Research Article

Stabilization for Networked Control Systems with Random Sampling Periods

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This paper investigates the stabilization of networked control systems (NCSs) with random delays and random sampling periods. Sampling periods can randomly switch between three cases according to the high, low, and medium types of network load. The sensor-to-controller (*S-C*) random delays and random sampling periods are modeled as Markov chains. The transition probabilities of Markov chains do not need to be completely known. A state feedback controller is designed via the iterative linear matrix inequality (LMI) approach. It is shown that the designed controller is two-mode dependent and depends on not only the current *S-C* delay but also the most recent available sampling period at the controller node. The resulting closed-loop systems are special discrete-time jump linear systems with two modes. The sufficient conditions for the stochastic stability are established. An example of the cart and inverted pendulum is given to illustrate the effectiveness of the theoretical result.

1. Introduction

Networked control systems (NCSs) are feedback control systems whose feedback paths are implemented by a real-time network. Recently, much attention has been paid to the study of NCSs, due to their low cost, reduced weight and power requirements, simple installation and maintenance, high reliability, and so on [1, 2].

A basic problem in NCSs is the stability of the systems. In real-time control systems, three main issues are bandwidth and packet size constraints, time delays, and packet losses, which will degrade the performance of control systems and even make systems unstable; so it is significant to overcome the adverse influences of time delays, packet losses, and dynamic bandwidth [3–7].

On the other hand, the Markov chain, a discrete-time stochastic process with the Markov property, can be effectively used to model NCSs with random time delays and random sampling periods, which are modeled as the Markovian jump linear systems (MJLSs). Recently, there have

been considerable research efforts into NCSs [8–12]. For example, Xiao et al. present an *V-K* iteration algorithm to design stabilizing controllers for specially structured discrete-time jump linear systems with random but bounded delays in the feedback loop [8]. Zhang et al. propose a promising two-mode-dependent-state feedback scheme to stabilize section with the current *S-C* delay (τ_k) and the previous controller-to-actuator (*C-A*) delay (d_{k-1}) modeled as two Markov chains [9]. Shi and Yu investigate the output feedback stabilization and robust mixed H_2/H_∞ control of NCSs, in which it is assumed that at each sampling instant, the current *S-C* delay (τ_k) and the most recent available *C-A* delay ($d_{k-1-\tau_k}$) can be obtained [10, 11]. Huang and Nguang discuss the stabilization problem for a class of linear uncertain continuous NCSs, in which it is assumed that random communication *S-C* delays ($\tau(t)$) and random communication *C-A* delays ($\rho(t)$) are modeled two continuous time-discrete-state Markov processes [12].

In the above references, the transition probabilities are assumed completely accessible and considered as the

available knowledge for analyzing and designing the NCSs. In practice, this kind of information including the variation of time delays and packet losses is hard to obtain. The problems of partly unknown transition probabilities were investigated [13–16]. Zhang et al. investigate the stability of Markovian jump linear systems with partly unknown transition probabilities [13, 14]. Wang et al. study the partially mode-dependent H_∞ filtering problem for discrete-time Markovian jump systems with partly unknown transition probabilities [15]. Sun and Qin investigate the stability and stabilization problems of a class of NCSs with bounded packet dropout, in which the transition probabilities are partly unknown due to the complexity of network [16]. However, the developed controllers of these references [13–16] only are either mode independent or one-mode dependent, and the design problem can thus be readily converted into a standard MJLS problem. To the best of the authors' knowledge, if the transition probabilities are assumed partly accessible, designing the two-mode-dependent controller that simultaneously depends on both the current S-C delay (τ_k^{sc}), and the most recent available sampling period ($h_{k-\tau_k^{sc}}$) has not been fully investigated, which is the focus of this work. When considering both τ_k^{sc} and $h_{k-\tau_k^{sc}}$, the resulting closed-loop system can be transformed to a special MJLS, and thus the well-developed results on MJLSs with partly unknown transition probabilities cannot be directly applied [13–16].

In this paper, the stochastic stability of NCSs with random time delays and random sampling periods is studied, in which time delays and sampling periods are driven by two finite-state Markov chains. This paper is organized as follows. In Section 2, the NCSs model with random S-C time delays and random sampling periods is made, which is equivalent to a class of special discrete-time jump linear systems with two modes. Sufficient and necessary conditions of stochastic stability with completely known transition probabilities for the foregoing model are considered in Section 3. Sufficient conditions of stochastic stability with partly unknown transition probabilities for the foregoing model are considered in Section 4. Section 5 is an illustrative example, and our work in this paper is summarized in Section 6.

Notation. In this paper, $\mathbf{R}$ is the set of all real numbers, $\mathbf{R}^n$ denotes the n-dimensional Euclidean space. (Ω, F, P) denotes the probability space. $\mathbf{A}^T$ and $\mathbf{A}^{-1}$ denote the transpose and the inverse of a matrix $\mathbf{A}$, respectively. $\mathbf{A} > \mathbf{0}$ ($\mathbf{A} < \mathbf{0}$) means that $\mathbf{A}$ is positive definite (negative definite). $\mathbf{0}$ and $\mathbf{I}$ are the zero and identity matrices with appropriate dimensions, respectively. In symmetric block matrices, we use an asterisk ($*$) to represent a term that is induced by symmetry.

2. Problem Formulation

The structure of the considered NCSs is shown in Figure 1, where the plant is described by the following linear system model:

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1}$$

where $x(t) \in \mathbf{R}^n, u(t) \in \mathbf{R}^m$ are the system state vector and control input vector, respectively; A, B are known constant
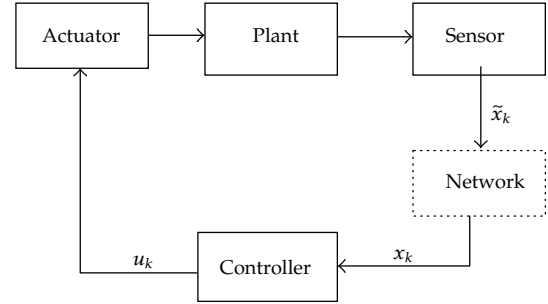

FIGURE 1: Diagram of NCSs.

matrices of appropriate dimensions. Suppose the sensor is clock-driven, the controller and the actuator are event-driven.

Suppose bounded random delays only exist in the link from sensor to controller, as shown in Figure 1. Here, $0 \leq \tau_k^{sc} \leq \tau^{sc}$ represents the S-C delay. The state feedback controller is to be designed.

For NCSs, the shorter the sampling period, the better system performance; however, the short sampling period will increase the possibility of network congestion. If the constant sampling period is adopted, the sampling period should be large enough to avoid network congestion, so network bandwidth cannot be sufficiently used when the network is idle. In [17–19], the variable sampling method is used, and the sampling periods are assumed to switch in a finite discrete set. But when the systems switch too fast, it is apt to cause oscillation and instability of the system. In the actual network, the size of sampling period is closely related to the network load. However, the network load usually is random [20]. As a result of the above, network load can be high load, low load, and medium load. Correspondingly, sampling periods can randomly switch between three cases of maximum, medium, and minimum in this paper. In the following, we consider that sample periods randomly switch between three cases and make NCSs model with random time delays and random sample periods. Thus, not only the network bandwidth can be fully used, but also the conservativeness of the stabilization conditions of NCSs can be reduced.

Suppose h_k is the length of the kth sampling period, if the network is idle, define the sampling period as $h_{\max}$, and if the network is occupied by the most users, define the sampling period as $h_{\min}$, otherwise define the sampling period as $(1/2)(h_{\max} + h_{\min})$. Then, the sampling period $h_k \in \{h_{\min}, (1/2)(h_{\max} + h_{\min}), h_{\max}\}$.

The discrete-time expression of the system (1) is as follows:

$$x(k+1) = \Phi(h_k) x(k) + \Gamma(h_k) u(k), \tag{2}$$

where $\Phi(h_k) = e^{Ah_k}$, and $\Gamma(h_k) = \int_0^{h_k} e^{At} dt B$.

In this paper, τ_k^{sc} and h_k are modeled as two homogeneous Markov chains defined in (Ω, F, P) that take values in $\varphi = \{0, 1, \ldots, \tau^{sc}\}$ and $\phi = \{1, 2, 3\}$, and their transition

probability matrices are $\Lambda = [\lambda_{ij}]$ and $\Pi = [\pi_{rs}]$, respectively, and

$$\lambda_{ij} = \Pr\{\tau_{k+1}^{sc} = j \mid \tau_k^{sc} = i\},$$
$$\pi_{rs} = \Pr\{h_{k+1} = s \mid h_k = r\}, \tag{3}$$

where $\lambda_{ij}, \pi_{rs} \geq 0$, and $\sum_{j=0}^{\tau^{sc}} \lambda_{ij} = 1, \sum_{s=1}^{3} \pi_{rs} = 1$, for all $i, j \in \varphi$ and $r, s \in \phi$.

Remark 1. In this paper, we assume that the controller will always use the most recent data [8]. Thus, if we have $x(k - \tau_k^{sc})$ at sampling instant t_k, then at sampling time t_{k+1}, even if there are delays longer than 1 or package loss, we still have $x(k - \tau_k^{sc})$ to use. So in our model of the system in Figure 1, the delay τ_k^{sc} can increase at most 1 each step, $\Pr\{\tau_{k+1}^{sc} > \tau_k^{sc} + 1\} = 0$.

In NCSs, to reduce the conservativeness of the stabilization conditions, it is desirable to consider not only the time delay but also the sample period information in the controller design. For the controller node, at time instant t_k, τ_k^{sc} can be obtained by comparing the current time and the time-stamp of the sensor information received. However, this sample period information cannot be received by the controller immediately because it needs to be transmitted through the network from sensor to controller. So if the time delay τ_k^{sc} exists, the information $h_{k-\tau_k^{sc}}$ of time instant t_k would be known at the controller node. Consequently, it is desirable to design the state feedback controller that simultaneously depends on both τ_k^{sc} and $h_{k-\tau_k^{sc}}$. The mode-dependent state feedback control law is

$$u(k) = K\left(\tau_k^{sc}, h_{k-\tau_k^{sc}}\right)\hat{x}(k) = K\left(\tau_k^{sc}, h_{k-\tau_k^{sc}}\right)x\left(k - \tau_k^{sc}\right), \tag{4}$$

where $\hat{x}(k)$ is the input vector of the state feedback controller, and K is the controller gain to be designed.

