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Research Article

Several Dynamic Properties of Solutions to a Generalized Camassa-Holm Equation

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For a nonlinear generalization of the Camassa-Holm equation, we investigate the dynamic properties of solutions for the equation under the assumption that the initial value $u_0(x)$ lies in the space $H^1(R)$. A one-sided upper bound estimate on the first-order spatial derivative, L^p bound estimate, and a space-time higher-norm estimate for the solutions are obtained.

1. Introduction

Hakkaev and Kirchev [1] investigated the following generalized Camassa-Holm equation:

$$u_t - u_{txx} + \left(2ku + \frac{n+2}{2}u^{n+1}\right)_x = \left(nu^{n-1}\frac{u_x^2}{2} + u^n u_{xx}\right)_x, \tag{1}$$

where n is an integer. When $n = 1$, (1) becomes the Camassa-Holm model (see [2]). The local well-posedness in the Sobolev space H^s with $s > 3/2$ is established, and sufficient conditions for the stability and instability of the solitary wave solutions are given in [1]. However, the L^p estimate of strong solutions and one-sided upper bound estimate on the first-order spatial derivative for the solutions are not discussed in [1]. This constitutes the objective of this work.

Like the Camassa-Holm equation (see [1, 2]), (1) has the conservation law

$$\int_R \left(u^2(t,x) + u_x^2(t,x)\right)dx = \int_R \left(u^2(0,x) + u_x^2(0,x)\right)dx, \tag{2}$$

which plays an important role in our further investigations.

In fact, many scholars have paid their attentions to the study of the Camassa-Holm equation. The existence of global weak solutions is established in Constantin and Escher [3], Constantin and Molinet [4], Xin and Zhang [5], and Coclite et al. [6]. It was shown in Constantin and Escher [7] that the blowup occurs in the form of breaking waves. Namely, the solution remains bounded, but its slope becomes unbounded in finite time. After wave breaking, the solution is continued uniquely either as a global conservative weak solution [8, 9] or as a global dissipative solution [10, 11]. Exact traveling wave solutions for the Camassa-Holm equation are presented in [12]. For other methods to investigate the problems involving various dynamic properties of the Camassa-Holm equation, the reader is referred to [13–16] and the references therein.

In this paper, we investigate several dynamic properties of strong solutions for the generalized Camassa-Holm equation (1) in the case where n is an odd natural number and the assumption $u_0(x) \in H^1(R)$. The results obtained in this work include a one-sided upper bound estimate on the first-order derivatives of the solution, a space-time higher-norm estimate, and the $L^p(2 \le p < \infty)$ bound estimate.

The rest of this paper is organized as follows. Section 2 states the main result. Several lemmas are given in Section 3 where the proof of main result is completed.

2. Main Result

Let m be a nonnegative integer and $n = 2m + 1$. In this case, the Cauchy problem for (1) is written in the form

$$u_t - u_{txx} + \partial_x\left(2ku + \frac{2m+3}{2}u^{2m+2}\right)$$

$$= \frac{1}{2m+2}u^{2m+2}\partial_x^3 u^{2m+2} - \frac{2m+1}{2}\partial_x\left(u^{2m}u_x^2\right),$$
$$u(0,x) = u_0(x),$$
(3)

which is equivalent to

$$u_t + u^{2m+1}u_x + \frac{\partial H}{\partial x} = 0,$$
$$\frac{\partial H}{\partial x} = \Lambda^{-2}\partial_x\left[2ku + \frac{2m^2+5m+2}{2(m+1)}u^{2m+2} + \frac{2m+1}{2}u^{2m}u_x^2\right],$$
$$u(0,x) = u_0(x),$$
(4)

where the operator $\Lambda^2 = 1 - \partial^2/\partial x^2$ and

$$\Lambda^{-2}f(x) = \frac{1}{2}\int_R e^{-|x-y|}f(y)\,dy \quad \text{for } f(x) \in L^2(R). \quad (5)$$

We introduce a result presented in [1] for problem (3).

