

## Research Article

# HAM-Based Adaptive Multiscale Meshless Method for Burgers Equation

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Based on the multilevel interpolation theory, we constructed a meshless adaptive multiscale interpolation operator (MAMIO) with the radial basis function. Using this operator, any nonlinear partial differential equations such as Burgers equation can be discretized adaptively in physical spaces as a nonlinear matrix ordinary differential equation. In order to obtain the analytical solution of the system of ODEs, the homotopy analysis method (HAM) proposed by Shijun Liao was developed to solve the system of ODEs by combining the precise integration method (PIM) which can be employed to get the analytical solution of linear system of ODEs. The numerical experiences show that HAM is not sensitive to the time step, and so the arithmetic error is mainly derived from the discrete in physical space.

## 1. Introduction

Burgers equation is a typical nonlinear partial differential equation, which was constructed to describe a kind of hydromechanical phenomenon. As there is a sharp wave in the solution of Burgers whose grads can vary with the time parameter, it is often employed to test the merits of the numerical algorithm. In recent years, there are many analytical methods for nonlinear problems that have been proposed in recent years. In these methods, the homotopy analysis method (HAM) proposed by Liao [1–6] is constantly being developed and applied to solve various nonlinear problems [7–13]. In order to further improve the properties of HAM, Liao [14] and some other researchers [15, 16] studied the choice rules about the auxiliary parameter  $\hbar$  and the auxiliary function  $H(t)$  on different nonlinear problems, which requires the users to have high-level skill about it. However, an effective way to develop HAM to solve the matrix differential equations is decoupling which is complicated obviously. In our research, we try to develop HAM to solve the matrix ODEs without decoupling by combining the precise integration method (PIM) and meshless method, which can be used to solve the nonlinear PDEs.

Meshless methods eliminate some or all of the traditional mesh-based view of the computational domain and rely on

a particle (either Lagrangian or Eulerian) view of the field problem. Compared with the traditional numerical method for PDEs such as the finite element method (FEM) and finite difference methods (FDM), meshless method is better suited to the problems associated with extremely large deformation and problems associated with frequency remeshing. During the past several decades, there are about 10 different meshless methods that have been developed, such as the Smooth Particle Hydrodynamics (SPH) [17], the Element-Free Galerkin (EFG) method [18], the Reproducing Kernel Particle (RKP) method [19], the Finite Point (FP) method [20], the hp clouds method [21], Meshless Local Petrov-Galerkin (MLPG) [22], Local Boundary Integral Equation (LBIE) [23, 24], and several others.

Collocation method and Galerkin method are the two discretization methods which have been dominant in existing meshless methods, in which the radial basis function is often employed to construct the interpolation operator. Radial basis functions are functions which have no preferred direction but only depend on norms in space. As a powerful tool in both of pure and applied mathematics, multi-scale analysis theory has been developed rapidly in recent years. Multilevel systems are by now used widely in many fields of science and technology such as signal analysis, data compression,

pattern recognition, and solutions of the partial differential equations [25, 26]. The major advantage of the multiscale analysis method is that it can improve the algorithm efficiency effectively. In recent years, many effective numerical methods for PDEs based on wavelets have been proposed, such as the wavelet Galerkin method (WGM) [27, 28], the wavelet finite element method [29], and the wavelet collocation method [30, 31]. So, construction of multi-scale interpolation operator with RBF is meaningful for meshless method.

After the discretization to the nonlinear PDEs, a nonlinear system of ODEs can be obtained. HAM has been developed to solve the system of ODEs by Sami, Noorani, and Hashim, which can be employed to solve PDEs combining the various numerical discretization methods mentioned above.

The purpose of this study is to construct a HAM-based multi-scale meshless method for Burgers equation. First, we construct a multi-scale interpolation operator with RBF, which can discretize PDEs adaptively into a system of nonlinear ODEs. Then, we develop HAM to solve the system of nonlinear ODEs obtained in the first step, in which the adaptability was introduced to the HAM. So, a new adaptive numerical-analytical method for nonlinear PDEs was obtained. At last, we test the algorithm by comparing with other algorithms. The numerical experiments show that this is an efficient second-order time-marching solver for time-dependent problems as long as a factorization of the differential operator is available.

## 2. Construction of Meshless Adaptive Multiscale Interpolation Operator

Let  $\phi(x)$  denote the radial basis function, which can be expressed as

$$\phi(x) = \exp\left(-r^2 \frac{x^2}{\sigma^2 - x^2}\right). \quad (1)$$

The corresponding first- and second-order derivative of the radial basis function can be derived as follows:

$$\begin{aligned} \frac{d}{dx}\phi(x) &= \left[ -\frac{2r^2x}{\sigma^2 - x^2} - \frac{2r^2x^3}{(\sigma^2 - x^2)^2} \right] \phi(x), \\ \frac{d^2}{dx^2}\phi(x) &= \begin{cases} -\frac{2r^2}{\sigma^2}, & x = 0 \\ 0, & x = \pm\sigma \\ -\frac{2r^2\sigma^2}{(x^2 - \sigma^2)^4} \\ \quad \times (-\sigma^4 - 2\sigma^2x^2 + 3x^4 + 2r^2x^2\sigma^2) \\ \quad \times \phi(x), & \text{otherwise.} \end{cases} \end{aligned} \quad (2)$$

Obviously, RBF is a compact support function with the interpolation property.  $[-\sigma, \sigma]$  is the support domain;  $r$  and  $\sigma$  are constants.