In the following, we make the closed-loop system model. Substituting the formula (4) into (2), we can obtain the following closed-loop NCSs:

$$x(k+1) = \Phi(h_k)x(k) + \Gamma(h_k)K\left(\tau_k^{sc}, h_{k-\tau_k^{sc}}\right)x\left(k - \tau_k^{sc}\right). \tag{5}$$

Augment the plant's state variable as

$$X(k) = \left[x^T(k)\ \ x^T(k-1)\ \ \ldots\ \ x^T\left(k - \tau^{sc}\right)\right]^T, \tag{6}$$

where $X(k) \in \mathbf{R}^{(\tau^{sc}+1)n}$, then we have

$$X(k+1) = \left(\overline{A}(h_k) + \overline{B}(h_k)K\left(\tau_k^{sc}, h_{k-\tau_k^{sc}}\right)\overline{C}\left(\tau_k^{sc}\right)\right)X(k)$$
$$= \widetilde{A}\left(\tau_k^{sc}, h_{k-\tau_k^{sc}}, h_k\right)X(k), \tag{7}$$

where

$$\overline{A}(h_k) = \begin{bmatrix} \Phi(h_k) & 0 & \cdots & 0 & 0 \\ I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{bmatrix},$$

$$\overline{B}(h_k) = \begin{bmatrix} \Gamma(h_k) \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$\overline{C}\left(\tau_k^{sc}\right) = [0 \cdots\cdots 0\ I\ 0 \cdots 0],$$

$$\widetilde{A}\left(\tau_k^{sc}, h_{k-\tau_k^{sc}}, h_k\right) = \overline{A}(h_k) + \overline{B}(h_k)K\left(\tau_k^{sc}, h_{k-\tau_k^{sc}}\right)\overline{C}\left(\tau_k^{sc}\right), \tag{8}$$

and $\overline{C}(\tau_k^{sc})$ has all elements being zero except for the $\tau_k^{sc} + 1$th block being an identity matrix.

Remark 2. Because the controller in (4) is two-mode dependent, the resulting closed-loop system in (7) depends on τ_k^{sc}, h_k, $h_{k-\tau_k^{sc}}$ and can be transformed to a special MJLS. In addition, $h_{k-\tau_k^{sc}}$ is related to both τ_k^{sc} and h_k.

It can be seen that the closed-loop system in (7) is a jump linear system with two modes modeled by different homogeneous Markov chains. The objective of this paper is to design the state feedback controller (4) to guarantee the stochastic stability of the NCSs in (7). For the stochastic stability, we adopt the following definition [9].

Definition 3. The system in (7) is stochastically stable if for every finite matrix $X_0 = X(0)$, initial mode $\tau_0^{sc} = \tau^{sc}(0) \in \varphi$, and $h_{-\tau_0^{sc}} = h(-\tau_0^{sc}) \in \phi$, there exists a finite matrix $W > 0$ such that the following holds:

$$\varepsilon\left\{\sum_{k=0}^{\infty}\|X(k)\|^2 \mid X_0, \tau_0^{sc}, h_{-\tau_0^{sc}}\right\} < X_0^T W X_0. \tag{9}$$

3. The Case of Completely Known Transition Probabilities

In this section, we first give the sufficient and necessary conditions for the state feedback stabilization of NCSs with completely known transition probabilities in (7) and then derive the equivalent conditions of LMIs with nonconvex constraints.

Because $h_{k-\tau_k^{sc}}$ is to be considered into the controller design, the multistep jump problem of Markov chains is involved in the system design. According to the Chapman-Kolmogorov (C-K) equation of stochastic process [21], we give the multistep transition probability as follows.

Lemma 4. *If the transition probability matrix from $h_k = s_1$ to $h_{k+1} = s$ is $\Pi = (\pi_{s_1 s})$, $s, s_1 \in \phi$, then the transition probability*

matrix from $h_{k-i} = r$ to $h_k = s_1$ is $\Pi^{(i)} = (\pi^{(i)}_{rs_1})$, $r, s_1 \in \phi$, which is a multistep transition probability matrix of the Markov chain. Particularly, when $i = 0$, the transition probability matrix is $\Pi^{(0)} = I$.

Proof. The transition probability from $h_{k-i} = r$ to $h_k = s_1$ is $\pi^{(i)}_{rs_1}$, then

$$
\begin{aligned}
\pi^{(i)}_{rs_1} &= P\left\{h_k = s_1 \mid h_{k-i} = r\right\} \\
&= P\left\{h_{(k-i)+1+(i-1)} = s_1 \mid h_{k-i} = r\right\} \\
&= \sum_{l_1 \in \phi} P\left\{h_{(k-i)+1} = l_1 \mid h_{k-i} = r\right\} \\
&\quad \times P\left\{h_{(k-i)+1+(i-1)} = s_1 \mid h_{k-i} = r, h_{(k-i)+1} = l_1\right\} \\
&= \sum_{l_1 \in \phi} P\left\{h_{(k-i)+1} = l_1 \mid h_{k-i} = r\right\} \\
&\quad \times P\left\{h_{(k-i)+1+(i-1)} = s_1 \mid h_{(k-i)+1} = l_1\right\} \\
&= \sum_{l_1 \in \phi} \pi_{rl_1} \pi^{(i-1)}_{l_1 s_1} \\
&= \sum_{l_1 \in \phi} \pi_{rl_1} \sum_{l_2 \in \phi} P\left\{h_{(k-i)+2} = l_2 \mid h_{(k-i)+1} = l_1\right\} \\
&\quad \times P\left\{h_{(k-i)+2+(i-2)} = s_1 \mid h_{(k-i)+2} = l_2\right\} \\
&= \sum_{l_1 \in \phi} \pi_{rl_1} \sum_{l_2 \in \phi} \pi_{l_1 l_2} \pi^{(i-2)}_{l_2 s_1} \\
&= \cdots = \sum_{l_1 \in \phi} \sum_{l_2 \in \phi} \cdots \sum_{l_{i-1} \in \phi} \pi_{rl_1} \pi_{l_1 l_2} \cdots \pi_{l_{i-1} s_1}.
\end{aligned}
\tag{10}
$$

From matrix multiplication, the transition probability matrix from h_{k-i} to h_k is $\Pi^{(i)}$. $\qquad\square$

Remark 5. It is noted that (3) to (4) are based on the ineffectiveness theory of Markov chain. With Definition 3, the necessary and sufficient conditions on the stochastic stability of closed-loop system in (7) can be obtained.

Theorem 6. *The closed-loop system with completely known transition probabilities in (7) is stochastically stable if and only if there exist the matrices $P(i, r) > 0$ such that the following matrix inequalities:*

$$
\begin{aligned}
L(i, r) &= \sum_{j=0}^{\tau^{sc}} \sum_{s_2=1}^{3} \sum_{s_1=1}^{3} \lambda_{ij} \pi^{(i+1-j)}_{rs_2} \pi^{(i)}_{rs_1} \\
&\quad \times \left(\overline{A}(s_1) + \overline{B}(s_1) K(i, r) \overline{C}(i)\right)^T P(j, s_2) \\
&\quad \times \left(\overline{A}(s_1) + \overline{B}(s_1) K(i, r) \overline{C}(i)\right) \\
&\quad - P(i, r) < 0
\end{aligned}
\tag{11}
$$

hold for all $i \in \varphi$ and $r \in \phi$.

Proof (sufficiency). For the closed-loop system in (7), consider the Lyapunov function

$$
V(X(k), k) = X^T(k) P\left(\tau^{sc}_k, h_{k-\tau^{sc}_k}\right) X(k).
\tag{12}
$$

Then

$$
\begin{aligned}
&\varepsilon\left\{\Delta V(X(k), k)\right\} \\
&= \varepsilon\left\{V(X(k+1), k+1) \right. \\
&\quad \left. -V(X(k), k) \mid X(k), \tau^{sc}_k = i, h_{k-\tau^{sc}_k} = r\right\} \\
&= \varepsilon\left\{X^T(k+1) P\left(\tau^{sc}_{k+1}, h_{k+1-\tau^{sc}_{k+1}}\right) \right. \\
&\quad \left. \times X(k+1) \mid X(k), \tau^{sc}_k = i, h_{k-\tau^{sc}_k} = r\right\} \\
&\quad - X^T(k) P(i, r) X(k).
\end{aligned}
\tag{13}
$$

Define $\tau^{sc}_{k+1} = j$, $h_k = s_1$, and $h_{k+1-\tau^{sc}_{k+1}} = s_2$. To evaluate the first term in (13), we need to apply the three probability transition matrices:

$$
\begin{aligned}
\tau^{sc}_k &\longrightarrow \tau^{sc}_{k+1} : \Lambda, \\
h_{k-\tau^{sc}_k} &\longrightarrow h_k : \Pi^{(i)}, \\
h_{k-\tau^{sc}_k} &\longrightarrow h_{k+1-\tau^{sc}_{k+1}} : \Pi^{(i+1-j)}.
\end{aligned}
\tag{14}
$$

Then, (13) can be written as

$$
\begin{aligned}
&\varepsilon\left\{\Delta V(X(k), k) \mid X(k), \tau^{sc}_k = i, \ h_{k-\tau^{sc}_k} = r\right\} \\
&= X^T(k) \left(\sum_{j=0}^{\tau^{sc}} \sum_{s_2=1}^{3} \sum_{s_1=1}^{3} \lambda_{ij} \pi^{(i+1-j)}_{rs_2} \pi^{(i)}_{rs_1} \right. \\
&\quad \times \left(\overline{A}(s_1) + \overline{B}(s_1) K(i, r) \overline{C}(i)\right)^T \\
&\quad \times P(j, s_2) \\
&\quad \times \left(\overline{A}(s_1) + \overline{B}(s_1) K(i, r) \overline{C}(i)\right) \\
&\quad \left. -P(i, r) \right) X(k) \\
&= X^T(k) L(i, r) X(k) < 0.
\end{aligned}
\tag{15}
$$

Thus, if $L(i, r) < 0$, then

$$
\begin{aligned}
&\varepsilon\left\{\Delta V(X(k), k) \mid X(k), \tau^{sc}_k = i, h_{k-\tau^{sc}_k} = r\right\} \\
&= \varepsilon\left\{V(X(k+1), k+1) \mid X(k), \tau^{sc}_k = i, h_{k-\tau^{sc}_k} = r\right\} \\
&\quad - V(X(k), k) \\
&\leq -\lambda_{\min}(-L(i, r)) X(k)^T X(k) \\
&\leq -\beta X^T(k) X(k) \\
&= -\beta \|X(k)\|^2,
\end{aligned}
\tag{16}
$$

where $\beta = \inf\{\lambda_{\min}(-L(i,r)), i \in \varphi, r \in \phi\} > 0$. From inequality (16), we can see that for any $T \geq 1$,

$$
\varepsilon\{V(X(T+1), T+1)\} - \varepsilon\{V(X_0, 0)\}
$$
$$
\leq -\beta\varepsilon\left\{\sum_{t=0}^{T}\|X(t)\|^2 \mid X_0, \tau_0^{sc}, h_{-\tau_0^{sc}}\right\}. \tag{17}
$$

Furthermore

$$
\varepsilon\left\{\sum_{t=0}^{T}\|X(t)\|^2 \mid X_0, \tau_0^{sc}, h_{-\tau_0^{sc}}\right\}
$$
$$
\leq \frac{1}{\beta}\left(\varepsilon\{V(X_0, 0)\} - \varepsilon\{V(X(T+1), T+1)\}\right) \tag{18}
$$
$$
\leq \frac{1}{\beta}\varepsilon\{V(X_0, 0)\}.
$$

Let $T \to \infty$, then

$$
\varepsilon\left\{\sum_{t=0}^{\infty}\|X(t)\|^2 \mid X_0, \tau_0^{sc}, h_{-\tau_0^{sc}}\right\} \leq \frac{1}{\beta}\varepsilon\{V(X_0, 0)\}
$$
$$
= \frac{1}{\beta}X^T(0) P\left(\tau_0^{sc}, h_{-\tau_0^{sc}}\right) X(0). \tag{19}
$$

From Definition 3, the closed-loop system in (7) is stochastically stable.