Lemma 1 (see [1]). *Suppose that $u_0 \in H^s$ with constant $s > 3/2$. Then there is a real number $T > 0$ such that the problem (3) has a unique solution $u(t,x)$ satisfying*

$$u(t,x) \in C\left([0,T); H^s(R)\right) \bigcap C^1\left([0,T); H^{s-1}(R)\right). \quad (6)$$

Now we state the main result of this paper.

Theorem 2. *Let $u_0 \in H^s(R)$ with $s > 3$. Then the solution of problem (3) has the following properties.*

(a) *There exists a positive constant c_0 depending on m and $\|u_0\|_{H^1(R)}$ such that the one-sided L^∞ norm estimate on the first-order spatial derivative holds*

$$\frac{\partial u(t,x)}{\partial x} \le c_0(1+t), \quad \text{for } (t,x) \in [0,\infty)\times R. \quad (7)$$

(b) *Let $0 < \gamma < 1$, $T > 0$, and $a, b \in R$, $a < b$. Then there exists a positive constant c_1 depending only on $\|u_0\|_{H^1(R)}$, γ, T, a, b, and m such that the following estimate holds:*

$$\int_0^t\int_a^b u^{2m}\left|\frac{\partial u(t,x)}{\partial x}\right|^{2+\gamma}dx \le c_1. \quad (8)$$

(c) *There exists a constant c_2 depending only on $\|u_0\|_{H^1(R)}$, and m, p such that*

$$\|u\|_{L^p(R)} \le c_2(1+t), \quad t \in [0,\infty), \ 2 \le p < \infty. \quad (9)$$

3. Proof of Main Result

From the conservation law (2), we have

$$\|u\|_{L^\infty(R)} \le \|u\|_{H^1(R)} = \|u_0\|_{H^1(R)}. \quad (10)$$

Differentiating the first equation of problem (4) with respect to x and writing $\partial u/\partial x = q$, we obtain

$$\frac{\partial q}{\partial t} + u^{2m+1}\frac{\partial q}{\partial x} + \frac{2m+1}{2}u^{2m}q^2 = 2ku + \frac{2m^2+5m+2}{2(m+1)}u^{2m+2} - \Lambda^{-2}\left[2ku + \frac{2m^2+5m+2}{2(m+1)}u^{2m+2} + \frac{2m+1}{2}u^{2m}q^2\right] = Q(t,x).$$
(11)

Lemma 3. *Let $0 < \gamma < 1$, $T > 0$, and $a, b \in R$, $a < b$. Then there exists a positive constant c_1 depending only on $\|u_0\|_{H^1(R)}$, γ, T, a, b, and m, such that the space higher integrability estimate holds*

$$\int_0^T\int_a^b u^{2m}\left|\frac{\partial u(t,x)}{\partial x}\right|^{2+\gamma}dx \le c_1, \quad (12)$$

where $u = u(t,x)$ is the unique solution of problem (3).

Proof. The proof is a variant of the proof presented in Xin and Zhang [5] (or see Coclite et al. [6]). Let $\phi \in C^\infty(R)$ be a cut-off function such that $0 < \phi < 1$ and

$$\phi(x) = \begin{cases} 1, & \text{if } x \in [a,b], \\ 0, & \text{if } x \in (-\infty, a-1] \bigcup [b+1,\infty). \end{cases} \quad (13)$$

Letting $\theta(\xi) = \xi(1+|\xi|)^\gamma$, $\xi \in R$, and $0 < \gamma < 1$ yields

$$\theta'(\xi) = \left(1+(1+\gamma)|\xi|\right)(1+|\xi|)^{\gamma-1}, \theta''(\xi) = \gamma\,\mathrm{sign}(\xi)(1+|\xi|)^{\gamma-2}\left(2+(1+\gamma)|\xi|\right) = \gamma(1+\gamma)\,\mathrm{sign}(\xi)(1+|\xi|)^{\gamma-1} + (1-\gamma)\gamma\,\mathrm{sign}(\xi)(1+|\xi|)^{\gamma-2},$$
(14)

from which we get

$$|\theta(\xi)| \le |\xi| + |\xi|^{1+\gamma}, \qquad |\theta'(\xi)| \le 1 + (1+\gamma)|\xi|, \qquad |\theta''(\xi)| \le 2\gamma,$$
(15)

$$\xi\theta(\xi) - \frac{1}{2}\xi^2\theta'(\xi) = \frac{1-\gamma}{2}\xi^2(1+|\xi|)^\gamma + \frac{\gamma}{2}\xi^2(1+|\xi|)^{\gamma-1} \ge \frac{1-\gamma}{2}\xi^2(1+|\xi|)^\gamma.$$
(16)