A discrete point sequence of  $\phi(x)$  is defined as

$$\begin{aligned} \varphi_{j,k}(x) &= \phi(2^j x - k), \\ x &\in [x_{\min}, x_{\max}], \quad j \in Z, \quad k = 0, 1, 2, \dots, 2^j. \end{aligned} \quad (3)$$

The subspaces of  $L^2(0, 1)$  constructed from these discrete point sequence are defined as

$$V_j := \text{span} \{ \phi_{j,k} \mid k = 0, 1, 2, \dots, 2^j \} \subset L^2(0, 1), \quad j \in Z, \quad (4)$$

$\phi_{j,k}(x)$  possesses the interpolation property, such as  $\phi_{j,k}(n2^{-j}) = \delta_{n,k}$ , so an interpolation operator  $I_j : C^0(0, 1) \rightarrow V_j$  can be defined as

$$I_j f = \sum_{k=0}^{2^j} f(x_{j,k}) \varphi_{j,k}, \quad x_{j,k} = x_{\min} + k \frac{x_{\max} - x_{\min}}{2^j}. \quad (5)$$

Without loss of generality, we set  $\psi_{j,k} = \phi_{j+1,2k+1}$  which has interpolation property as well; that is,

$$\begin{aligned} \psi_{j,k}(y_{j,n}) &= \delta_{k,n}, \\ \psi_{j,k}(y_{j',n}) &= 0, \quad \forall j' < j, \end{aligned} \quad (6)$$

where  $y_{j,k} = x_{j+1,2k+1}$ . Based on the interpolation theory, it is possible to define an interpolating transform mapping any continuous function  $f$  into a sequence of its coefficients  $\{\beta_{j_0,k}\}, \{\alpha_{j_0,k}\}, \{\alpha_{j_0+1,k}\}, \dots$ . Any function  $f \in C^0(0, 1)$  may be reconstructed from its transform by means of

$$f \approx f_j = \sum_{k=0}^{2^{j_0}} \beta_{j_0,k} \phi_{j_0,k} + \sum_{j \geq j_0} \sum_{k=0}^{2^j} \alpha_{j,k} \psi_{j,k}. \quad (7)$$

Under a minimal regularity assumption the coefficients  $\beta_{j,k}$  and  $\alpha_{j,k}$  have the following meaning:

$$\beta_{j_0,k} = f(x_{j_0,k}), \quad \alpha_{j,k} = f(y_{j,k}) - I_j f(y_{j,k}). \quad (8)$$

So, the multi-scale interpolating operator can be deduced as follows:

$$I_i(x) = \sum_{k=0}^{2^{j_0}} R_{k_0,i}^{j_0,J} \phi_{j_0,k}(x) + \sum_{j=j_0}^{J-1} \sum_{k \in Z^j} C_{k,i}^{j,J} \psi_{j,k}(x), \quad (9)$$

where

$$\begin{aligned} Z^j &= \{0, 1, 2, \dots, 2^j\}, \\ C_{k,n}^{j,J} &= R_{2k+1,n}^{j+1,J} - \sum_{k_0=0}^{2^{j_0}} R_{k_0,n}^{j_0,J} \phi_{j_0,k_0}(x_{j+1,2k+1}) \\ &\quad - \sum_{j_1=j_0}^{j-1} \sum_{k_1=0}^{2^{j_1-1}} C_{k_1,n}^{j_1,J} \psi_{j_1,k_1}(x_{j+1,2k+1}) \end{aligned} \quad (10)$$

$$j = j_0 + 1, \quad j_0 + 2, \dots, J-1, \quad k = 0, 1, 2, \dots, 2^j - 1.$$

The operator  $C_{k,n}^{j,J}$  maps a set of functional values at the  $J$  level of resolution into a set of interpolation transform coefficients at  $j$  level. For  $j = j_0$

$$C_{k,i}^{j_0,J} = R_{2k+1,i}^{j_0+1,J} - \sum_{k_0=0}^{2^{j_0}} R_{k_0,i}^{j_0,J} \theta_{j_0,k_0}(x_{j_0+1,2k+1}), \quad (11)$$

where the operator  $R_{i,m}^{l,j}$  is the restriction operator defined as

$$R_{i,m}^{l,j} = \begin{cases} 1, & x_i^l = x_m^j, \\ 0, & \text{otherwise.} \end{cases} \quad (12)$$

Since the restriction operator is known, it is easy to calculate the numerical solution of the interpolation operator from (9)~(11). The  $m$ th derivative of  $I_i(x)$  is

$$D_i^{(m)}(x) = \sum_{k_0=0}^{2^{j_0}} R_{k_0,i}^{j_0,J} \phi_{j_0 k_0}^{(m)}(x) + \sum_{j=j_0}^{J-1} \sum_{k \in Z^j} C_{k,i}^{j,J} \psi_{j,k}^{(m)}(x). \quad (13)$$

For collocation method, it is necessary to setup a threshold  $\varepsilon$  so that the collocation points corresponding to interpolation transform coefficients less than  $\varepsilon$  can be neglected. For convenience, we define two integer subsets of integers  $Z_{\geq}^j$  and  $Z^c$ .  $Z_{\geq}^j \subset Z^j$  is the set of the indices of collocation points whose interpolation transform coefficients are larger than a certain threshold  $\varepsilon$  at the  $j$  level of resolution, and  $Z^c \subset Z^J$  is the set of the indices of the ultimate set of collocation points, which are larger than a certain threshold used in the interpolation, that is,

$$Z^c = Z_{\geq}^1 \cup Z_{\geq}^2 \cup \dots \cup Z_{\geq}^J. \quad (14)$$

So, the adaptive interpolating operator and its  $m$ th derivative can be written as

$$\begin{aligned} I_i(x) &= \sum_{k_0=0}^{2^{j_0}} R_{k_0,i}^{j_0,J} \phi_{j_0 k_0}(x) \\ &+ \sum_{j=j_0}^{J-1} \sum_{k \in Z_{\geq}^j \cap (k \in [N_1, N_2])} C_{k,i}^{j,J} \psi_{j,k}(x), \quad i \in Z^c, \end{aligned} \quad (15)$$

where

$$\begin{aligned} C_{k,n}^{j,J} &= R_{2k+1,n}^{j+1,J} - \sum_{k_0=0}^{2^{j_0}} R_{k_0,n}^{j_0,J} \phi_{j_0 k_0}(x_{j+1,2k+1}) \\ &- \sum_{j_1=j_0}^{j-1} \sum_{k_1 \in Z_{\geq}^{j_1} \cap (k_1 \in [N_1, N_2])} C_{k_1,n}^{j_1,J} \psi_{j_1,k_1}(x_{j+1,2k+1}), \\ j &= j_0 + 1, \quad j_0 + 2, \dots, J-1, \quad k \in Z_{\geq}^j, \quad n \in Z^c. \end{aligned} \quad (16)$$

