad j \in \mathbb{Z}_+, \\ y_0 &= 0. \end{aligned} \tag{21}$$

The solutions of (20) and (21) are (resp.) given by $x_j = T(j)b$ and $y_j = \sum_{k=0}^{j} T(j-k)x(k)$. Here $T(k) := A^k$.

From Theorem 5, the following two statements are equivalent.

(1) For each $b \in X$ the solution of (20) decays exponentially, or, equivalently, there exist two positive constants K and ν such that

$$\|T(j)x\| \leq Ke^{-\nu j}\|x\| \quad \forall x \in X. \tag{22}$$

(2) For each $f \in AP_0(\mathbb{Z}_+, X)$ the solution of (21) belongs to $AP_0(\mathbb{Z}_+, X)$.

In fact, we can state a more general result concerning q-periodic discrete evolution families. To establish this result, we recall that a family $\mathcal{U} = \{U(n,m) : n \geq m \in \mathbb{Z}_+\} \subset \mathcal{B}(X)$ is said to be q-periodic discrete evolution family if it satisfies the following properties.

(i) $U(n,n) = I$ and $U(n,m)U(m,r) = U(n,r)$, for all $n,m,r \in \mathbb{Z}_+$ with $n \geq m \geq r \in \mathbb{Z}_+$, where I is the identity operator on X.

(ii) $U(n+q, m+q) = U(n,m)$, for all $n \geq m \in \mathbb{Z}_+$.

It is said to be uniformly exponentially stable if there exist the positive constants K and ν such that

$$\|U(n,m)\| \leq Ke^{-\nu(n-m)} \quad \forall m \geq n \in \mathbb{Z}_+. \tag{23}$$

Also, the family $\mathcal{U}$ is said to be $AP_0(\mathbb{Z}_+, X)$ exact admissible, if for every $z \in AP_0(\mathbb{Z}_+, X)$ the sequence $(\sum_{k=0}^{n} U(n,k)z(k))_{n\in\mathbb{Z}_+}$ belongs to $AP_0(\mathbb{Z}_+, X)$.

The discrete evolution semigroup $\mathcal{T} = \{\mathcal{T}(n), n \in \mathbb{Z}_+\}$ associated with the evolution family $\mathcal{U}$ on $AP_0(\mathbb{Z}_+, X)$ is defined by

$$(\mathcal{T}(n)z)(r) = \begin{cases} U(r, r-n)z(r-n), & \forall r \geq n, \\ 0, & \text{otherwise.} \end{cases} \tag{24}$$

As in Lemma 1 it can be proved that it acts on $AP_0(\mathbb{Z}_+, X)$.

Theorem 7. *Let $\mathcal{U} = \{U(n,m) : n \geq m \in \mathbb{Z}_+\}$ be a q-periodic evolution family of bounded linear operators on X. The following statements are equivalent:*

(1) *$\mathcal{U}$ is uniformly exponentially stable.*

(2) *The evolution semigroup $\mathcal{T}$ associated with $\mathcal{U}$ is uniformly exponentially stable.*

(3) *$\mathcal{U}$ is $AP_0(\mathbb{Z}_+, X)$ exact admissible.*

(4) $\sup_{n\in\mathbb{Z}_+} \| \sum_{k=0}^{n} U(n,k)h(k)\| < \infty$, *for all $h \in AP_0(\mathbb{Z}_+, X)$.*

The proofs of (1) ⇒ (2) ⇒ (3) ⇒ (4) are similar to those in the semigroup case. For the proof of (4) ⇒ (1) we use the following result from [13].

If for every $\mu \in \mathbb{R}$ and every $z \in P_q^0(\mathbb{Z}_+, X)$, one has

$$\sup_{n\in\mathbb{Z}_+} \left\| \sum_{k=0}^{n} e^{i\mu k} U(n,k)z(k) \right\| := M(\mu, z) < \infty, \tag{25}$$

then the family $\mathcal{U}$ is uniformly exponentially stable.

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