Multiplying (11) by $\phi\theta'(q)$, using the chain rule, and integrating over $\pi_T := [0,T]\times R$, we have

$$\begin{aligned}
&(2m+1)\int_{\pi_T}\phi(x)\,u^{2m}q\theta(q)\,dt\,dx\\
&\quad-\frac{2m+1}{2}\int_{\pi_T}\phi(x)\,u^{2m}q^2\theta'(q)\,dt\,dx\\
&\quad=\int_R\phi(x)\left(\theta(q(t,x))-\theta(q(0,x))\right)dx\\
&\quad-\int_{\pi_T}u^{2m+1}\phi'(x)\,\theta(q)\,dt\,dx\\
&\quad-\int_{\pi_T}Q(t,x)\,\phi(x)\,\theta'(q)\,dt\,dx.
\end{aligned}\tag{17}$$

From (16), we get

$$\begin{aligned}
&(2m+1)\int_{\pi_T}\phi(x)\,u^{2m}q\theta(q)\,dt\,dx\\
&\quad-\frac{2m+1}{2}\int_{\pi_T}\phi(x)\,u^{2m}q^2\theta'(q)\,dt\,dx\\
&\quad=\int_{\pi_T}(2m+1)\,\phi(x)\,u^{2m}\left(q\theta(q)-\frac{1}{2}q^2\theta'(q)\right)dt\,dx\\
&\quad\geq\frac{(1-\gamma)}{2}\int_{\pi_T}\phi(x)\,u^{2m}q^2(1+|q|)^{\gamma}dt\,dx.
\end{aligned}\tag{18}$$

Using the Hölder inequality, (2), and (15) yields

$$\begin{aligned}
&\left|\int_R\phi(x)\,\theta(q)\,dx\right|\\
&\leq\int_R\phi(x)\left(|q|^{1+\gamma}+|q|\right)dx\\
&\leq\|\phi\|_{L^{2/(1-\gamma)}(R)}\|q\|^{1+\gamma}_{L^2(R)}+\|\phi\|_{L^2(R)}\|q\|_{L^2(R)}\\
&\leq(b-a+2)^{(1-\gamma)/2}\|u_0\|^{1+\gamma}_{H^1(R)}+(b-a+2)^{1/2}\|u_0\|_{H^1(R)},\\
&\left|\int_{\pi_T}u^{2m+1}\phi'(x)\,\theta(q)\,dt\,dx\right|\\
&\leq\int_{\pi_T}|u|^{2m+1}\left|\phi'(x)\right|\left(|q|^{1+\gamma}+|q|\right)dt\,dx\\
&\leq\|u_0\|^{2m+1}_{H^1(R)}\int_0^T\Big(\left\|\phi'\right\|_{L^{2/(1-\gamma)}(R)}\|q\|^{1+\gamma}_{L^2(R)}\\
&\qquad+\left\|\phi'\right\|_{L^2(R)}\|q\|_{L^2(R)}\Big)dt\\
&\leq cT\left(\left\|\phi'\right\|_{L^{2/(1-\gamma)}(R)}\|u_0\|^{1+\gamma}_{H^1(R)}+\left\|\phi'\right\|_{L^2(R)}\|u_0\|_{H^1(R)}\right).
\end{aligned}\tag{19}$$