The  $m$ th derivative of  $I_i(x)$  is

$$\begin{aligned} D_i^{(m)}(x) &= \sum_{k_0=0}^{2^{j_0}} R_{k_0,i}^{j_0,J} \phi_{j_0 k_0}^{(m)}(x) \\ &+ \sum_{j=j_0}^{J-1} \sum_{k \in Z_{\geq}^j \cap (k \in [N_1, N_2])} C_{k,i}^{j,J} \psi_{j,k}^{(m)}(x), \quad i \in Z^c. \end{aligned} \quad (17)$$

### 3. Coupling Technique of HAM and Adaptive Multilevel Interpolator on Nonlinear PDEs

HAM is an analytic approximation method for highly nonlinear equations in science, finance, and engineering. It was first proposed by Dr. Shijun LIAO in 1992 in his Ph.D. dissertation and modified and developed by Dr. Liao with his team and researchers in many other countries. As a result, the HAM overcomes the restrictions of all other analytic approximation methods mentioned above and is valid for highly nonlinear problems.

Consider the Burgers equation:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \frac{1}{\text{Re}} \frac{\partial^2 u}{\partial x^2}, \quad x \in [0, 2], \quad t \geq 0, \quad (18)$$

with the initial and boundary conditions

$$u(x, 0) = \sin(\pi x), \quad u(0, t) = u(2, t) = 0, \quad (19)$$

where  $t$  denotes time and  $\text{Re}$  denotes Reynolds number.

Following the classical collocation approach and the definition of the operator  $I_i(x)$  in (15), the approximating formulation  $u_j(x)$  of a function  $u(x)$  can be written as

$$u_j(x) = \sum_{i \in Z^c} I_i(x) u_{ji}. \quad (20)$$

Substituting (20) into (18) leads to a system of nonlinear ordinary differential equations as follows:

$$\begin{aligned} \sum_{n \in Z^c} u_j(x_n, t) \left[ \frac{1}{\text{Re}} D_n''(x_k) - u_j(x_k, t) D_n'(x_k) \right] \\ = \frac{\partial u_j(x_k, t)}{\partial t}, \end{aligned} \quad (21)$$

where  $k \in Z^c$ . The corresponding vector expression is

$$\frac{\partial}{\partial t} V_J = M_0 V_J + M_1(V_J) V_J, \quad (22)$$

where

$$\begin{aligned} V_J &= (u_j(x_0, t), u_j(x_1, t), \dots, u_j(x_{2^J}, t))^T, \\ M_0(k, n) &= m_{k,n}^0 = \frac{1}{\text{Re}} D_n''(x_k), \quad k, n \in Z^c, \\ M_1(k, n) &= m_{k,n}^1 = -u_j(x_k, t) D_n'(x_k), \quad k, n \in Z^c. \end{aligned} \quad (23)$$

According HAM, we can choose the auxiliary linear operator as follows:

$$\mathcal{L}[\Phi(t; q)] = \frac{\partial \Phi(t; q)}{\partial t} - (M_0 + M_1(A)) \Phi(t; q), \quad (24)$$

where  $q \in [0, 1]$  is the embedded variable,  $\Phi(t; q)$  is the function with respect to  $t$  and  $q$ , and  $A$  is the known initial value  $\Phi(0; q)$ . Based on (22), the nonlinear operator can be defined as follows:

$$\begin{aligned} \mathcal{N}[\Phi(t; q)] &= \frac{\partial \Phi(t; q)}{\partial t} \\ &- (M_0 + M_1(\Phi(t; q))) \Phi(t; q). \end{aligned} \quad (25)$$

$\hbar \neq 1$  and  $H(t) \neq 0$  denote the auxiliary parameter and the auxiliary function, respectively.

According to the homotopy analysis theory, we can construct the 0th order deformed equation as follows:

$$(1 - q) \mathcal{L} [\Phi(t; q) - \Phi(t; 0)] = \hbar q H(t) \mathcal{N} [\Phi(t; q)] \quad (26)$$

that is,

$$\begin{aligned} (1 - q) \left[ \frac{\partial \Phi(t; q)}{\partial t} - (M_0 + M_1(A)) \Phi(t; q) \right] \\ = \hbar q H(t) \left[ \frac{\partial \Phi(t; q)}{\partial t} - (M_0 + M_1(\Phi(t; q))) \Phi(t; q) \right], \end{aligned} \quad (27)$$

where  $\Phi(t; 0) = V_j^0(t)$ ,  $\Phi(t; 1) = V_j(t)$ . Let

$$V_j^m(t) = \frac{1}{m!} \frac{\partial^m \Phi(t; q)}{\partial q^m} \Big|_{q=0}. \quad (28)$$

According to the Taylor series theory,  $\Phi(t; q)$  can be expressed as the following power series:

$$\begin{aligned} \Phi(t; q) = \Phi(t; 0) + \sum_{m=1}^{\infty} \frac{1}{m!} \frac{\partial^m \Phi(t; q)}{\partial q^m} \Big|_{q=0} \\ q^m = V_j^0(t) + \sum_{m=1}^{\infty} V_j^m(t) q^m. \end{aligned} \quad (29)$$

For simplification, (27) can be rewritten as

$$\begin{aligned} \left[ \frac{\partial \Phi(t; q)}{\partial t} - (M_0 + M_1(A)) \Phi(t; q) \right] [1 - q - \hbar q H(t)] \\ + \hbar q H(t) [M_1(\Phi(t; q)) - M_1(A)] \Phi(t; q) = 0 \end{aligned} \quad (30)$$

which can be rewritten as

$$\begin{aligned} \frac{\partial \Phi(t; q)}{\partial t} = (M_0 + M_1(A)) \Phi(t; q) - \frac{\hbar q H(t)}{1 - q - \hbar q H(t)} \\ \times [M_1(\Phi(t; q)) - M_1(A)] \Phi(t; q). \end{aligned} \quad (31)$$