Necessity. Assume that the closed-loop system in (7) is stochastically stable. Then, we have

$$
\varepsilon\left\{\sum_{k=0}^{\infty}\|X(k)\|^2 \mid X_0, \tau_0^{sc}, h_{-\tau_0^{sc}}\right\} < X_0^T W X_0. \tag{20}
$$

Define

$$
X^T(t)\widetilde{P}\left(T-t, \tau_t^{sc}, h_{t-\tau_t^{sc}}\right) X(t)
$$
$$
\triangleq \varepsilon\left\{\sum_{k=t}^{T}X^T(k)Q\left(\tau_k^{sc}, h_{k-\tau_k^{sc}}\right)X(k) \mid X_t, \tau_t^{sc}, h_{t-\tau_t^{sc}}\right\}, \tag{21}
$$

with $Q(\tau_k^{sc}, h_{k-\tau_k^{sc}}) > 0$. Assume that $X(k) \neq 0$, since $Q(\tau_k^{sc}, h_{k-\tau_k^{sc}}) > 0$, as T increases, $X^T(t)\widetilde{P}(T - t, \tau_t^{sc}, h_{t-\tau_t^{sc}})X(t)$ is monotonically increasing, for $k \geq t$. From (20), $X^T(t)\widetilde{P}(T - t, \tau_t^{sc}, h_{t-\tau_t^{sc}})X(t)$ is upper bounded. Furthermore, its limit exists and can be represented as

$$
X^T(t) P(i,r) X(t)
$$
$$
\triangleq \lim_{T \to \infty} X^T(t)\widetilde{P}\left(T-t, \tau_t^{sc} = i, h_{t-\tau_t^{sc}} = r\right) X(t)
$$
$$
= \lim_{T \to \infty}\varepsilon\left\{\sum_{k=t}^{T}X^T(k)Q\left(\tau_k^{sc}, h_{k-\tau_k^{sc}}\right) \right. \tag{22}
$$
$$
\left. \times X(k) \mid X_t, \tau_t^{sc} = i, h_{t-\tau_t^{sc}} = r\right\}.
$$

Since this is valid for any $X(t)$, we have

$$
P(i,r) = \lim_{T \to \infty}\widetilde{P}\left(T-t, \tau_t^{sc} = i, h_{t-\tau_t^{sc}} = r\right). \tag{23}
$$

From (22), we have $P(i,r) > 0$, since $Q(\tau_k^{sc}, h_{k-\tau_k^{sc}}) > 0$. Consider

$$
\varepsilon\left\{X^T(t)\widetilde{P}\left(T-t, \tau_t^{sc}, h_{t-\tau_t^{sc}}\right) X(t) - X^T(t+1)\right.
$$
$$
\times \widetilde{P}\left(T-t-1, \tau_{t+1}^{sc}, h_{t+1-\tau_{t+1}^{sc}}\right)
$$
$$
\left. \times X(t+1) \mid X_t, \tau_t^{sc} = i, h_{t-\tau_t^{sc}} = r\right\} \tag{24}
$$
$$
= X^T(k) Q(i,r) X(k).
$$

By using Lemma 4, the second term in (24) can be written as

$$
\varepsilon\left\{X^T(t+1)\widetilde{P}\left(T-t-1, \tau_{t+1}^{sc}, h_{t+1-\tau_{t+1}^{sc}}\right) X(t+1) \mid X_t,\right.
$$
$$
\left. \tau_t^{sc} = i, h_{t-\tau_t^{sc}} = r\right\}
$$
$$
= X^T(t)\left\{\sum_{j=0}^{\tau^{sc}}\sum_{s_2=1}^{3}\sum_{s_1=1}^{3}\lambda_{ij}\pi_{rs_2}^{(i+1-j)}\pi_{rs_1}^{(i)}\right.
$$
$$
\times \left(\overline{A}(s_1) + \overline{B}(s_1) K(i,r)\overline{C}(i)\right)^T
$$
$$
\times \widetilde{P}(T-t-1, j, s_2)
$$
$$
\left. \times \left(\overline{A}(s_1) + \overline{B}(s_1) K(i,r)\overline{C}(i)\right)\right\} X(t). \tag{25}
$$

Substituting (25) into (24), we can get

$$
X^T(t)\left\{\widetilde{P}\left(T-t, \tau_t^{sc}, h_{t-\tau_t^{sc}}\right)\right.
$$
$$
- \sum_{j=0}^{\tau}\sum_{s_2=1}^{3}\sum_{s_1=1}^{3}\lambda_{ij}\pi_{rs_2}^{(i+1-j)}\pi_{rs_1}^{(i)}
$$
$$
\times \left(\overline{A}(s_1) + \overline{B}(s_1) K(i,r)\overline{C}(i)\right)^T
$$
$$
\times P(T-t-1, j, s_2)
$$
$$
\left. \times \left(\overline{A}(s_1) + \overline{B}(s_1) K(i,r)\overline{C}(i)\right)\right\} X(t)
$$
$$
= X^T(t) Q(i,r) X(t). \tag{26}
$$

Letting $T \to \infty$ and noticing (22) and $Q(i, r) > 0$, we have

$$
\begin{aligned}
P(i, r) - \sum_{j=0}^{\tau} \sum_{s_2=1}^{3} \sum_{s_1=1}^{3} \lambda_{ij} \pi_{rs_2}^{(i+1-j)} \pi_{rs_1}^{(i)} \\
\times \left(\overline{A}(s_1) + \overline{B}(s_1) K(i, r) \overline{C}(i) \right)^T \\
\times P(j, s_2) \left(\overline{A}(s_1) \right. \\
\left. + \overline{B}(s_1) K(i, r) \overline{C}(i) \right) > 0.
\end{aligned}
\tag{27}
$$

This completes the proof. $\qquad \square$

Remark 7. Although the proof of Theorem 6 is similar to the literature [10], the considered problem is not the same. The *S-C* random delays and random sampling periods are considered in this paper; however, the *S-C* and *C-A* random delays are considered in [10]. In addition, the continuous systems are considered in this paper and the discrete systems are considered in [10].

Theorem 6 gives the sufficient and necessary conditions on the existence of the state feedback controller, when transition probabilities are completely accessible. However, the conditions in (11) are nonlinear in the controller matrices. In the following theorem, the equivalent LMI conditions to (11) are given.

Theorem 8. *There exists a controller* (4) *such that the closed-loop system in* (7) *is stochastically stable if and only if there exist matrices $K(i, r)$ and $M(j, s_2) > 0$, $P(i, r) > 0$ such that the following matrix inequalities*

$$
\begin{bmatrix} -P(i, r) & V^T(i, r) \\ * & -Y(i, r) \end{bmatrix} < 0,
\tag{28}
$$

$$
P(j, s_2) M(j, s_2) = I
$$

with

$$
V(i, r) = \begin{bmatrix} V_0^T(i, r) & V_1^T(i, r) & \cdots & V_{\tau^{sc}}^T(i, r) \end{bmatrix}^T,
$$

$$
V_j(i, r) = \begin{bmatrix} V_{j1}^T(i, r) & V_{j2}^T(i, r) & V_{j3}^T(i, r) \end{bmatrix}^T,
$$

$$
V_{js_2}(i, r)
$$

$$
= \begin{bmatrix} \left(\lambda_{ij} \pi_{rs_2}^{(i+1-j)} \pi_{r1}^{(i)} \right)^{1/2} \left[\overline{A}(1) + \overline{B}(1) K(i, r) \overline{C}(i) \right] \\ \left(\lambda_{ij} \pi_{rs_2}^{(i+1-j)} \pi_{r2}^{(i)} \right)^{1/2} \left[\overline{A}(2) + \overline{B}(2) K(i, r) \overline{C}(i) \right] \\ \left(\lambda_{ij} \pi_{rs_2}^{(i+1-j)} \pi_{r3}^{(i)} \right)^{1/2} \left[\overline{A}(3) + \overline{B}(3) K(i, r) \overline{C}(i) \right] \end{bmatrix},
$$

$$
Y(i, r) = \text{diag} \left\{ Y_0(i, r) \quad Y_1(i, r) \quad \cdots \quad Y_{\tau^{sc}}(i, r) \right\},
$$

$$
Y_j(i, r) = \text{diag} \left\{ Y_{j1}(i, r) \quad Y_{j2}(i, r) \quad Y_{j3}(i, r) \right\},
$$

$$
Y_{js_2}(i, r) = \text{diag} \left\{ M(j, s_2) \quad M(j, s_2) \quad M(j, s_2) \right\}
\tag{29}
$$

hold for all $i, j \in \varphi$ and r, s_1, and $s_2 \in \phi$.

Proof. By applying the Schur complement and letting $M(j, s_2) = P^{-1}(j, s_2)$, the proof can be readily completed. $\qquad \square$

Remark 9. The conditions in Theorem 8 are a set of LMIs with nonconvex constraints. This can be solved by several existing iterative LMI algorithms. Using a cone complementarity approach [22–24], we can solve this nonconvex feasibility problem by formulating it into an optimisation problem subject to LMI constraints.

Now, using a cone complementarity approach, we suggest the following nonlinear minimisation problem involving LMI conditions instead of the original nonconvex feasibility problem formulated in Theorem 8.
Minimise

$$
\text{Tr} \left(\sum_{j=0}^{\tau^{sc}} \sum_{s_2=1}^{3} P_{js_2} M_{js_2} \right)
\tag{30}
$$

subject to (28) and

$$
\begin{bmatrix} P(j, s_2) & I \\ I & M(j, s_2) \end{bmatrix} \geq 0.
\tag{31}
$$

Algorithm 10. (1) Find a feasible point $(P_{js_2}(0), M_{js_2}(0))$ satisfying (30). If there are none, set $k = 0$.

(2) Solve the following LMI problem

$$
\text{Min Tr} \left(\sum_{j=0}^{\tau^{sc}} \sum_{s_2=1}^{3} \left(P_{js_2}^k M_{js_2} + P_{js_2} M_{js_2}^k \right) \right) \text{ subject to } (31).
\tag{32}
$$

Set $P_{js_2}^{k+1} = P_{js_2}, M_{js_2}^{k+1} = M_{js_2}$.

(3) If the conditions in (31) are satisfied, then return to (1). If the conditions in (31) are not satisfied within a specified number $(k_{\max})$ of iterations, then exit. Otherwise, set $k = k+1$ and go to (2).

4. The Case of Partly Known Transition Probabilities

In the above section, the transition probabilities are assumed completely accessible and considered as the available knowledge for analyzing the NCSs. In practice, this kind of information including the variation of time delay and actual sampling period is hard to obtain. This variation can degrade the control performance and even make the system unstable.