Using (10), we obtain

$$\begin{aligned}
&\left\|2ku+\frac{2m^2+5m+2}{2(m+1)}u^{2m+2}\right\|_{L^\infty}\leq c,\\
&\left\|\Lambda^{-2}\left[2ku+\frac{2m^2+5m+2}{2(m+1)}u^{2m+2}\right]\right\|_{L^\infty}\leq c.
\end{aligned}\tag{20}$$

Applying (10), the Hölder inequality, and $\int_R e^{-|x-y|}dy=2$, we have

$$\left|\Lambda^{-2}\left[u^{2m}q^2\right]\right|=\left|\int_R\frac{1}{2}e^{-|x-y|}u^{2m}u_y^2dy\right|\leq c\|u\|_{H^1(R)}\leq c,\tag{21}$$

where c is a constant depending on $\|u_0\|_{H^1(R)}$ and m.

From (20) and (21), we deduce that there exists a positive constant c depending on $\|u_0\|_{H^1(R)}$ and m, such that

$$\|Q(t,x)\|_{L^\infty(R)}\leq c,\tag{22}$$

from which we get

$$\begin{aligned}
&\left|\int_{\pi_T}Q(t,x)\,\phi(x)\,\theta'(q)\,dt\,dx\right|\\
&\leq c\int_{\pi_T}\left|\phi(x)\right|\left((1+\gamma)|q|+1\right)dt\,dx\\
&\leq cT\Big((1+\gamma)\left\|\phi(x)\right\|_{L^2(R)}\|u_0\|_{H^1(R)}\\
&\qquad+\int_R\left|\phi(x)\right|dx\Big).
\end{aligned}\tag{23}$$

From inequalities (18)–(19) and (23), we obtain (12). □

Lemma 4. *There exists a positive constant c depending only on $\|u_0\|_{H^1(R)}$ and m such that*

$$\|Q(t,\cdot)\|_{L^\infty(R)}\leq c,\qquad\left\|\frac{\partial H(t,\cdot)}{\partial x}\right\|_{L^\infty(R)}\leq c.\tag{24}$$

If $k=0$, it holds that

$$\|Q(t,\cdot)\|_{L^1(R)}\leq c,\qquad\left\|\frac{\partial H(t,\cdot)}{\partial x}\right\|_{L^1(R)}\leq c,\tag{25}$$

$$\|Q(t,\cdot)\|_{L^2(R)}\leq c,\qquad\left\|\frac{\partial H(t,\cdot)}{\partial x}\right\|_{L^2(R)}\leq c.\tag{26}$$

Proof. We have

$$\begin{aligned}
&Q(t,x)\\
&\quad=2ku+\frac{2m^2+5m+2}{2(m+1)}u^{2m+2}\\
&\qquad-\Lambda^{-2}\left[2ku+\frac{2m^2+5m+2}{2(m+1)}u^{2m+2}+\frac{2m+1}{2}u^{2m}q^2\right],\\
&\frac{\partial H}{\partial x}=\Lambda^{-2}\partial_x\left[2ku+\frac{2m^2+5m+2}{2(m+1)}u^{2m+2}+\frac{2m+1}{2}u^{2m}u_x^2\right].
\end{aligned}\tag{27}$$

The first inequality of (24) is proved in Lemma 3 (see (22)). For the second inequality in (24), we have