In order to obtain the analytical solution of this nonlinear matrix differential equation, it is necessary to choose a suitable basis function to approximate  $\Phi(t; q)$  based on the homotopy analysis method. It is well known that the matrix differential equation is quite different from the common differential equation, as the matrix one needs decoupling. To avoid the complicated decoupling, without loss of generality, the term  $\hbar q H(t)/(1 - q - \hbar q H(t))$  in (31) can be identified as

$$p = \frac{\tau}{t} = \frac{\hbar q H(t)}{1 - q - \hbar q H(t)}, \quad \tau \in [0, t], \quad \varepsilon \in [0, 1]. \quad (32)$$

Then

$$\Phi(t; q) \equiv V_j(t). \quad (33)$$

Equation (31) can be rewritten as

$$\begin{aligned} \frac{dV_j}{dt} = M_0 V_j + M_1(A) V_j \\ - \varepsilon [M_1(A) V_j - M_1(V_j) V_j]. \end{aligned} \quad (34)$$

So,  $V_j(t)$  can be expressed as the Taylor series with respect to  $t$  as follows:

$$V_j(t) = \sum_{m=0}^{\infty} \frac{1}{m!} \frac{d^m V_j(t)}{dt^m}. \quad (35)$$

Substituting (35) into (31) and rearranging based on powers of  $p$ -terms, we have

$$p^0 : \frac{dV_j^0}{dt} = (M_0 + M_1(A)) V_j^0, \quad (36)$$

$$\begin{aligned} p^1 : \frac{dV_j^1}{dt} = [M_0 + M_1(A)] V_j^1 \\ + [M_1(V_j^0) - M_1(A)] V_j^0, \end{aligned} \quad (37)$$

$$\begin{aligned} p^2 : \frac{dV_j^2}{dt} = [M_0 + M_1(A)] V_j^2 \\ + [M_1(V_j^0) - M_1(A)] V_j^1 + M_1(V_j^1) V_j^0. \end{aligned} \quad (38)$$

Equation (36) is the system of homogeneous linear ODEs and its general solution is

$$V_j^0(t) = e^{Ht} A, \quad (39)$$

where  $H = M_0 + M_1(A)$ . Both (37) and (38) are the system of inhomogeneous linear ODEs, and the general solutions are

$$\begin{aligned} V_j^1(t) = e^{Ht} (H^{-1} r_0) - H^{-1} r_0, \\ V_j^2(t) = e^{Ht} (H^{-1} r_1) - H^{-1} r_1 \end{aligned} \quad (40)$$

respectively, where  $r_0 = (M_1(V_j^0) - M_1(A)) V_j^0$ ,  $r_1 = (M_1(V_j^0) - M_1(A)) V_j^1 + M_1(V_j^1) V_j^0$ . Substituting (39) and (40) into (35), the numerical solution of (18) can be obtained. To improve the computational accuracy, the time interval  $[0, t]$  can be divided evenly as

$$t_0 = 0, \quad t_1 = \tau, \dots, \quad t_k = k \cdot \tau, \dots \quad (41)$$

Taking the solution at time  $t_k$  instead of “ $A$ ” as the initial value in (40), the recurrence formula can be obtained. The matrix exponential function  $e^{Ht}$  can be calculated accurately as follows:

$$\exp(H \cdot \tau) = \left[ \exp \left( H \cdot \frac{\tau}{2^N} \right) \right]^{2^N}. \quad (42)$$

Let  $\Delta t = \tau/2^N$ , where  $N$  is a positive integer (usually take  $N = 20$ , and then  $\Delta t = \tau/1048576$ ). Taking account of that  $\tau$  is a small time interval,  $\Delta t$  is a very small value, and then

$$\exp(H \cdot \Delta t) = I + T_a = I + H\Delta t + \frac{(H\Delta t)^2 [I + (H\Delta t)/3 + (H\Delta t)^2/12]}{2} \quad (43)$$

which is the Taylor series expansion of  $\exp(H\Delta t)$ . In order to calculate the matrix  $\exp(H \cdot \tau)$  more accurately, it is necessary to factorize the matrix as

$$[\exp(H \cdot \Delta t)]^{2^N} = (I + T_a)^{2^{(N-1)}} (I + T_a)^{2^{(N-1)}}. \quad (44)$$

After  $N$  times of factorization as above, a more accurate solution can be obtained.

The mesh parameter in the extrapolation method is usually taken as  $\tau_i = 2^{-i}\tau$ , where  $\tau = t_{k+1} - t_k$ . It is easily noticed from (42) to (44) that the precise algorithm of exponent matrix is a  $2^N$ -type algorithm. Let  $T(\tau) = [\exp(H \cdot \Delta t)]^{2^N}$ , and then

$$T(\tau_i) = [\exp(H \cdot \Delta t)]^{2^{N-i}}. \quad (45)$$

Equation (45) can be described in program statement as

$$\text{for (iter = 0; iter < N; iter ++)} \quad T_a = 2 * T_a + T_a * T_a. \quad (46)$$

At the end of the routine, it follows that

$$T = I + T_a. \quad (47)$$

Obviously, the sum of the resultant matrix  $T_a$  at the end of  $n$ th routine and the identity matrix  $I$  is the matrix  $T(\tau_{N-n})$ . And so, for calculating all  $T(\tau_i)$ , it needs merely to rewrite the above program statement as

$$\begin{aligned} &\text{for (iter = 0; iter < N; iter ++)} \{ \\ &\quad T_a = 2T_a + T_a * T_a; \\ &\quad T(N - \text{iter}) = T_a + I; \\ &\} \end{aligned} \quad (48)$$

The amount of calculation is the same as the original program for  $T(\tau_i)$ . It should be pointed out that the computation accuracy is remained due to the decreasing of the time step  $\tau_i$  although the cycle index in calculating the matrix  $T(\tau_i)$  is fewer than  $N$ .

The calculation of the exponent matrix  $T(ih)$  at different time steps is needed in solving nonlinear equations through iteration based on the precise integration method, and the algorithm of the matrix  $T(ih)$  presented here can obtain all the matrices at different time steps for once.