In this section, the transition probabilities are assumed partially accessible. The transition probabilities in (3) are considered to be partially available; namely, some elements in matrices Π and Λ are time invariant but unknown. For instance,

$$
\Pi = \begin{bmatrix} \pi_{11} & ? & ? \\ ? & \pi_{22} & ? \\ \pi_{31} & \pi_{32} & \pi_{33} \end{bmatrix}, \qquad \Lambda = \begin{bmatrix} \lambda_{11} & ? & \lambda_{13} & ? \\ ? & \lambda_{22} & ? & \lambda_{24} \\ ? & ? & \lambda_{33} & \lambda_{34} \\ \lambda_{41} & \lambda_{42} & \lambda_{43} & \lambda_{44} \end{bmatrix},
\tag{33}
$$

where ? represents the unavailable elements. For notation clarity, $i \in \varphi, r \in \phi$, we denote

$$\varphi_\chi^i = \{j : \text{if } \lambda_{ij} \text{ is known}\},$$

$$\varphi_{\mu\chi}^i = \{j : \text{if } \lambda_{ij} \text{ is unknown}\},$$

$$\phi_\chi^r = \{s : \text{if } \pi_{rs} \text{ is known}\},$$

$$\phi_{\mu\chi}^r = \{s : \text{if } \pi_{rs} \text{ is unknown}\}.$$

(34)

Moreover, if $\varphi_\chi^i \neq 0, \phi_\chi^r \neq 0$, it is further described as

$$\varphi_\chi^i = \{\chi_0^i, \chi_1^i, \ldots, \chi_m^i\}, \quad 0 \leq m \leq \tau^{sc};$$

$$\phi_\chi^r = \{\chi_1^r, \ldots, \chi_n^r\}, \quad 1 \leq n \leq 3,$$

$$\varphi_{\mu\chi}^i = \{\overline{\chi}_1^i, \overline{\chi}_2^i, \ldots, \overline{\chi}_{\tau^{sc}-m}^i\}, \quad 0 \leq m \leq \tau^{sc},$$

$$\phi_{\mu\chi}^r = \{\overline{\chi}_1^r, \ldots, \overline{\chi}_{3-n}^r\}, \quad 1 \leq n \leq 3,$$

(35)

where χ_m^i represents the mth known element with the index in the ith row of matrix Λ, $\overline{\chi}_j^i$ $(j = 1, 2, \ldots, \tau^{sc} - m)$ represents the jth unknown element with the index in the ith row of matrix Λ, χ_n^r represents the nth known element with the index in the rth row of matrix Π, and $\overline{\chi}_j^r$ $(j = 1, 2, \ldots, 3 - n)$ represents the jth unknown element with the index in the rth row of matrix Π.

Remark 11. Note that the expression of (33)–(35) was first introduced for regular state-space Markovian systems [13, 14], and it covers completely known and completely unknown probabilities. If $\varphi_\chi^i = \varphi, \varphi_{\mu\chi}^i = 0$ or $\phi_\chi^r = \phi$, and $\phi_{\mu\chi}^r = 0$, the transition probabilities are completely available. If $\varphi_\chi^i = 0, \varphi_{\mu\chi}^i = \varphi$ or $\phi_\chi^r = 0$, and $\phi_{\mu\chi}^r = \phi$, the transition probabilities are completely unavailable.

Now, the following theorem presents a sufficient condition for the stochastic stability of the system described by (7) with partially known transition probabilities (33).

Theorem 12. *The system described by (7) with partially known transition probabilities (33) is stochastically stable if there exist matrices $P(i, r) > 0$ such that the following matrix inequalities*

$$\left(\sum_{j \in \varphi_\chi^i, s_1 \in \phi_\chi^r} \left(\sum_{s_2 \in \phi_\chi^r} \lambda_{ij} \pi_{rs_2}^{(i+1-j)} \pi_{rs_1}^{(i)} + \sum_{s_2 \in \phi_{\mu\chi}^r} \lambda_{ij} \pi_r^{(i+1-j)} \pi_{rs_1}^{(i)} \right) \right.$$

$$\left. + \sum_{j \in \varphi_\chi^i, s_1 \in \phi_{\mu\chi}^r} \left(\sum_{s_2 \in \phi_\chi^r} \lambda_{ij} \pi_{rs_2}^{(i+1-j)} \pi_r^{(i)} + \sum_{s_2 \in \phi_{\mu\chi}^r} \lambda_{ij} \pi_r^{(i+1-j)} \pi_r^{(i)} \right) \right.$$

$$+ \sum_{j \in \varphi_{\mu\chi}^i, s_1 \in \phi_\chi^r} \left(\sum_{s_2 \in \phi_\chi^r} \lambda_i \pi_{rs_2}^{(i+1-j)} \pi_{rs_1}^{(i)} + \sum_{s_2 \in \phi_{\mu\chi}^r} \lambda_i \pi_r^{(i+1-j)} \pi_{rs_1}^{(i)} \right)$$

$$\left. + \sum_{j \in \varphi_{\mu\chi}^i, s_1 \in \phi_{\mu\chi}^r} \left(\sum_{s_2 \in \phi_\chi^r} \lambda_i \pi_{rs_2}^{(i+1-j)} \pi_r^{(i)} + \sum_{s_2 \in \phi_{\mu\chi}^r} \lambda_i \pi_r^{(i+1-j)} \pi_r^{(i)} \right) \right)$$

$$* \widetilde{A}^T(i, r, s_1) P(j, s_2) \widetilde{A}(i, r, s_1) - P(i, r) < 0, \quad i = 0, 1, 2,$$

(36)

$$\left(\sum_{j \in \varphi_\chi^{i-2}, s_1=1}^{3} \left(\sum_{s_2 \in \phi_\chi^r} \lambda_{ij} + \sum_{s_2 \in \phi_{\mu\chi}^r} \lambda_{ij} \right) \right.$$

$$+ \sum_{j \in \varphi_{\mu\chi}^{i-2}, s_1=1}^{3} \left(\sum_{s_2 \in \phi_\chi^r} \lambda_i + \sum_{s_2 \in \phi_{\mu\chi}^r} \lambda_i \right)$$

$$+ \sum_{j \in \overline{\varphi}_\chi^{i-2}, s_1=1}^{3} \left(\sum_{s_2 \in \phi_\chi^r} \lambda_{ij} \pi_{rs_2}^{(i+1-j)} + \sum_{s_2 \in \phi_{\mu\chi}^r} \lambda_{ij} \pi_r^{(i+1-j)} \right)$$

$$\left. + \sum_{j \in \overline{\varphi}_{\mu\chi}^{i-2}, s_1=1}^{3} \left(\sum_{s_2 \in \phi_\chi^r} \lambda_i \pi_{rs_2}^{(i+1-j)} + \sum_{s_2 \in \phi_{\mu\chi}^r} \lambda_i \pi_r^{(i+1-j)} \right) \right)$$

$$* \widetilde{A}^T(i, r, s_1) P(j, s_2) \widetilde{A}(i, r, s_1) - P(i, r) < 0, \quad 3 \leq i \leq \tau^{sc}$$

(37)

hold for all $i \in \varphi$ and $r \in \phi$, where

$$\widetilde{A}(i, r, s_1) = \overline{A}(s_1) + \overline{B}(s_1) K(i, r) \overline{C}(i),$$

$$\lambda_i = 1 - \sum_{j \in \varphi_\chi^i} \lambda_{ij}, \quad \pi_r^{(i+1-j)} = 1 - \sum_{s_2 \in \phi_\chi^r} \pi_{rs_2}^{(i+1-j)},$$

$$\pi_r^{(i)} = 1 - \sum_{s_1 \in \phi_\chi^r} \pi_{rs_1}^{(i)},$$

$$\varphi_\chi^{i-2} = \{j : \text{if } \lambda_{ij} \text{ is known and } j \leq i - 2\},$$

$$\varphi_{\mu\chi}^{i-2} = \{j : \text{if } \lambda_{ij} \text{ is unknown and } j \leq i - 2\},$$

$$\overline{\varphi}_\chi^{i-2} = \{j : \text{if } \lambda_{ij} \text{ is known and } j > i - 2\},$$

$$\overline{\varphi}_{\mu\chi}^{i-2} = \{j : \text{if } \lambda_{ij} \text{ is unknown and } j > i - 2\}.$$

(38)

Proof. From Theorem 6, we know

$$L(i, r) = \sum_{j=0}^{\tau^{sc}} \sum_{s_2=1}^{3} \sum_{s_1=1}^{3} \lambda_{ij} \pi_{rs_2}^{(i+1-j)} \pi_{rs_1}^{(i)} \widetilde{A}^T(i, r, s_1) P(j, s_2)$$

$$\times \widetilde{A}(i, r, s_1) - P(i, r) < 0.$$

(39)

However, from (30), we know that some elements are inaccessible in Λ and Π. According to the property of Markov chains, we obtain

$$\overline{P}(j,r) = \sum_{s_2=1}^{3} \pi_{rs_2}^{(i+1-j)} P(j,s_2)$$

$$= \sum_{s_2 \in \phi_\chi^r} \pi_{rs_2}^{(i+1-j)} P(j,s_2) + \sum_{s_2 \in \phi_{\mu\chi}^r} \pi_{rs_2}^{(i+1-j)} P(j,s_2)$$

$$\leq \sum_{s_2 \in \phi_\chi^r} \pi_{rs_2}^{(i+1-j)} P(j,s_2) + \sum_{s_2 \in \phi_{\mu\chi}^r} \pi_r^{(i+1-j)} P(j,s_2),$$

$$(40)$$

$$\widetilde{P}(i,j,r) = \sum_{s_1=1}^{3} \pi_{rs_1}^{(i)} \widetilde{A}^T(i,r,s_1) \overline{P}(j,r) \widetilde{A}(i,r,s_1)$$

$$= \sum_{s_1 \in \phi_\chi^r} \pi_{rs_1}^{(i)} \widetilde{A}^T(i,r,s_1) \overline{P}(j,r) \widetilde{A}(i,r,s_1)$$

$$+ \sum_{s_1 \in \phi_{\mu\chi}^r} \pi_{rs_1}^{(i)} \widetilde{A}^T(i,r,s_1) \overline{P}(j,r) \widetilde{A}(i,r,s_1)$$

$$\leq \sum_{s_1 \in \phi_\chi^r} \pi_{rs_1}^{(i)} \widetilde{A}^T(i,r,s_1) \overline{P}(j,r) \widetilde{A}(i,r,s_1)$$

$$+ \sum_{s_1 \in \phi_{\mu\chi}^r} \pi_r^{(i)} \widetilde{A}^T(i,r,s_1) \overline{P}(j,r) \widetilde{A}(i,r,s_1),$$

$$(41)$$

$$\sum_{j=0}^{\tau^{sc}} \lambda_{ij} \widetilde{P}(i,j,r) = \sum_{j \in \varphi_\chi^i} \lambda_{ij} \widetilde{P}(i,r,j) + \sum_{j \in \varphi_{\mu\chi}^i} \lambda_{ij} \widetilde{P}(i,r,j)$$

$$\leq \sum_{j \in \varphi_\chi^i} \lambda_{ij} \widetilde{P}(i,r,j) + \sum_{j \in \varphi_{\mu\chi}^i} \lambda_i \widetilde{P}(i,r,j),$$

$$(42)$$

where $\widetilde{A}(i,r,s_1), \lambda_i, \pi_r^{(i+1-j)}, \pi_r^{(i)}$ are defined in (38).

