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*Research Article*

# On Some Symmetric Systems of Difference Equations

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Here we show that the main results in the papers by Yalcinkaya (2008), Yalcinkaya and Cinar (2010), and Yalcinkaya, Cinar, and Simsek (2008), as well as a conjecture from the last mentioned paper, follow from a slight modification of a result by G. Papaschinopoulos and C. J. Schinas. We also give some generalizations of these results.

## 1. Introduction

Studying difference equations and systems which possess some kind of symmetry attracted some attention recently (see, e.g., [1–25] and the related references therein).

Paper [23] studied the following system of difference equations:

$$x_{n+1} = \frac{y_n x_{n-1} + a}{y_n + x_{n-1}}, \qquad y_{n+1} = \frac{x_n y_{n-1} + a}{x_n + y_{n-1}}, \quad n \in \mathbb{N}_0. \tag{1}$$

In [24], authors claim that they study the system

$$x_{n+1} = \frac{y_n + x_{n-1}}{y_n x_{n-1} + a}, \qquad y_{n+1} = \frac{x_n + y_{n-1}}{x_n y_{n-1} + a}, \quad n \in \mathbb{N}_0, \tag{2}$$

while in [25], the authors studied the system

$$x_{n+1} = \frac{x_n y_{n-1} + a}{x_n + y_{n-1}}, \qquad y_{n+1} = \frac{y_n x_{n-1} + a}{y_n + x_{n-1}}, \quad n \in \mathbb{N}_0, \tag{3}$$

where $a > 0$.

Since $a > 0$, it is clear that the change of variables

$$(x_n, y_n) \longrightarrow (\sqrt{a} x_n \sqrt{a} y_n), \tag{4}$$

reduces systems (1) and (3) to the case $a = 1$. The authors of [24] claim that the change of variables (4) reduces (2) to the case $a = 1$ too; however by using the change system (2) becomes

$$x_{n+1} = \frac{y_n + x_{n-1}}{a(y_n x_{n-1} + 1)}, \qquad y_{n+1} = \frac{x_n + y_{n-1}}{a(x_n y_{n-1} + 1)}, \quad n \in \mathbb{N}_0. \tag{5}$$

Therefore, in fact, [24] studied only system (2) for the case $a = 1$.

Based on this observation we may, and will, assume that $a = 1$ in systems of difference equations (1)–(3).

In the main results in [23–25] it is proved that when $a = 1$, the positive equilibrium point $(\overline{x}, \overline{y}) = (1, 1)$ of systems (1)–(3) is globally asymptotically stable.

The authors of [25] finish their paper by the statement that they believe that the results therein can be conveniently extended to the following higher order system of difference equations:

$$x_{n+1} = \frac{x_n y_{n-l} + a}{x_n + y_{n-l}}, \qquad y_{n+1} = \frac{y_n x_{n-l} + a}{y_n + x_{n-l}}, \quad n \in \mathbb{N}_0, \tag{6}$$

when $l \in \mathbb{N} \setminus \{1\}$.

Here, among others, we show that all the results and conjectures mentioned above follow from a slight modification of a result in the literature published before papers [23–25]. For related systems see also [2, 5–10, 12, 17–20].

## 2. Main Results

Let $\mathbb{R}_+ = (0, +\infty)$ and $\mathbb{R}_+^n$ be the set of all positive $n$-dimensional vectors. The following theorem was proved in [4].

**Theorem A.** *Let $(M, d)$ be a complete metric space, where $d$ denotes a metric and $M$ is an open subset of $\mathbb{R}^n$, and let $T : M \to M$ be a continuous mapping with the unique equilibrium $x^* \in M$. Suppose that for the discrete dynamic system*

$$x_{n+1} = T x_n, \quad n \in \mathbb{N}_0, \tag{7}$$

*there is a $k \in \mathbb{N}$ such that for the kth iterate of $T$, the following inequality holds:*

$$d\left(T^k x, x^*\right) < d\left(x, x^*\right), \tag{8}$$

*for all $x \neq x^*$. Then $x^*$ is globally asymptotically stable with respect to metric $d$.*

The part-metric (see [21]) is a metric defined on $\mathbb{R}_+^n$ by

$$p\ (X, Y) = -\log \min_{1 \le i \le n} \left\{ \frac{x_i}{y_i}, \frac{y_i}{x_i} \right\}, \tag{9}$$

for arbitrary vectors $X = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}_+^n$ and $Y = (y_1, y_2, \ldots, y_n)^T \in \mathbb{R}_+^n$.

It is known that the part-metric $p$ is a continuous metric on $\mathbb{R}_+^n$, $(\mathbb{R}_+^n, p)$ is a complete metric space, and that the distances induced by the part-metric and by the Euclidean norm are equivalent on $\mathbb{R}_+^n$ (see, e.g., [4]).