It should be noticed that the nonlinear term in (22) can be expressed in Taylor series directly, and then another iterative

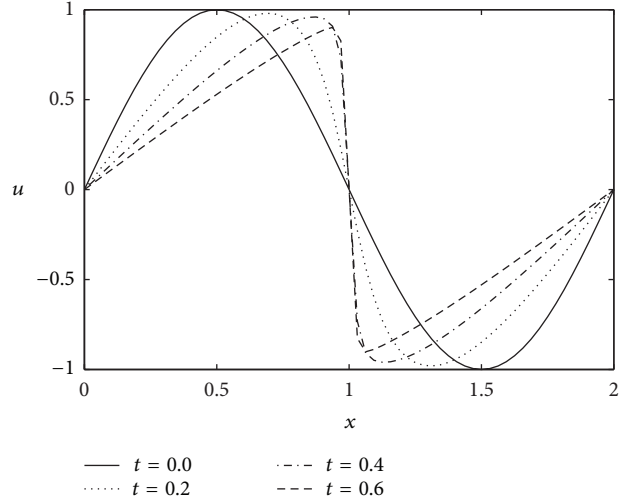


FIGURE 1: Analytical solutions of the Burgers equation at different times.

format can be obtained. For comparison, we give this iterative format as follows directly:

$$\begin{aligned} V_j^{k+1} = & T \left[ V_j^k + M_0^{-1} (s_0 + M_0^{-1} s_1) \right] \\ & - M_0^{-1} (s_0 + M_0^{-1} s_1 + s_1 \tau), \end{aligned} \quad (49)$$

where  $s_0 = M_1 V_j$ ,  $s_1 = ds_0/dt$ .

#### 4. Numerical Result and Discussion

The Burgers equation has analytical solution as follows:

$$\begin{aligned} u(x, t) = & \frac{4\pi}{\text{Re}} \\ & \cdot \left( \sum_{n=1}^{\infty} n \exp \left( -\frac{n^2 \pi^2 t}{\text{Re}} \right) I_n \left( \frac{\text{Re}}{2\pi} \right) \sin(n\pi x) \right) \\ & \times \left( \left( 2 \sum_{n=1}^{\infty} n \exp \left( -\frac{n^2 \pi^2 t}{\text{Re}} \right) \right. \right. \\ & \left. \left. \times I_n \left( \frac{\text{Re}}{2\pi} \right) \cos(n\pi x) + I_0 \left( \frac{\text{Re}}{2\pi} \right) \right) \right)^{-1}, \end{aligned} \quad (50)$$

where  $I_n(x)$  is the  $n$ th order modified Bessel function of the first kind.

In this section, the adaptive method which is proposed in this paper is used to calculate Burgers equation (18) with Reynolds number  $\text{Re} = 1000$ . The figures of the analytic solution with  $\text{Re} = 100$  at  $t = 0.0, 0.4, 0.6$  are shown in Figure 1, respectively. It is easy to find that the analytic solution evolves into a shock wave near  $t = 1$ , and the gradient becomes larger and larger with the increasing of  $\text{Re}$ . So the Burgers equation is often used to test the validity of numerical methods.

In our method, RBF was employed to construct the adaptive meshless interpolation operator and the regular width

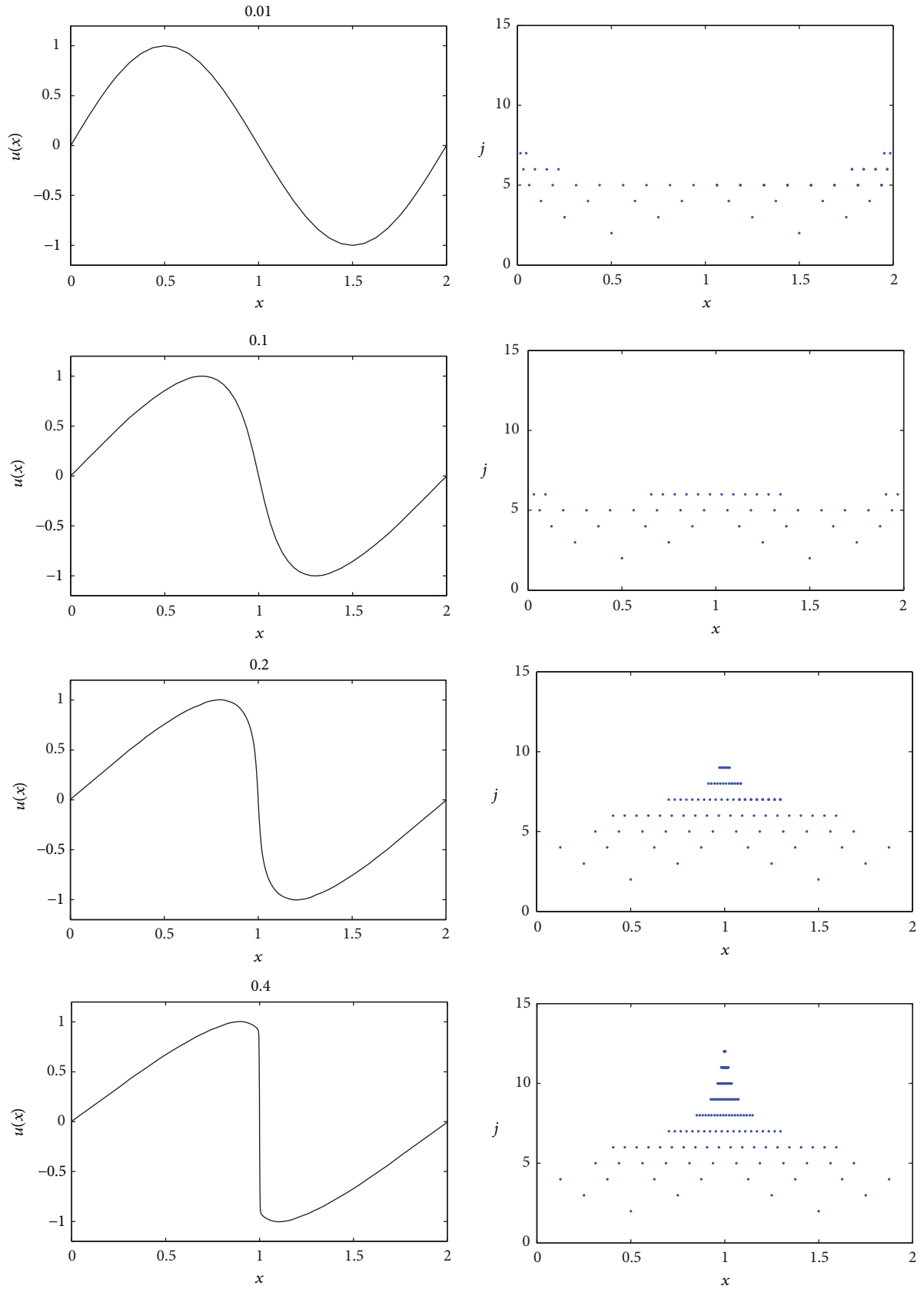


FIGURE 2: Evolution of the solution and collocation points for the solution of the Burgers equation with HAM-based MAMIO.