According to Lemma 4, $\pi_{rs}^{(3)} = \sum_{l_2} \sum_{l_1} \pi_{rl_1} \pi_{l_1 l_2} \pi_{l_2 s}$. We obtain $\pi_{rs}^{(3)}$ (for all $r, s \in \phi$), that is inaccessible in $\Pi^{(3)}$, even if there is an inaccessible element in Π. So we obtain $\pi_{rs}^{(i)}$ (for all $r, s \in \phi, i \geq 3$), that is inaccessible in $\Pi^{(i)}$, and

$$\sum_{s \in \phi_{\mu\chi}^r} \pi_{rs}^{(i)} = \sum_{s \in \phi} \pi_{rs}^{(i)} = 1. \quad (43)$$

Substituting the formulae (40)–(43) into (39), we can obtain (36) and (37). This completes the proof. $\square$

Theorem 12 gives the sufficient conditions on the existence of the state feedback controller, when transition probabilities are partly accessible. However, the conditions in (36) and (37) are nonlinear in the controller matrices. In the following theorem, the equivalent LMI conditions to (36) and (37) are given.

Theorem 13. *Consider the system described by* (7) *with partially known transition probabilities* (30), *the corresponding system is stochastically stable if there exist matrices* $K(i,r)$ *and* $X(\chi_c^i, \chi_a^r) > 0$, $X(\overline{\chi}_d^i, \overline{\chi}_b^r) > 0$, $X(\chi_c^i, \overline{\chi}_b^r) > 0$, $X(\overline{\chi}_d^i, \chi_a^r) > 0$, $X(\chi_g^{i-2}, \chi_a^r) > 0$, $X(\chi_g^{i-2}, \overline{\chi}_b^r) > 0$, $X(\chi_e^{i-2}, \chi_a^r) > 0$, $X(\chi_e^{i-2}, \overline{\chi}_b^r) > 0$, $X(\overline{\chi}_f^{i-2}, \chi_a^r) > 0$, $X(\overline{\chi}_f^{i-2}, \overline{\chi}_b^r) > 0$, $X(\overline{\chi}_h^{i-2}, \chi_a^r) > 0$, $X(\overline{\chi}_h^{i-2}, \overline{\chi}_b^r) > 0$, $P(\chi_c^i, \chi_a^r) > 0$, $P(\overline{\chi}_d^i, \overline{\chi}_b^r) > 0$, $P(\chi_c^i, \overline{\chi}_b^r) > 0$, $P(\overline{\chi}_d^i, \chi_a^r) > 0$, $P(\chi_g^{i-2}, \chi_a^r) > 0$, $P(\chi_g^{i-2}, \overline{\chi}_b^r) > 0$, $P(\chi_e^{i-2}, \chi_a^r) > 0$, $P(\chi_e^{i-2}, \overline{\chi}_b^r) > 0$, $P(\overline{\chi}_f^{i-2}, \chi_a^r) > 0$, $P(\overline{\chi}_f^{i-2}, \overline{\chi}_b^r) > 0$, $P(\overline{\chi}_h^{i-2}, \chi_a^r) > 0$, $P(\overline{\chi}_h^{i-2}, \overline{\chi}_b^r) > 0$, *such that the following matrix inequalities:*

$$\begin{bmatrix} -P(i,r) & V^T(i,r) & \widetilde{V}^T(i,r) \\ * & -Y(i,r) & 0 \\ * & * & -\widetilde{Y}(i,r) \end{bmatrix} < 0, \quad (44)$$

$$P\left(\chi_c^i, \chi_a^r\right) X\left(\chi_c^i, \chi_a^r\right) = I,$$

$$P\left(\chi_c^i, \chi_a^r\right) X\left(\chi_c^i, \overline{\chi}_b^r\right) = I, \quad (45)$$

$$P\left(\overline{\chi}_d^i, \chi_a^r\right) X\left(\overline{\chi}_d^i, \chi_a^r\right) = I,$$

$$P\left(\overline{\chi}_d^i, \overline{\chi}_b^r\right) X\left(\overline{\chi}_d^i, \overline{\chi}_b^r\right) = I, \quad (46)$$

$$P\left(\chi_g^{i-2}, \chi_a^r\right) X\left(\chi_g^{i-2}, \chi_a^r\right) = I,$$

$$P\left(\chi_g^{i-2}, \overline{\chi}_b^r\right) X\left(\chi_g^{i-2}, \overline{\chi}_b^r\right) = I, \quad (47)$$

$$P\left(\chi_e^{i-2}, \chi_a^r\right) X\left(\chi_e^{i-2}, \chi_a^r\right) = I,$$

$$P\left(\chi_e^{i-2}, \overline{\chi}_b^r\right) X\left(\chi_e^{i-2}, \overline{\chi}_b^r\right) = I, \quad (48)$$

$$P\left(\overline{\chi}_f^{i-2}, \chi_a^r\right) X\left(\overline{\chi}_f^{i-2}, \chi_a^r\right) = I,$$

$$P\left(\overline{\chi}_f^{i-2}, \overline{\chi}_b^r\right) X\left(\overline{\chi}_f^{i-2}, \overline{\chi}_b^r\right) = I, \quad (49)$$

$$P\left(\overline{\chi}_h^{i-2}, \chi_a^r\right) X\left(\overline{\chi}_h^{i-2}, \chi_a^r\right) = I,$$

$$P\left(\overline{\chi}_h^{i-2}, \overline{\chi}_b^r\right) X\left(\overline{\chi}_h^{i-2}, \overline{\chi}_b^r\right) = I \quad (50)$$

with

$$i = 0, 1, 2,$$

$$V^T(i,r) = \begin{bmatrix} V_1^T(i,r) & V_2^T(i,r) \end{bmatrix},$$

$$\widetilde{V}^T(i,r) = \begin{bmatrix} \widetilde{V}_1^T(i,r) & \widetilde{V}_2^T(i,r) \end{bmatrix},$$

$$V_i(i,r) = \begin{bmatrix} V_{i\chi_0^i}^T(i,r) & V_{i\chi_1^i}^T(i,r) & \cdots & V_{i\chi_m^i}^T(i,r) \end{bmatrix}^T,$$

$$\widetilde{V}_i(i,r) = \begin{bmatrix} \widetilde{V}_{i\overline{\chi}_1^i}^T(i,r) & \widetilde{V}_{i\overline{\chi}_2^i}^T(i,r) & \cdots & \widetilde{V}_{i\overline{\chi}_{\tau^{sc}-m}^i}^T(i,r) \end{bmatrix}^T,$$

$$V_{1\chi_c^i}(i,r) = \begin{bmatrix} V_{1\chi_c^i \chi_1^r}^T(i,r) & V_{1\chi_c^i \chi_2^r}^T(i,r) & \cdots & V_{1\chi_c^i \chi_n^r}^T(i,r) \end{bmatrix}^T,$$

$$V_{2\chi_a^r}(i,r) = \begin{bmatrix} V_{2\chi_c^i \overline{\chi}_1^r}^T(i,r) & V_{2\chi_c^i \overline{\chi}_2^r}^T(i,r) & \cdots & V_{2\chi_c^i \overline{\chi}_{3-n}^r}^T(i,r) \end{bmatrix}^T,$$

$$\widetilde{V}_{1\overline{\chi}_d^i}(i,r) = \left[\widetilde{V}_{1\overline{\chi}_d\chi_1^r}^T(i,r) \ \ \widetilde{V}_{1\overline{\chi}_d\chi_2^r}^T(i,r) \ \cdots \ \widetilde{V}_{1\overline{\chi}_d\chi_n^r}^T(i,r) \right]^T,$$

$$\widetilde{V}_{2\overline{\chi}_d^i}(i,r) = \left[\widetilde{V}_{2\overline{\chi}_d\overline{\chi}_1^r}^T(i,r) \ \widetilde{V}_{2\overline{\chi}_d\overline{\chi}_2^r}^T(i,r) \ \cdots \ \widetilde{V}_{2\overline{\chi}_d\overline{\chi}_{3-n}^r}^T(i,r) \right]^T,$$

$$V_{1\chi_c^i\chi_a^r}(i,r) = \begin{bmatrix} \left(\lambda_{i\chi_c^i}\pi_{r\chi_0^r}^{(i+1-\chi_c^i)}\pi_{r\chi_a^r}^{(i)}\right)^{1/2}\widetilde{A}(i,r,\chi_a^r) \\ \vdots \\ \left(\lambda_{i\chi_c^i}\pi_{r\chi_m^r}^{(i+1-\chi_c^i)}\pi_{r\chi_a^r}^{(i)}\right)^{1/2}\widetilde{A}(i,r,\chi_a^r) \\ \left(\lambda_{i\chi_c^i}\pi_r^{(i+1-\chi_c^i)}\pi_{r\chi_a^r}^{(i)}\right)^{1/2}\widetilde{A}(i,r,\chi_a^r) \\ \vdots \\ \left(\lambda_{i\chi_c^i}\pi_r^{(i+1-\chi_c^i)}\pi_{r\chi_a^r}^{(i)}\right)^{1/2}\widetilde{A}(i,r,\chi_a^r) \end{bmatrix},$$

$$V_{2\chi_c^i\overline{\chi}_b^r}(i,r) = \begin{bmatrix} \left(\lambda_{i\chi_c^i}\pi_{r\chi_0^r}^{(i+1-\chi_c^i)}\pi_r^{(i)}\right)^{1/2}\widetilde{A}(i,r,\overline{\chi}_b^r) \\ \vdots \\ \left(\lambda_{i\chi_c^i}\pi_{r\chi_m^r}^{(i+1-\chi_c^i)}\pi_r^{(i)}\right)^{1/2}\widetilde{A}(i,r,\overline{\chi}_b^r) \\ \left(\lambda_{i\chi_c^i}\pi_r^{(i+1-\chi_c^i)}\pi_r^{(i)}\right)^{1/2}\widetilde{A}(i,r,\overline{\chi}_b^r) \\ \vdots \\ \left(\lambda_{i\chi_c^i}\pi_r^{(i+1-\chi_c^i)}\pi_r^{(i)}\right)^{1/2}\widetilde{A}(i,r,\overline{\chi}_b^r) \end{bmatrix},$$