$$\begin{aligned}
&\left|\frac{\partial H(t,\cdot)}{\partial x}\right| \\
&= \left| \frac{1}{2}\int_R e^{-|x-y|} \right. \\
&\qquad \times \left[2ku + \frac{2m^2+5m+2}{2(m+1)}u^{2m+2} \right. \\
&\qquad \left. \left. + \frac{2m+1}{2}u^{2m}u_y^2 \right]_y dy \right| \\
&= \left| \frac{1}{2}e^{-x}\int_{-\infty}^{x} e^{y} \left[2ku + \frac{2m^2+5m+2}{2(m+1)}u^{2m+2} \right. \right. \\
&\qquad \left. \left. + \frac{2m+1}{2}u^{2m}u_y^2 \right]_y dy \right. \\
&\quad + \frac{1}{2}e^{x}\int_{x}^{\infty} e^{-y} \left[2ku + \frac{2m^2+5m+2}{2(m+1)}u^{2m+2} \right. \\
&\qquad \left. \left. + \frac{2m+1}{2}u^{2m}u_y^2 \right]_y dy \right| \qquad (28) \\
&= \left| -\frac{1}{2}e^{-x}\int_{-\infty}^{x} e^{y} \left[2ku + \frac{2m^2+5m+2}{2(m+1)}u^{2m+2} \right. \right. \\
&\qquad \left. \left. + \frac{2m+1}{2}u^{2m}u_y^2 \right] dy \right. \\
&\quad + \frac{1}{2}e^{x}\int_{x}^{\infty} e^{-y} \left[2ku + \frac{2m^2+5m+2}{2(m+1)}u^{2m+2} \right. \\
&\qquad \left. \left. + \frac{2m+1}{2}u^{2m}u_y^2 \right] dy \right| \\
&\le \int_R e^{-|x-y|} \left| 2ku + \frac{2m^2+5m+2}{2(m+1)}u^{2m+2} \right. \\
&\qquad \left. + \frac{2m+1}{2}u^{2m}u_y^2 \right| dy \\
&\le c\|u\|_{H^1(R)} \\
&\le c\|u_0\|_{H^1(R)},
\end{aligned}$$

which together with (22) results in (24).

In fact, we have

$$\begin{aligned}
&\left\|u^{2m+2}\right\|_{L^1(R)} \le c\,\|u\|_{L^2(R)}^2 \le c\|u\|_{H^1(R)}^2 \le c\|u_0\|_{H^1(R)}^2, \\
&\left\|\Lambda^{-2}\left[\frac{2m^2+5m+2}{2(m+1)}u^{2m+2} + \frac{2m+1}{2}u^{2m}q^2\right]\right\|_{L^1(R)} \\
&\qquad \le c\|u\|_{H^1(R)}^2 \le c\|u_0\|_{H^1(R)}^2,
\end{aligned}$$

$$\begin{aligned}
\left\|\Lambda^{-2}[\partial_x u^{2m+2}]\right\|_{L^1(R)} &\le \iint_R e^{-|x-y|}u^{2m+2}\,dx\,dy \\
&\le c\|u\|_{H^1(R)}^2 \le c\|u_0\|_{H^1(R)}^2, \\
\left\|\Lambda^{-2}\partial_x[u^{2m}u_x^2]\right\|_{L^1(R)} &\le \iint_R e^{-|x-y|}u^{2m}u_y^2\,dx\,dy \\
&\le c\|u\|_{H^1(R)}^2 \le c\|u_0\|_{H^1(R)}^2.
\end{aligned}$$

From (3)–(31), we obtain (25).

Using

$$\begin{aligned}
&\|Q(t,\cdot)\|_{L^2(R)}^2 \le \|Q(t,\cdot)\|_{L^\infty}\|Q(t,\cdot)\|_{L^1(R)} \le c, \\
&\left\|\frac{\partial H(t,\cdot)}{\partial x}\right\|_{L^2(R)}^2 \le \left\|\frac{\partial H(t,\cdot)}{\partial x}\right\|_{L^\infty}\left\|\frac{\partial H(t,\cdot)}{\partial x}\right\|_{L^1(R)} \le c
\end{aligned} \qquad (29)$$

completes the proof of (26). □

From Lemma 1, we know that for any $u_0 \in H^s(R)$ with $s > 3/2$, there exist a maximal $T = T(u_0) > 0$ and a unique strong solution u to problem (3) such that

$$u \in C\left([0,T); H^s(R)\right) C^1\left([0,T); H^{s-1}(R)\right). \qquad (30)$$

Consider the differential equation

$$\begin{aligned}
\xi_t &= u^{2m+1}(t,\xi), \quad t \in [0,T), \\
\xi(0,x) &= x.
\end{aligned} \qquad (31)$$