TABLE 1: Iteration times at different evolution time domain of the Burgers equation.

Time domain	0–0.13	0.13–0.22	0.22–0.31	0.31–0.39	0.39–0.4
Iteration times	2	3	3	4	3

parameter  $\sigma = 3\Delta$ , scale parameter  $j = 15$ , time step  $\tau = 0.02$ , and threshold parameter  $\epsilon = 0.005$ .

The evolution of the solution of Burgers equation from the uniformly smooth distribution to the shock structure causes the growth of the interpolation coefficients corresponding to the smaller scales, which in turn results in the refinement of the grids. Figure 2 shows the progressive refinement of the irregular grid with the decreasing of the shock thickness. This illustrates that the meshless adaptive multi-level interpolation operator (MAMIO) constructed with RBF is effective, which is helpful to improve the efficiency of the algorithm.

Next, we will discuss the efficiency of the HAM-based adaptive precise integration method on nonlinear dynamics equations. Let  $\epsilon$  denote the maximum difference between the numerical solutions derived from the time steps  $\Delta t$  and  $\Delta t/2$ , respectively; the condition of stop iteration is  $|\epsilon| < 1 \times 10^{-6}$ . Table 1 shows the iteration times of the method. It is obvious that iteration times are adaptive with the evolution of Burgers equation. That is, we do not have to set the smallest time step to satisfy the precision requirement, which is helpful to decrease the computation costs effectively.

The numerical method for nonlinear PDEs constructed by combining RBF collocation method and Runge-Kutta method is a more common method. To test the validity of the multi-level adaptive PIM, these two methods are being compared in the following.

The computational error curves of them are shown in Figures 3(a) and 3(b), respectively. It should be pointed that the error obtained by HAM-based MAMIO occurs merely near  $x = 1$  and that at other points is almost equal to zero; however, the error obtained by Runge-Kutta method-based MAMIO is evident at all points, especially near the boundary and  $x = 1$ . Furthermore, it should be noted that the error that occurs near the boundary is also evident even if the multistep methods, such as the Adams-Bashforth-Moulton method, are used (see Figure 4). The largest errors of these methods at different times are listed in Table 2. It can be seen that the computational precision of the HAM-based MAMIO is higher than that of the other two methods.

The comparison of the classical adaptive precise integration method (APIM) and the HAM-based APIM is shown in Figure 5.

Figure 5(b) shows that APIM is failed when the time step  $\tau$  is larger ( $\tau = 0.02$ ). However, the HAM-based APIM has enough precision even when the time step  $\tau$  increases to 0.2 and  $t$  is more than 0.4 (Figure 5(c)).

As the time step  $\tau$  becomes small enough (Figure 5,  $\tau = 0.01$ ), both of the APIM and the HAM-based APIM have higher precision however, the latter is still better evidently than the APIM.

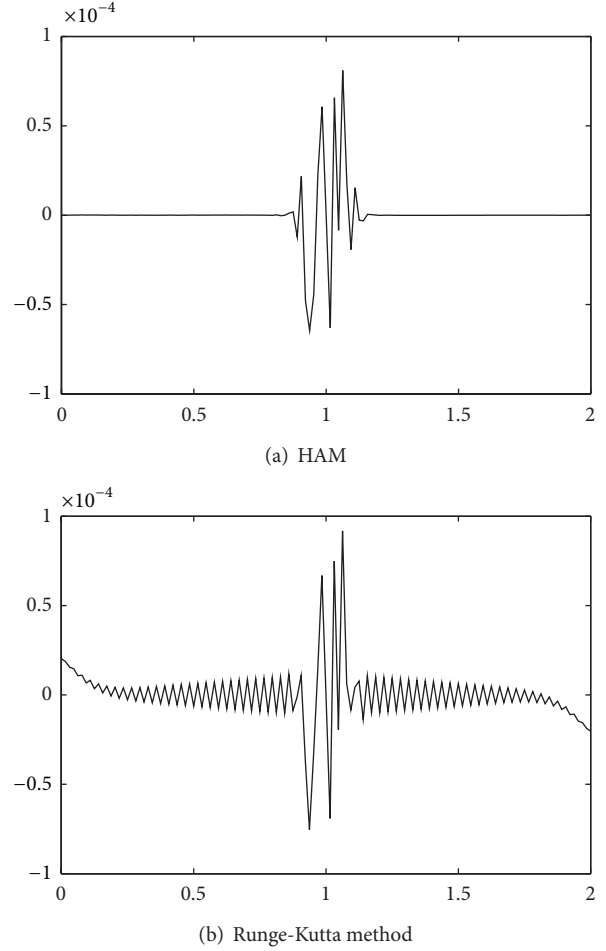
FIGURE 3: The computational errors of HAM the Runge-Kutta method ( $t = 0.4$ ).

TABLE 2: Largest computational errors of the Burgers equation.

$t$ (s)	HAM	Runge-Kutta method	Adams-Bashforth-Moulton method
0.4	$8.0949 \times 10^{-5}$	$9.1795 \times 10^{-5}$	$6.8626 \times 10^{-5}$
0.8	$1.7884 \times 10^{-4}$	$1.8425 \times 10^{-4}$	$1.4890 \times 10^{-4}$
1.2	$8.4061 \times 10^{-6}$	$1.1917 \times 10^{-5}$	$3.9004 \times 10^{-5}$
1.6	$1.6539 \times 10^{-6}$	$6.2157 \times 10^{-6}$	$3.4486 \times 10^{-5}$
2.0	$3.2393 \times 10^{-7}$	$9.6786 \times 10^{-6}$	$9.5912 \times 10^{-6}$

The above comparisons show that the HAM-based APIM is not sensitive to the time step compared to the APIM, which can improve the efficiency of the algorithm to some extent.