Based on these properties and Theorem A, the following corollary follows.

**Corollary 1.** *Let $T : \mathbb{R}_+^n \to \mathbb{R}_+^n$ be a continuous mapping with a unique equilibrium $x^* \in \mathbb{R}_+^n$. Suppose that for the discrete dynamic system (7), there is some $k \in \mathbb{N}$ such that for the part-metric $p$ inequality*

$$p\ \left(T^k x, x^*\right) < p\ \left(x, x^*\right) \tag{10}$$

*holds for all $x \neq x^*$. Then $x^*$ is globally asymptotically stable.*

Some applications of various part-metric-related inequalities and some asymptotic methods in studying difference equations related to symmetric ones can be found, for example, in [1, 3–5, 10, 11, 13–16, 22] (see also the related references therein).

In Lemma 2.3 in [10], Papaschinopoulos and Schinas formulated a variant of the following result, without giving a proof. However, the part concerning the equality in inequality (12) below, is not mentioned, but it is crucial in applying Corollary 1 (see inequality (10)). For this reason, the completeness and the benefit of the reader we will give a complete proof of it.

**Proposition 2.** *Let $f : \mathbb{R}_+^{2m} \to \mathbb{R}_+$, $g : \mathbb{R}_+^{2m} \to \mathbb{R}_+$ be continuous functions. We suppose that the system of two difference equations,*

$$u_{n+m} = f\left(u_n, u_{n+1}, \ldots, u_{n+m-1}, v_n, v_{n+1}, \ldots, v_{n+m-1}\right), \qquad v_{n+m} = g\left(u_n, u_{n+1}, \ldots, u_{n+m-1}, v_n, v_{n+1}, \ldots, v_{n+m-1}\right), \tag{11}$$

*has a unique positive equilibrium $(w, w)$. Suppose also that there is an $r \in \mathbb{N}$ such that for any positive solution $(u_n, v_n)_{n \in \mathbb{N}_0}$ of system (11), the following inequalities:*

$$(u_n - u_{n+r})\left(\frac{w^2}{u_n} - u_{n+r}\right) \le 0, \qquad (v_n - v_{n+r})\left(\frac{w^2}{v_n} - v_{n+r}\right) \le 0, \quad n \in \mathbb{N}_0, \tag{12}$$

*hold, with the equalities if and only if $u_n = w$, for every $n \in \mathbb{N}_0$, and $v_n = w$, for every $n \in \mathbb{N}_0$, respectively. Then the equilibrium $(w, w)$ is globally asymptotically stable.*

*Proof.* First, we prove that for every $n \in \mathbb{N}_0$

$$\min\left\{\frac{u_{n+r}}{w}, \frac{v_{n+r}}{w}, \frac{w}{u_{n+r}}, \frac{w}{v_{n+r}}\right\} > \min\left\{\frac{u_n}{w}, \frac{v_n}{w}, \frac{w}{u_n}, \frac{w}{v_n}\right\}, \tag{13}$$

if and only if $(u_n, v_n) \neq (w, w)$.

To prove (13), it is enough to prove that

$$\min\left\{\frac{u_{n+r}}{w}, \frac{w}{u_{n+r}}\right\} > \min\left\{\frac{u_n}{w}, \frac{w}{u_n}\right\}, \tag{14}$$

if and only if $u_n \neq w$, and

$$\min\left\{\frac{v_{n+r}}{w}, \frac{w}{v_{n+r}}\right\} > \min\left\{\frac{v_n}{w}, \frac{w}{v_n}\right\}, \tag{15}$$

if and only if $v_n \neq w$.

The proofs of inequalities (14) and (15) are the same (up to the interchanging letters $u$ and $v$) so it is enough to prove (14).

Now note that if the equality holds in the first inequality in (12), then we have that

$$u_n = u_{n+r} \quad \text{or} \quad \frac{w}{u_n} = \frac{u_{n+r}}{w}, \tag{16}$$

from which, in both cases, it easily follows that

$$\min\left\{\frac{u_{n+r}}{w}, \frac{w}{u_{n+r}}\right\} = \min\left\{\frac{u_n}{w}, \frac{w}{u_n}\right\}. \tag{17}$$

On the other hand, if (17) holds, then we easily obtain that one of the equalities in (16) holds, and consequently it follows that the equality holds in the first inequality in (12). Hence, by one of the assumptions, we have that (17) holds if and only if $u_n = w$ for every $n \in \mathbb{N}_0$.

Now suppose that the first inequality in (12), is strict. Then, if $u_n > u_{n+r}$, directly follows that $w/u_{n+r} > w/u_n$, while from the first inequality in (12) it follows that $u_{n+r}/w > w/u_n$. Hence

$$\min\left\{\frac{u_{n+r}}{w}, \frac{w}{u_{n+r}}\right\} > \frac{w}{u_n}, \tag{18}$$

from which inequality (14) easily follows.