$$\widetilde{V}_{1\overline{\chi}_d^i\chi_a^r}(i,r) = \begin{bmatrix} \left(\lambda_i\pi_{r\chi_0^r}^{(i+1-\overline{\chi}_d^i)}\pi_{r\chi_a^r}^{(i)}\right)^{1/2}\widetilde{A}(i,r,\chi_a^r) \\ \vdots \\ \left(\lambda_i\pi_{r\chi_m^r}^{(i+1-\overline{\chi}_d^i)}\pi_{r\chi_a^r}^{(i)}\right)^{1/2}\widetilde{A}(i,r,\chi_a^r) \\ \left(\lambda_i\pi_r^{(i+1-\overline{\chi}_d^i)}\pi_{r\chi_a^r}^{(i)}\right)^{1/2}\widetilde{A}(i,r,\chi_a^r) \\ \vdots \\ \left(\lambda_i\pi_r^{(i+1-\overline{\chi}_d^i)}\pi_{r\chi_a^r}^{(i)}\right)^{1/2}\widetilde{A}(i,r,\chi_a^r) \end{bmatrix},$$

$$\widetilde{V}_{2\overline{\chi}_d^i\overline{\chi}_b^r}(i,r) = \begin{bmatrix} \left(\lambda_i\pi_{r\chi_0^r}^{(i+1-\overline{\chi}_d^i)}\pi_r^{(i)}\right)^{1/2}\widetilde{A}(i,r,\overline{\chi}_b^r) \\ \vdots \\ \left(\lambda_i\pi_{r\chi_m^r}^{(i+1-\overline{\chi}_d^i)}\pi_r^{(i)}\right)^{1/2}\widetilde{A}(i,r,\overline{\chi}_b^r) \\ \left(\lambda_i\pi_r^{(i+1-\overline{\chi}_d^i)}\pi_r^{(i)}\right)^{1/2}\widetilde{A}(i,r,\overline{\chi}_b^r) \\ \vdots \\ \left(\lambda_i\pi_r^{(i+1-\overline{\chi}_d^i)}\pi_r^{(i)}\right)^{1/2}\widetilde{A}(i,r,\overline{\chi}_b^r) \end{bmatrix},$$

$$Y(i,r) = \text{diag}\{Y_1(i,r), \ Y_2(i,r)\},$$

$$\widetilde{Y}(i,r) = \text{diag}\{\widetilde{Y}_1(i,r), \ \widetilde{Y}_2(i,r)\},$$

$$Y_i(i,r) = \text{diag}\{Y_{i\chi_0^i}(i,r),$$
$$Y_{i\chi_1^i}(i,r),\ldots,Y_{i\chi_m^i}(i,r)\},$$

$$\widetilde{Y}_i(i,r) = \text{diag}\{\widetilde{Y}_{i\overline{\chi}_1^i}(i,r),$$
$$\widetilde{Y}_{i\overline{\chi}_2^i}(i,r),\ldots,\widetilde{Y}_{i\overline{\chi}_{\tau^{sc}-m}^i}(i,r)\},$$

$$Y_{1\chi_c^i}(i,r) = \text{diag}\{Y_{1\chi_c^i\chi_1^r}(i,r),$$
$$Y_{1\chi_c^i\chi_2^r}(i,r),\ldots,Y_{1\chi_c^i\chi_n^r}(i,r)\},$$

$$Y_{2\chi_c^r}(i,r) = \text{diag}\{Y_{2\chi_c^i\overline{\chi}_1^r}(i,r),$$
$$Y_{2\chi_c^i\overline{\chi}_2^r}(i,r),\ldots,Y_{2\chi_c^i\overline{\chi}_{3-n}^r}(i,r)\},$$

$$Y_{1\chi_c^i\chi_a^r}(i,r) = Y_{2\chi_a^r\overline{\chi}_b^r}(i,r)$$
$$= \text{diag}\{X(\chi_c^i,\chi_0^r),\ldots,X(\chi_c^i,\chi_m^r),$$
$$X(\chi_c^i,\overline{\chi}_1^r),\ldots,X(\chi_c^i,\overline{\chi}_{\tau^{sc}-m}^r)\},$$

$$\widetilde{Y}_{1\overline{\chi}_d^i}(i,r) = \text{diag}\{\widetilde{Y}_{1\overline{\chi}_d\chi_1^r}(i,r),$$
$$\widetilde{Y}_{1\overline{\chi}_d\chi_2^r}(i,r),\ldots,\widetilde{Y}_{1\overline{\chi}_d\chi_n^r}(i,r)\},$$

$$\widetilde{Y}_{2\overline{\chi}_b^r}(i,r) = \text{diag}\{\widetilde{Y}_{2\overline{\chi}_d\overline{\chi}_1^r}(i,r),$$
$$\widetilde{Y}_{2\overline{\chi}_d\overline{\chi}_2^r}(i,r),\ldots,\widetilde{Y}_{2\overline{\chi}_d\overline{\chi}_{3-n}^r}(i,r)\},$$

$$\widetilde{Y}_{1\overline{\chi}_d^i\chi_a^r}(i,r) = \widetilde{Y}_{2\overline{\chi}_b^r\overline{\chi}_b^r}(i,r)$$
$$= \text{diag}\{X(\overline{\chi}_d^i,\chi_0^r),\ldots,X(\overline{\chi}_d^i,\chi_m^r),$$
$$X(\overline{\chi}_d^i,\overline{\chi}_1^r),\ldots,X(\overline{\chi}_d^i,\overline{\chi}_{\tau^{sc}-m}^r)\},$$

$$3 \le i \le \tau^{sc}$$

$$V^T(i,r) = \left[V_1^T(i,r) \ V_2^T(i,r)\right],$$

$$\widetilde{V}^T(i,r) = \left[\widetilde{V}_1^T(i,r) \ \widetilde{V}_2^T(i,r)\right],$$

$$V_1(i,r) = \left[V_{1\chi_0^{i-2}}^T(i,r) \ \cdots \ V_{1\chi_{\overline{m}}^{i-2}}^T(i,r)\right]^T,$$

$$V_2(i,r) = \left[V_{2\chi_{\overline{m}+1}^{i-2}}^T(i,r) \ \cdots \ V_{2\chi_m^{i-2}}^T(i,r)\right]^T,$$

$$\widetilde{V}_1(i,r) = \left[\widetilde{V}_{1\overline{\chi}_1^{i-2}}^T(i,r) \ \cdots \ \widetilde{V}_{1\overline{\chi}_{\overline{m}}^{i-2}}^T(i,r)\right]^T,$$

$$\widetilde{V}_2(i,r) = \left[\widetilde{V}_{2\overline{\chi}_{\overline{m}+1}^{i-2}}^T(i,r) \ \cdots \ \widetilde{V}_{2\overline{\chi}_{\tau^{sc}-m}^{i-2}}^T(i,r)\right]^T,$$

$$V_{1\chi_g^{i-2}}(i,r)$$
$$= \left[V_{1\chi_g^{i-2}1}^T(i,r) \ V_{1\chi_g^{i-2}2}^T(i,r) \ V_{1\chi_g^{i-2}3}^T(i,r)\right]^T,$$

$$V_{2\overline{\chi}_h^{i-2}}(i,r)$$
$$= \left[V_{2\overline{\chi}_h^{i-2}1}^T(i,r) \ V_{2\overline{\chi}_h^{i-2}2}^T(i,r) \ V_{2\overline{\chi}_h^{i-2}3}^T(i,r)\right]^T,$$

$$\widetilde{V}_{1\chi_e^{i-2}}(i,r)$$
$$= \left[\widetilde{V}_{1\chi_e^{i-2}1}^T(i,r) \;\; \widetilde{V}_{1\chi_e^{i-2}2}^T(i,r) \;\; \widetilde{V}_{1\chi_e^{i-2}3}^T(i,r) \right]^T,$$

$$\widetilde{V}_{2\overline{\chi}_f^{i-2}}(i,r)$$
$$= \left[\widetilde{V}_{2\overline{\chi}_f^{i-2}1}^T(i,r) \;\; \widetilde{V}_{2\overline{\chi}_f^{i-2}2}^T(i,r) \;\; \widetilde{V}_{2\overline{\chi}_f^{i-2}3}^T(i,r) \right]^T,$$

$$V_{1\chi_g^{i-2}s_1}(i,r) = \begin{bmatrix} \left(\lambda_{i\chi_g^{i-2}}\right)^{1/2}\widetilde{A}(i,r,s_1) \\ \vdots \\ \left(\lambda_{i\chi_g^{i-2}}\right)^{1/2}\widetilde{A}(i,r,s_1) \\ \left(\lambda_{i\chi_g^{i-2}}\right)^{1/2}\widetilde{A}(i,r,s_1) \\ \vdots \\ \left(\lambda_{i\chi_g^{i-2}}\right)^{1/2}\widetilde{A}(i,r,s_1) \end{bmatrix},$$

$$V_{2\overline{\chi}_h^{i-2}s_1}(i,r) = \begin{bmatrix} (\lambda_i)^{1/2}\widetilde{A}(i,r,s_1) \\ \vdots \\ (\lambda_i)^{1/2}\widetilde{A}(i,r,s_1) \\ (\lambda_i)^{1/2}\widetilde{A}(i,r,s_1) \\ \vdots \\ (\lambda_i)^{1/2}\widetilde{A}(i,r,s_1) \end{bmatrix},$$

$$\widetilde{V}_{1\chi_e^{i-2}s_1}(i,r) = \begin{bmatrix} \left(\lambda_{i\chi_e^{i-2}}\pi_{r\chi_1^r}^{(i+1-\chi_e^{i-2})}\right)^{1/2}\widetilde{A}(i,r,s_1) \\ \vdots \\ \left(\lambda_{i\chi_e^{i-2}}\pi_{r\chi_m^r}^{(i+1-\chi_e^{i-2})}\right)^{1/2}\widetilde{A}(i,r,s_1) \\ \left(\lambda_{i\chi_e^{i-2}}\pi_r^{(i+1-\chi_e^{i-2})}\right)^{1/2}\widetilde{A}(i,r,s_1) \\ \vdots \\ \left(\lambda_{i\chi_e^{i-2}}\pi_r^{(i+1-\chi_e^{i-2})}\right)^{1/2}\widetilde{A}(i,r,s_1) \end{bmatrix},$$

$$\widetilde{V}_{2\overline{\chi}_f^{i-2}s_1}(i,r) = \begin{bmatrix} \left(\lambda_i\pi_{r\chi_0^{sc}}^{(i+1-\overline{\chi}_f^{i-2})}\right)^{1/2}\widetilde{A}(i,r,s_1) \\ \vdots \\ \left(\lambda_i\pi_{r\chi_m^{sc}}^{(i+1-\overline{\chi}_f^{i-2})}\right)^{1/2}\widetilde{A}(i,r,s_1) \\ \left(\lambda_i\pi_r^{(i+1-\overline{\chi}_f^{i-2})}\right)^{1/2}\widetilde{A}(i,r,s_1) \\ \vdots \\ \left(\lambda_i\pi_r^{(i+1-\overline{\chi}_f^{i-2})}\right)^{1/2}\widetilde{A}(i,r,s_1) \end{bmatrix},$$

$$Y(i,r) = \mathrm{diag}\left\{ Y_1(i,r), Y_2(i,r) \right\},$$
$$\widetilde{Y}_1(i,r) = \mathrm{diag}\left\{ \widetilde{Y}_1(i,r), \widetilde{Y}_2(i,r) \right\},$$
$$Y_1(i,r) = \mathrm{diag}\left\{ Y_{1\chi_0^{i-2}}(i,r), \ldots, Y_{1\chi_{\overline{m}}^{i-2}}(i,r) \right\},$$