## 5. Conclusions

Coupling technique of HAM and the meshless multi-level adaptive interpolating operator constructed with RBF was developed to solve partial equations in a finite domain. In this method, the adaptive interpolating operator must be reconstructed at different moment because the amount and

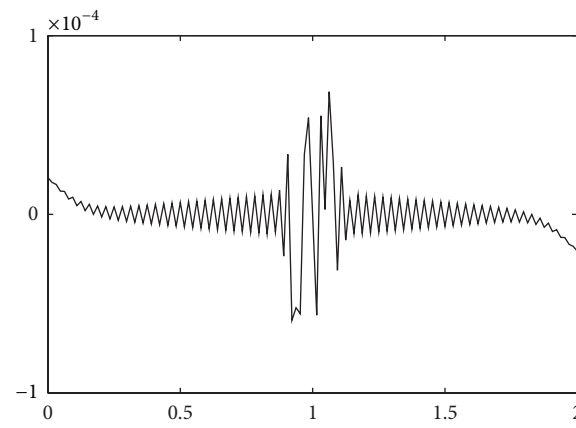
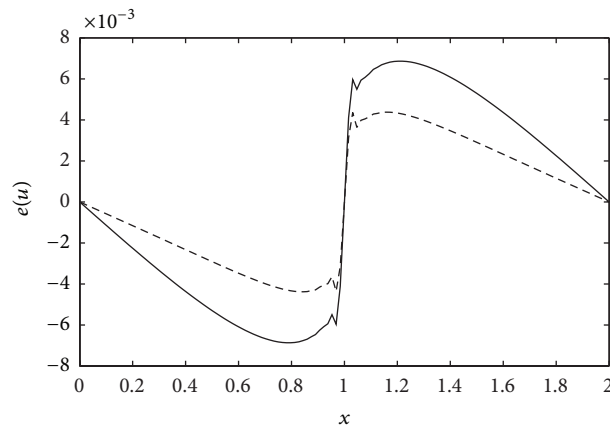
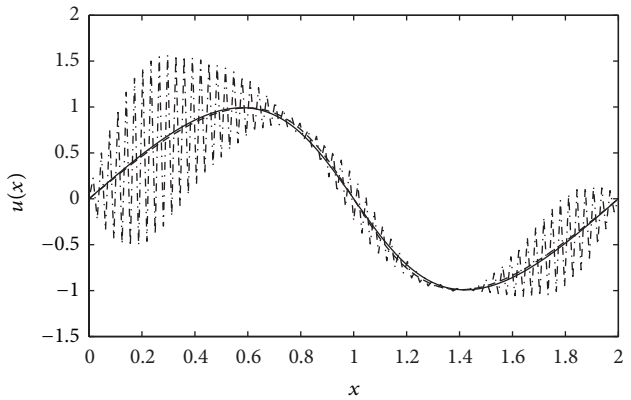


FIGURE 4: The computational error of the Adams-Bashforth-Moulton method ( $t = 0.4$ ).



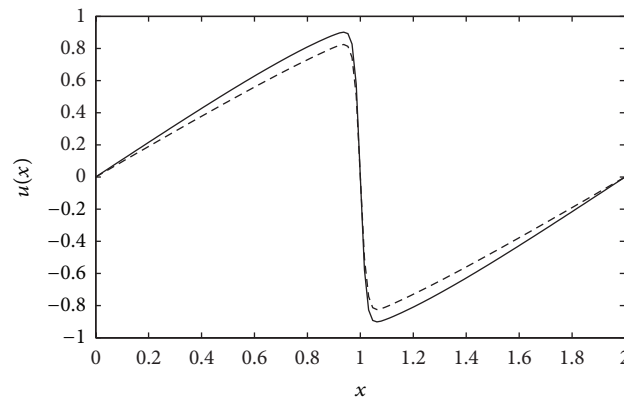
--- HAM  
— Iterative method

(a)  $t = 0.01, t = 0.6$



--- HAM  
-.- Iterative method  
— Exact solution

(b)  $\tau = 0.02, t = 0.08$



--- HAM  
— Exact solution

(c)  $\tau = 0.2, t = 0.6$

FIGURE 5: Numerical results of the classical APIM and the HPM-based APIM.



the position of collocation points are different at different time. The constructing efficiency of the interpolating operator is crucial to the efficiency of the adaptive method. Compared with multi-level wavelet operator, the dynamic operator proposed in this paper is more simple and efficient, as RBF possesses almost all the excellent numerical properties such as the compact support, analytical expression, and interpolation. This is helpful to improve calculation efficiency. Furthermore, the method becomes more efficient and accurate by means of HAM which possesses uniform convergence.