If $u_n < u_{n+r}$, then $u_{n+r}/w > u_n/w$, while from the first inequality in (12), it follows that $w/u_{n+r} > u_n/w$. From these two inequalities, we have that

$$\min\left\{\frac{u_{n+r}}{w}, \frac{w}{u_{n+r}}\right\} > \frac{u_n}{w}, \tag{19}$$

and consequently (14).

If (14) and (15) hold then if $u_n \neq w$ and $v_n \neq w$, inequality (13) immediately follows by using the following elementary implication: if $a > b$ and $c > d$, then $\min\{a, c\} > \min\{b, d\}$.

If $u_n \neq w$ and $v_n = w$, then from the second inequality in (12), we have that $v_{n+r} = v_n = w$. Hence

$$\min\left\{\frac{v_{n+r}}{w}, \frac{w}{v_{n+r}}\right\} = \min\left\{\frac{v_n}{w}, \frac{w}{v_n}\right\} = 1 > \min\left\{\frac{u_n}{w}, \frac{w}{u_n}\right\}, \tag{20}$$

which along with (14) implies (13). The case $u_n = w$ and $v_n \neq w$ directly follows from the case $u_n \neq w$ and $v_n = w$, by the symmetry.

Finally, note that if $u_n = v_n = w$, then from (12), we have that $u_{n+r} = u_n = w$ and $v_{n+r} = v_n = w$, so that the first equality in (20) holds and

$$\min\left\{\frac{u_{n+r}}{w}, \frac{w}{u_{n+r}}\right\} = \min\left\{\frac{u_n}{w}, \frac{w}{u_n}\right\} = 1, \tag{21}$$

from which it follows that both minima in (13) are equal, finishing the proof of the claim.

Now we define the map $T : \mathbb{R}_+^{2m} \to \mathbb{R}_+^{2m}$ as follows:

$$\begin{aligned} &T(x_1, x_2, \ldots, x_{m-1}, x_m, y_1, y_2, \ldots, y_{m-1}, y_m) \\ &\quad = (x_2, \ldots, x_m, f(x_1, \ldots, x_m, y_1, \ldots, y_m), \\ &\qquad y_2, \ldots, y_m, g(x_1, \ldots, x_m, y_1, \ldots, y_m)). \end{aligned} \tag{22}$$

Then we get

$$\begin{aligned} &T(u_n, u_{n+1}, \ldots, u_{n+m-2}, u_{n+m-1}, v_n, v_{n+1}, \ldots, v_{n+m-2}, v_{n+m-1}) \\ &\quad = (u_{n+1}, \ldots, u_{n+m-1}, u_{n+m}, v_{n+1}, \ldots, v_{n+m-1}, v_{n+m}), \end{aligned} \tag{23}$$

and by induction

$$\begin{aligned} &T^s(u_n, u_{n+1}, \ldots, u_{n+m-2}, u_{n+m-1}, v_n, v_{n+1}, \ldots, v_{n+m-2}, v_{n+m-1}) \\ &\quad = (u_{n+s}, \ldots, u_{n+m-2+s}, u_{n+m-1+s}, \\ &\qquad v_{n+s}, \ldots, v_{n+m-2+s}, v_{n+m-1+s}), \end{aligned} \tag{24}$$

for every $s \in \mathbb{N}$.

By using inequality (13) and the fact that the inequalities $1 \geq a_i > b_i$, $i \in I \subseteq \{1, \ldots, m\}$, $I \neq \emptyset$, along with equalities $a_i = b_i = 1$, $i \in \{1, \ldots, m\} \setminus I$, imply the inequality $\min_{1\leq i\leq m} a_i > \min_{1\leq i\leq m} b_i$, we have that for each vector $\vec{x} \in \mathbb{R}_+^{2m}$ such that $\vec{x} \neq (w, w, \ldots, w) =: \vec{w} \in \mathbb{R}_+^{2m}$,

$$\begin{aligned} p(T^r(\vec{x}), \vec{w}) &= -\log\min\left\{\frac{u_{n+r}}{w}, \frac{w}{u_{n+r}}, \ldots, \frac{u_{n+r+m-1}}{w},\right. \\ &\qquad \frac{w}{u_{n+r+m-1}}, \frac{v_{n+r}}{w}, \frac{w}{v_{n+r}}, \ldots, \\ &\qquad \left.\frac{v_{n+r+m-1}}{w}, \frac{w}{v_{n+r+m-1}}\right\} \\ &< -\log\min\left\{\frac{u_n}{w}, \frac{w}{u_n}, \ldots, \frac{u_{n+m-1}}{w}, \frac{w}{u_{n+m-1}},\right. \\ &\qquad \left.\frac{v_n}{w}, \frac{w}{v_n}, \ldots, \frac{v_{n+m-1}}{w}, \frac{w}{v_{n+m-1}}\right\} \\ &= p(\vec{x}, \vec{w}), \end{aligned} \tag{25}$$

from which the proof follows by Corollary 1. □

It is not difficult to see that the following extension of Proposition 2 can be proved by slight modifications of the proof of Proposition 2.