$$Y_2(i,r)$$
$$= \mathrm{diag}\left\{ Y_{2\chi_{\overline{m}+1}^{i-2}}(i,r), \ldots, Y_{2\chi_m^{i-2}}(i,r) \right\},$$

$$\widetilde{Y}_1(i,r)$$
$$= \mathrm{diag}\left\{ \widetilde{Y}_{1\overline{\chi}_1^{i-2}}(i,r), \ldots, \widetilde{Y}_{1\overline{\chi}_{\overline{m}}^{i-2}}(i,r) \right\},$$

$$\widetilde{Y}_2(i,r)$$
$$= \mathrm{diag}\left\{ \widetilde{Y}_{2\overline{\chi}_{\overline{m}+1}^{i-2}}(i,r), \ldots, \widetilde{Y}_{2\overline{\chi}_{\tau^{sc}-m}^{i-2}}(i,r) \right\},$$

$$Y_{1\chi_g^{i-2}}(i,r)$$
$$= \mathrm{diag}\left\{ Y_{1\chi_g^{i-2}1}(i,r), \right.$$
$$\left. Y_{1\chi_g^{i-2}2}(i,r), Y_{1\chi_g^{i-2}3}(i,r) \right\},$$

$$Y_{2\overline{\chi}_h^{i-2}}(i,r)$$
$$= \mathrm{diag}\left\{ Y_{2\overline{\chi}_h^{i-2}1}(i,r), \right.$$
$$\left. Y_{2\overline{\chi}_h^{i-2}2}(i,r), Y_{2\overline{\chi}_h^{i-2}3}(i,r) \right\},$$

$$\widetilde{Y}_{1\chi_e^{i-2}}(i,r)$$
$$= \mathrm{diag}\left\{ \widetilde{Y}_{1\chi_e^{i-2}1}(i,r), \right.$$
$$\left. \widetilde{Y}_{1\chi_e^{i-2}2}(i,r), \widetilde{Y}_{1\chi_e^{i-2}3}(i,r) \right\},$$

$$\widetilde{Y}_{2\overline{\chi}_f^{i-2}}(i,r)$$
$$= \mathrm{diag}\left\{ \widetilde{Y}_{2\overline{\chi}_f^{i-2}1}(i,r), \right.$$
$$\left. \widetilde{Y}_{2\overline{\chi}_f^{i-2}2}(i,r), \widetilde{Y}_{2\overline{\chi}_f^{i-2}3}(i,r) \right\},$$

$$Y_{1\chi_g^{i-2}s_1}(i,r)$$
$$= \mathrm{diag}\left\{ X\left(\chi_g^{i-2},\chi_1^r\right), \ldots, X\left(\chi_g^{i-2},\chi_n^r\right), \right.$$
$$\left. X\left(\chi_g^{i-2},\overline{\chi}_1^r\right), \ldots, X\left(\chi_g^{i-2},\overline{\chi}_{3-n}^r\right) \right\},$$

$$Y_{2\overline{\chi}_h^{i-2}s_1}(i,r)$$
$$= \mathrm{diag}\left\{ X\left(\overline{\chi}_h^{i-2},\chi_1^r\right), \ldots, X\left(\overline{\chi}_h^{i-2},\chi_n^r\right), \right.$$
$$\left. X\left(\overline{\chi}_h^{i-2},\overline{\chi}_1^r\right), \ldots, X\left(\overline{\chi}_{\tau^{sc}-m}^{i-2},\overline{\chi}_{3-n}^r\right) \right\},$$

$$\widetilde{Y}_{1\chi_e^{i-2}s_1}(i,r)$$
$$= \mathrm{diag}\left\{ X\left(\chi_e^{i-2},\chi_1^r\right), \ldots, X\left(\chi_e^{i-2},\chi_n^r\right), \right.$$
$$\left. X\left(\chi_e^{i-2},\overline{\chi}_1^r\right), \ldots, X\left(\chi_e^{i-2},\overline{\chi}_{3-n}^r\right) \right\},$$

$$\widetilde{Y}_{2\overline{\chi}_f^{i-2}s_1}(i,r)$$

$$= \text{diag}\left\{X\left(\overline{\chi}_f^{i-2}, \chi_1^r\right), \ldots, X\left(\overline{\chi}_f^{i-2}, \chi_n^r\right),\right.$$

$$\left. X\left(\overline{\chi}_f^{i-2}, \overline{\chi}_1^r\right), \ldots, X\left(\overline{\chi}_f^{i-2}, \overline{\chi}_{3-n}^r\right)\right\} \tag{51}$$

hold for all $i, j \in \varphi$, $r, s_1, s_2 \in \phi$, $\chi_a^r \in \phi_\chi^r$, $\overline{\chi}_b^r \in \phi_{\mu\chi}^r$, $\chi_c^i \in \varphi_\chi^i$, $\overline{\chi}_d^i \in \varphi_{\mu\chi}^i$, $\chi_g^{i-2} \in \varphi_\chi^{i-2}$, and $\overline{\chi}_h^{i-2} \in \varphi_{\mu\chi}^{i-2}$, where

$$\varphi_\chi^{i-2} = \left\{\chi_0^{i-2}, \chi_1^{i-2}, \ldots, \chi_{\overline{m}}^{i-2}\right\}, \quad 0 \le \overline{m} \le m,$$

$$\overline{\varphi}_\chi^{i-2} = \left\{\chi_{\overline{m}+1}^{i-2}, \chi_{\overline{m}+2}^{i-2}, \ldots, \chi_m^{i-2}\right\}, \quad 0 \le m \le \tau^{sc},$$

$$\varphi_{\mu\chi}^{i-2} = \left\{\overline{\chi}_1^{i-2}, \overline{\chi}_2^{i-2}, \ldots, \overline{\chi}_{\overline{m}}^{i-2}\right\}, \quad 1 \le \overline{m} \le \tau^{sc} - m,$$

$$\overline{\varphi}_{\mu\chi}^{i-2} = \left\{\overline{\chi}_{\overline{m}+1}^{i-2}, \overline{\chi}_{\overline{m}+2}^{i-2}, \ldots, \overline{\chi}_{\tau^{sc}-m}^{i-2}\right\}, \quad 0 \le m \le \tau^{sc},$$

$$\pi_r^{(i+1-\chi_e^{i-2})} = 1 - \sum_{s_2 \in \phi_\chi^r} \pi_{rs_2}^{(i+1-\chi_e^{i-2})},$$

$$\pi_r^{(i+1-\overline{\chi}_f^{i-2})} = 1 - \sum_{s_2 \in \phi_\chi^r} \pi_{rs_2}^{(i+1-\overline{\chi}_f^{i-2})}, \tag{52}$$

$$\pi_r^{(i+1-\chi_c^i)} = 1 - \sum_{s_2 \in \phi_\chi^r} \pi_{rs_2}^{(i+1-\chi_c^i)},$$

$$\pi_r^{(i+1-\overline{\chi}_d^i)} = 1 - \sum_{s_2 \in \phi_\chi^r} \pi_{rs_2}^{(i+1-\overline{\chi}_d^i)},$$

$$\forall \chi_e^{i-2} \in \overline{\varphi}_\chi^{i-2}, \ \overline{\chi}_f^{i-2} \in \overline{\varphi}_{\mu\chi}^{i-2}, \ \chi_c^i \in \varphi_\chi^i, \ \overline{\chi}_d^i \in \varphi_{\mu\chi}^i,$$

and $\widetilde{A}(i, r, s_1), \lambda_i$, and $\pi_r^{(i)}$ are defined in (38).

Proof. By applying the Schur complement and letting $X(\chi_c^i, \chi_a^r) = P^{-1}(\chi_c^i, \chi_a^r)$, $X(\overline{\chi}_d^i, \overline{\chi}_b^r) = P^{-1}(\overline{\chi}_d^i, \overline{\chi}_b^r)$, $X(\chi_c^i, \chi_a^r) = P^{-1}(\chi_c^i, \chi_a^r)$, $X(\overline{\chi}_d^i, \overline{\chi}_b^r) = P^{-1}(\overline{\chi}_d^i, \overline{\chi}_b^r)$, $X(\chi_c^i, \overline{\chi}_b^r) = P^{-1}(\chi_c^i, \overline{\chi}_b^r)$, $X(\overline{\chi}_d^i, \chi_a^r) = P^{-1}(\overline{\chi}_d^i, \chi_a^r)$, $X(\chi_g^{i-2}, \chi_a^r) = P^{-1}(\chi_g^{i-2}, \chi_a^r)$, $X(\chi_g^{i-2}, \overline{\chi}_b^r) = P^{-1}(\chi_g^{i-2}, \overline{\chi}_b^r)$, $X(\chi_e^{i-2}, \chi_a^r) = P^{-1}(\chi_e^{i-2}, \chi_a^r)$, $X(\chi_e^{i-2}, \overline{\chi}_b^r) = P^{-1}(\chi_e^{i-2}, \overline{\chi}_b^r)$, $X(\overline{\chi}_f^{i-2}, \chi_a^r) = P^{-1}(\overline{\chi}_f^{i-2}, \chi_a^r)$, $X(\overline{\chi}_f^{i-2}, \overline{\chi}_b^r) = P^{-1}(\overline{\chi}_f^{i-2}, \overline{\chi}_b^r)$, $X(\overline{\chi}_h^{i-2}, \overline{\chi}_b^r) = P^{-1}(\overline{\chi}_h^{i-2}, \overline{\chi}_b^r)$, and $X(\overline{\chi}_h^{i-2}, \chi_a^r) = P^{-1}(\overline{\chi}_h^{i-2}, \chi_a^r)$, the proof can be readily completed. □

Remark 14. The nonconvex feasibility problem (45)–(50) can be formulated as an optimisation problem subject to LMI constraints by using Algorithm 10.