## References

- [1] S. Liao, *Beyond Perturbation: Introduction to the Homotopy Analysis Method*, Chapman and Hall, CRC Press, Boca Raton, Fla, USA, 2003.
- [2] S. Liao, "On the homotopy analysis method for nonlinear problems," *Applied Mathematics and Computation*, vol. 147, no. 2, pp. 499–513, 2004.
- [3] S.-J. Liao, "An explicit, totally analytic approximate solution for Blasius' viscous flow problems," *International Journal of Non-Linear Mechanics*, vol. 34, no. 4, pp. 759–778, 1999.
- [4] S. Liao, "A new branch of solutions of boundary-layer flows over an impermeable stretched plate," *International Journal of Heat and Mass Transfer*, vol. 48, no. 12, pp. 2529–3259, 2005.
- [5] S. Liao, "An optimal homotopy-analysis approach for strongly nonlinear differential equations," *Communications in Nonlinear Science and Numerical Simulation*, vol. 15, no. 8, pp. 2003–2016, 2010.
- [6] S. J. Liao, *Homotopy Analysis Method in Nonlinear Differential Equations*, Springer & Higher Education Press, Heidelberg, Germany, 2012.
- [7] F. Guerrero, F. J. Santonja, and R. J. Villanueva, "Solving a model for the evolution of smoking habit in Spain with homotopy analysis method," *Nonlinear Analysis: Real World Applications*, vol. 14, no. 1, pp. 549–558, 2013.
- [8] M. Ayub, A. Rasheed, and T. Hayat, "Exact flow of a third grade fluid past a porous plate using homotopy analysis method," *International Journal of Engineering Science*, vol. 41, no. 18, pp. 2091–2103, 2003.
- [9] T. Hayat and M. Khan, "Homotopy solutions for a generalized second-grade fluid past a porous plate," *Nonlinear Dynamics*, vol. 42, no. 4, pp. 395–405, 2005.
- [10] S. Abbasbandy, "The application of homotopy analysis method to nonlinear equations arising in heat transfer," *Physics Letters A*, vol. 360, no. 1, pp. 109–113, 2006.
- [11] A. B. Bataineh, M. S. M. Noorani, and I. Hashim, "Solving systems of ODEs by homotopy analysis method," *Communications in Nonlinear Science and Numerical Simulation*, vol. 13, no. 10, pp. 2060–2070, 2008.
- [12] F. Awawdeh, A. Adawi, and Z. Mustafa, "Solutions of the SIR models of epidemics using HAM," *Chaos, Solitons and Fractals*, vol. 42, no. 5, pp. 3047–3052, 2009.
- [13] H. Khan, R. N. Mohapatra, K. Vajravelu, and S. J. Liao, "The explicit series solution of SIR and SIS epidemic models," *Applied Mathematics and Computation*, vol. 215, no. 2, pp. 653–669, 2009.
- [14] S. Liao, "An optimal homotopy-analysis approach for strongly nonlinear differential equations," *Communications in Nonlinear Science and Numerical Simulation*, vol. 15, no. 8, pp. 2315–2332, 2010.
- [15] M. Ghoreishi, A. I. B. Ismail, A. K. Alomari, and A. S. Bataineh, "The comparison between homotopy analysis method and optimal homotopy asymptotic method for nonlinear age-structured population models," *Communications in Nonlinear Science and Numerical Simulation*, vol. 17, no. 3, pp. 1163–1177, 2012.
- [16] S. Abbasbandy, E. Shivanian, and K. Vajravelu, "Mathematical properties of h-curve in the frame work of the homotopy analysis method," *Communications in Nonlinear Science and Numerical Simulation*, vol. 16, no. 11, pp. 4268–4275, 2011.
- [17] R. A. Gingold and J. J. Moraghan, "Smoothed particle hydrodynamics: theory and applications to non-spherical stars," *Monthly Notices of the Royal Astronomical Society*, vol. 181, pp. 375–389, 1977.
- [18] T. Belytschko, Y. Y. Lu, and L. Gu, "Element-free Galerkin methods," *International Journal for Numerical Methods in Engineering*, vol. 37, no. 2, pp. 229–256, 1994.
- [19] W. K. Liu, S. Jun, and Y. F. Zhang, "Reproducing kernel particle methods," *International Journal for Numerical Methods in Fluids*, vol. 20, no. 8-9, pp. 1081–1106, 1995.
- [20] E. Onate, S. Idelsohn, O. C. Zienkiewicz, and R. L. Taylor, "A finite point method in computational mechanics. Applications to convective transport and fluid flow," *International Journal for Numerical Methods in Engineering*, vol. 39, no. 22, pp. 3839–3866, 1996.
- [21] T. J. Liszka, C. A. M. Duarte, and W. W. Tworzydło, "hp-Meshless cloud method," *Computer Methods in Applied Mechanics and Engineering*, vol. 139, no. 1–4, pp. 263–288, 1996.
- [22] S. N. Atluri and T.-L. Zhu, "Meshless local Petrov-Galerkin (MLPG) approach for solving problems in elasto-statics," *Computational Mechanics*, vol. 25, no. 2, pp. 169–179, 2000.
- [23] S. N. Atluri and T. Zhu, "New concepts in meshless methods," *International Journal for Numerical Methods in Engineering*, vol. 47, no. 1–3, pp. 537–556, 2000.
- [24] S. N. Atluri, J. Sladek, V. Sladek, and T. Zhu, "Local boundary integral equation (LBIE) and its meshless implementation for linear elasticity," *Computational Mechanics*, vol. 25, no. 2, pp. 180–198, 2000.
- [25] S. L. Mei, Q. S. Lu, S. W. Zhang, and L. Jin, "Adaptive interval wavelet precise integration method for partial differential equations," *Applied Mathematics and Mechanics*, vol. 26, no. 3, pp. 364–371, 2005.
- [26] S.-L. Mei, H.-L. Lv, Q. Ma et al., "Construction of interval wavelet based on restricted variational principle and its application for solving differential equations," *Mathematical Problems in Engineering*, vol. 2008, Article ID 629253, 14 pages, 2008.
- [27] M. Mehra and R. B. V. Kumar, "Time accurate fast three-step wavelet-Galerkin method for partial differential equations," *International Journal of Wavelets, Multiresolution and Information Processing*, vol. 4, no. 1, pp. 65–79, 2006.
- [28] A. M. Abdel Sattar, J. R. Dickerson, and M. H. Chaudhry, "Wavelet-Galerkin solution to the water hammer equations," *Journal of Hydraulic Engineering*, vol. 135, no. 4, pp. 283–295, 2009.
- [29] B. Zhao and K. Wang, "The application of the wavelet finite element method on the temperature field and thermal stress analysis of the petroleum tank," *Journal of Information and Computational Science*, vol. 8, no. 7, pp. 1103–1111, 2011.

- [30] S. Bertoluzza and G. Naldi, "A wavelet collocation method for the numerical solution of partial differential equations," *Applied and Computational Harmonic Analysis*, vol. 3, no. 1, pp. 1–9, 1996.
- [31] H. Yan, "Adaptive wavelet precise integration method for nonlinear Black-Scholes model based on variational iteration method," *Abstract and Applied Analysis*, vol. 2013, Article ID 735919, 6 pages, 2013.