Lemma 5. *Assume $u_0 \in H^s$, $s > 3$, and let $T > 0$ be the maximal existence time of the solution to problem (3). Then there exists a unique solution $\xi \in C^1([0,T)\times R, R)$ to problem (31). In addition, the map $p(t,\cdot)$ is an increasing diffeomorphism of R with $\xi_x(t,x) > 0$ for $(t,x) \in [0,T)\times R$.*

Proof. Using Lemma 1, we obtain $u \in C^1([0,T); H^{s-1}(R))$ and $H^{s-1} \in C^1(R)$. Therefore, we know that functions $u(t,x)$ and $u_x(t,x)$ are bounded, Lipschitz in space, and C^1 in time. The existence and uniqueness theorem for differential equations guarantees that problem (31) has a unique solution $\xi \in C^1([0,T)\times R, R)$.

From (31), we get

$$\begin{aligned}
\frac{d}{dt}\xi_x &= (2m+1)\,u^{2m}u_\xi(t,\xi)\,\xi_x, \quad t \in [0,T), \\
p_x(0,x) &= 1.
\end{aligned} \qquad (32)$$

Furthermore,

$$\xi_x(t,x) = \exp\left(\int_0^t (2m+1)\,u^{2m}u_\xi(\tau,\xi(\tau,x))\,d\tau\right). \qquad (33)$$

For every $T' < T$, the Sobolev imbedding theorem gives rise to

$$\sup_{(\tau,x)\in[0,T')\times R} \left|u_x(\tau,x)\right| < \infty. \qquad (34)$$

Therefore, there exists a constant $K_0 > 0$ such that $\xi_x(t,x) \ge e^{-K_0 t}$ for $(t,x) \in [0,T)\times R$. The proof is completed. □

Using (11) and (32), we get

$$\begin{aligned}\frac{du_\xi(t,\xi(t,x))}{dt} &= \frac{du_\xi(t,\xi(t,x))}{dt} + \frac{du_\xi(t,\xi(t,x))}{d\xi}\xi_t \\ &= \frac{du_\xi(t,\xi(t,x))}{dt} + \frac{du_\xi(t,\xi(t,x))}{d\xi}u^{2m+1} \\ &= Q(t,\xi(t,x)) - \frac{2m+1}{2}u^{2m}q^2 \\ &\le \| Q(t,x) \|_{L^\infty} \\ &\le c.\end{aligned} \tag{35}$$

It follows from (35) that

$$u_\xi(t,\xi(t,x)) \le c + ct. \tag{36}$$

Using Lemma 5 and (36), we have

$$u_x(t,x) \le c + ct, \quad t \in [0,\infty). \tag{37}$$

Using the first equation of problem (4) and Lemma 4, for an arbitrary integer $K \ge 2$, we have

$$\begin{aligned}&\frac{d}{dt}\int_R u^{2K}dx \\ &= 2K\int_R u^{2K-1}\frac{du}{dt}dx \\ &= 2K\int_R u^{2K-1}\left[-u^{2m+1}u_x - \frac{\partial H}{\partial x}\right]dx \\ &= -2K\int_R u^{2K+2m}u_x dx - 2K\int_R u^{2K-1}\frac{\partial H}{\partial x}dx \\ &= -2K\int_R u^{2K-1}\frac{\partial H}{\partial x}dx \\ &\le 2K\|u\|^{2K-3}_{L^\infty(R)}\left\|\frac{\partial H}{\partial x}\right\|_{L^\infty(R)}\|u\|^2_{L^2(R)} \\ &\le c\|u_0\|_{H^1(R)} \\ &\le c,\end{aligned} \tag{38}$$

from which we get

$$\int_R u^{2K}dx \le ct + \int_R u_0^{2K}dx \le ct + c\|u_0\|_{L^\infty(R)}\|u_0\|^2_{H^1(R)}. \tag{39}$$

By the L^p interpolation theorem, for all $2 \le p < \infty$, we obtain

$$\|u\|_{L^p(R)} \le c(1+t), \quad t \in [0,\infty), \tag{40}$$

where c depends on $\|u_0\|_{H^1(R)}$ and p.

From Lemma 3 and (37) and (40), we complete the proof of Theorem 2.

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