5. Number Example

To illustrate the effectiveness of the proposed method, we consider the cart and inverted pendulum problem [8–10] in Figure 2, where m_1 is the cart mass, m_2 is the pendulum mass, L is the length from the point of rotation to the center of gravity of the pendulum, x is the cart position, θ is the
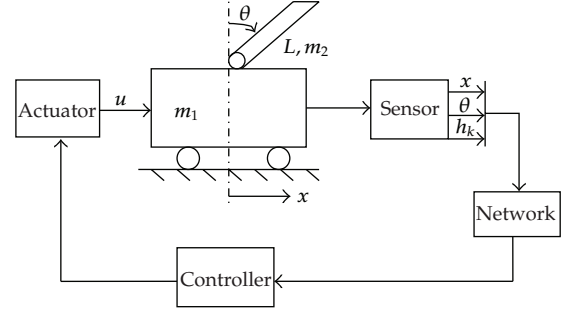

FIGURE 2: Cart and inverted pendulum system.

pendulum angular position, and u is the input force. This is a fourth-order unstable system. The state variables are $x_1 = x$, $x_2 = \dot{x}$, $x_3 = \theta$, and $x_4 = \dot{\theta}$. The parameters used are $m_1 = 1\,\text{kg}$, $m_2 = 0.5\,\text{kg}$, and $L = 1\,\text{m}$ and smooth surfaces. The state-space model is

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{53}$$

where

$$A = \begin{bmatrix} 1.0000 & 1.0000 & -0.0005 & 0.0005 \\ 0 & 1.0000 & -3.2663 & 0 \\ 0 & 0 & -0.0002 & 1.0000 \\ 0 & 0 & 19.6007 & -0.0002 \end{bmatrix},$$

$$B = \begin{bmatrix} -0.0003 \\ 0.8448 \\ -0.0003 \\ -1.3328 \end{bmatrix}. \tag{54}$$

Assume the minimum sampling period is $0.1s$, the maximum sampling period is $0.3s$, then $h_k \in \{0.1s, 0.2s, 0.3s\}$. Assume the case where random delays only exist from the sensor to the controller, and $\tau_k^{sc} \in \{0, 1\}$, and their transition probability matrices are given by

$$\Lambda = \begin{bmatrix} 0.4 & 0.6 \\ 0.5 & 0.5 \end{bmatrix}, \qquad \Pi = \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.2 & 0.5 & 0.3 \\ 0.1 & 0.4 & 0.5 \end{bmatrix}. \tag{55}$$

Figure 3 shows part of the simulated run of the S-C delays τ_k^{sc} and sampling periods h_k governed by their corresponding transition probability matrices, respectively.

In the above assumption, we discretize the system (53) and obtain

$$\Phi_1 = \begin{bmatrix} 1.1052 & 0.1105 & -0.0177 & -0.0005 \\ 0 & 1.1052 & -0.3546 & -0.0172 \\ 0 & 0 & 1.0996 & 0.1033 \\ 0 & 0 & 2.0247 & 1.0996 \end{bmatrix},$$

$$\Phi_2 = \begin{bmatrix} 1.2214 & 0.2443 & -0.0794 & -0.0049 \\ 0 & 1.2214 & -0.8165 & -0.0745 \\ 0 & 0 & 1.4182 & 0.2272 \\ 0 & 0 & 4.4527 & 1.4182 \end{bmatrix},$$
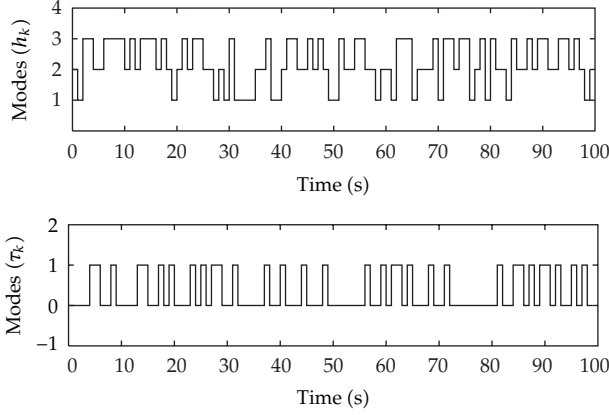
FIGURE 3: Modes with τ_k and h_k.

$$\Phi_3 = \begin{bmatrix} 1.3499 & 0.4050 & -0.2055 & -0.0184 \\ 0 & 1.3499 & -1.4816 & -0.1872 \\ 0 & 0 & 2.0194 & 0.3963 \\ 0 & 0 & 7.7676 & 2.0194 \end{bmatrix},$$

$$\Gamma_1 = \begin{bmatrix} 0.0045 \\ 0.0896 \\ -0.0068 \\ -0.1377 \end{bmatrix}, \qquad \Gamma_2 = \begin{bmatrix} 0.0196 \\ 0.1934 \\ -0.0285 \\ -0.3029 \end{bmatrix},$$

$$\Gamma_3 = \begin{bmatrix} 0.0482 \\ 0.3187 \\ -0.0694 \\ -0.5285 \end{bmatrix}.$$

$$(56)$$

The eigenvalues of A are 1, 1, 4.4272, and -4.4275, the eigenvalues of Φ_1 are 1.1052, 1.1052, 0.6423, and 1.5569, the eigenvalues of Φ_2 are 1.2214, 1.2214, 2.4240, and 0.4125, the eigenvalues of Φ_3 are 1.3499, 1.3499, 3.7739, and 0.2649. Hence, the system is unstable. The state trajectory of the open-loop system is shown in Figure 4.

By Theorem 8 and Algorithm 10, when transition probabilities are completely known, we can obtain the controllers with the following matrices $K(\tau_k^{sc}, h_{k-\tau_k^{sc}})$:

$$K01 = \begin{bmatrix} -0.1722 & -1.4488 & 12.0905 & 3.2261 \end{bmatrix},$$

$$K02 = \begin{bmatrix} -0.5648 & -1.8662 & 0.8811 & 4.2691 \end{bmatrix},$$

$$K03 = \begin{bmatrix} -0.5648 & -1.8662 & 0.8811 & 4.2690 \end{bmatrix},$$

$$K11 = \begin{bmatrix} -0.4764 & -1.6341 & 1.1288 & 3.6632 \end{bmatrix},$$

$$K12 = \begin{bmatrix} -0.5657 & -1.8687 & 0.8783 & 4.2466 \end{bmatrix},$$

$$K13 = \begin{bmatrix} -0.5655 & -1.8397 & 0.8806 & 4.2296 \end{bmatrix}.$$

$$(57)$$

By Theorem 13, when transition probabilities are partly known; that is,

$$\Lambda = \begin{bmatrix} 0.4 & 0.6 \\ ? & ? \end{bmatrix}, \qquad \Pi = \begin{bmatrix} 0.5 & ? & ? \\ ? & 0.5 & ? \\ 0.1 & 0.4 & 0.5 \end{bmatrix}, \qquad (58)$$
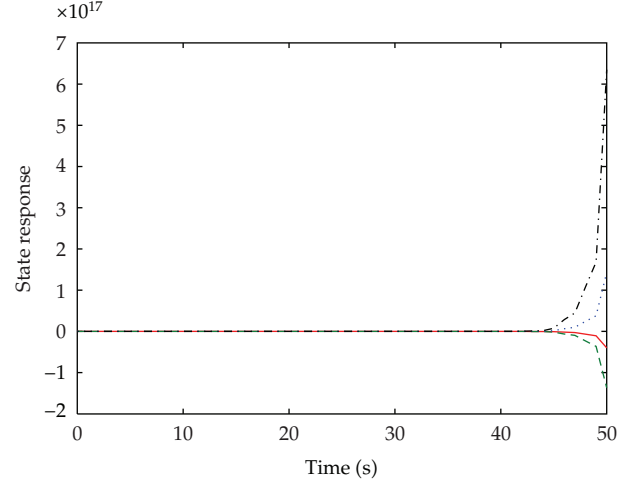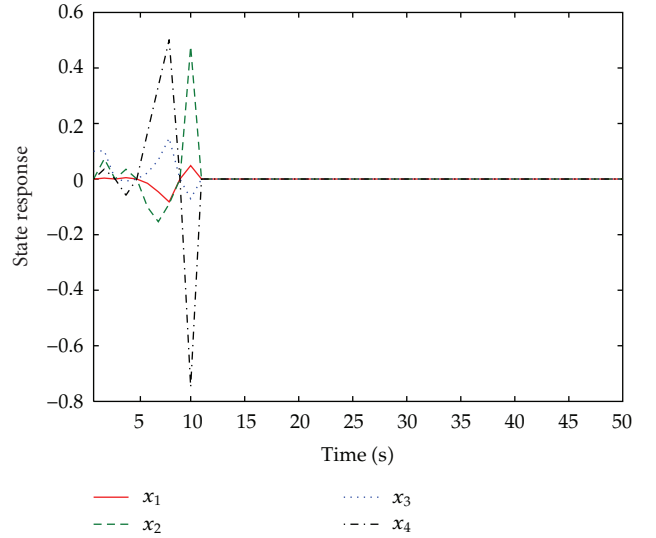

FIGURE 4: States of the open-loop system.

FIGURE 5: States of the closed-loop system (completely known).

we can obtain the controllers with the following matrices $K(\tau_k^{sc}, h_{k-\tau_k^{sc}})$:

$$K01 = \begin{bmatrix} -0.1534 & 1.5482 & 10.1902 & 2.2463 \end{bmatrix},$$

$$K02 = \begin{bmatrix} 0.5648 & -1.8662 & -0.5814 & 14.2531 \end{bmatrix},$$

$$K03 = \begin{bmatrix} -0.5648 & -1.8662 & 0.8811 & 3.9625 \end{bmatrix},$$

$$K11 = \begin{bmatrix} -0.3561 & -1.4026 & 1.4723 & 2.9558 \end{bmatrix},$$

$$K12 = \begin{bmatrix} -0.5659 & -1.8432 & 0.8791 & 4.2211 \end{bmatrix},$$

$$K13 = \begin{bmatrix} -0.5646 & -1.7649 & 0.8869 & 4.1893 \end{bmatrix}.$$

$$(59)$$

Assume that the initial condition for simulation is $\theta(0) = 0.1$ rad, and all other initial states are zero. The state trajectories of the closed-loop system caused by the discretized model are shown in Figures 5 and 6, respectively.
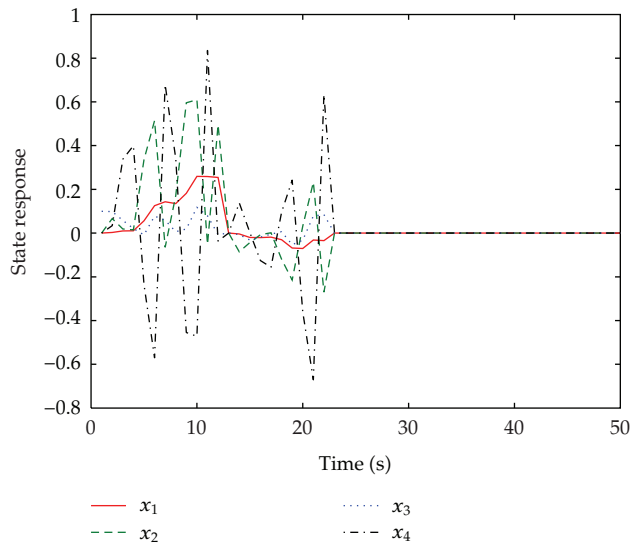
FIGURE 6: States of the closed-loop system (partly known).

6. Conclusions

This paper proposes a state feedback controller design method for NCSs with random time delays and random sampling periods. Sampling periods also can randomly switch between three cases according to the network load. The *S-C* random delays and random sampling periods are modeled as Markov chains. The transition probabilities do not need to be completely known. The closed-loop systems can be expressed as jump linear systems with two modes. Sufficient conditions of stochastic stability are obtained in terms of a set of LMIs with matrix inversion constraints. Further, the state feedback controller is designed via the iterative linear matrix inequality approach and the state feedback gain depends on the two modes.

Acknowledgments

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