**Proposition 3.** *Let $f_i : \mathbb{R}_+^{lm} \to \mathbb{R}_+, i = 1, \ldots, l$, be continuous functions. Suppose that the system of difference equations*

$$\begin{aligned} u_{n+m}^{(1)} &= f_1\left(u_n^{(1)}, u_{n+1}^{(1)}, \ldots, u_{n+m-1}^{(1)}, \ldots, u_n^{(l)}, u_{n+1}^{(l)}, \ldots, u_{n+m-1}^{(l)}\right), \\ &\vdots \\ u_{n+m}^{(i)} &= f_i\left(u_n^{(1)}, u_{n+1}^{(1)}, \ldots, u_{n+m-1}^{(1)}, \ldots, u_n^{(l)}, u_{n+1}^{(l)}, \ldots, u_{n+m-1}^{(l)}\right), \\ &\vdots \\ u_{n+m}^{(l)} &= f_l\left(u_n^{(1)}, u_{n+1}^{(1)}, \ldots, u_{n+m-1}^{(1)}, \ldots, u_n^{(l)}, u_{n+1}^{(l)}, \ldots, u_{n+m-1}^{(l)}\right) \end{aligned} \tag{26}$$

*has a unique positive equilibrium* $(w,\ldots,w)\in\mathbb{R}_+^l$, *and that there is an* $r\in\mathbb{N}$ *such that for any solution* $(u_n^{(1)},\ldots,u_n^{(l)})_{n\in\mathbb{N}_0}\subset\mathbb{R}_+^l$ *of system* (26), *the following inequalities:*

$$\left(u_n^{(i)}-u_{n+r}^{(i)}\right)\left(\frac{w^2}{u_n^{(i)}}-u_{n+r}^{(i)}\right)\le 0,\quad n\in\mathbb{N}_0,\ i=1,\ldots,l, \tag{27}$$

*hold, with the equalities if and only if* $u_n^{(i)}=w$, *for every* $n\in\mathbb{N}_0$, *and* $i=1,\ldots,l$. *Then the equilibrium* $(w,\ldots,w)$ *is globally asymptotically stable.*

Now we use Proposition 2 in proving the results in papers [23–25].

**Corollary 4.** *Let* $k,l\in\mathbb{N}_0$, $k\neq l$. *Consider the system*

$$x_{n+1}=\frac{x_{n-k}y_{n-l}+1}{x_{n-k}+y_{n-l}},\qquad y_{n+1}=\frac{y_{n-k}x_{n-l}+1}{y_{n-k}+x_{n-l}},\quad n\in\mathbb{N}_0. \tag{28}$$

*Then the positive equilibrium point* $(\overline{x},\overline{y})=(1,1)$ *of system* (28) *is globally asymptotically stable with respect to the set* $\mathbb{R}_+^m\times\mathbb{R}_+^m$, *where* $m=\max\{k,l\}$.

*Proof.* We may assume that $m=k$. From system (28), we have that

$$x_{n+1}-x_{n-k}=\frac{1-x_{n-k}^2}{y_{n-l}+x_{n-k}},\qquad x_{n+1}-\frac{1}{x_{n-k}}=\frac{y_{n-l}\left(x_{n-k}^2-1\right)}{x_{n-k}\left(y_{n-l}+x_{n-k}\right)}, \tag{29}$$

$$y_{n+1}-y_{n-k}=\frac{1-y_{n-k}^2}{x_{n-l}+y_{n-k}},\qquad y_{n+1}-\frac{1}{y_{n-k}}=\frac{x_{n-l}\left(y_{n-k}^2-1\right)}{y_{n-k}\left(x_{n-l}+y_{n-k}\right)} \tag{30}$$

from which it follows that

$$\left(x_{n+1}-x_{n-k}\right)\left(x_{n+1}-\frac{1}{x_{n-k}}\right)\le 0,\qquad \left(y_{n+1}-y_{n-k}\right)\left(y_{n+1}-\frac{1}{y_{n-k}}\right)\le 0 \tag{31}$$

so that condition (12) in Proposition 2 is fulfilled with $r=k+1$.

Clearly if

$$x_n=1=y_n\quad\text{for every } n\ge-\max\{l,k\}, \tag{32}$$

then in (31) equalities follow. On the other hand, if equality holds in the first inequality in (31), we have that

$$x_{n+1}=x_{n-k}\quad\text{or}\quad x_{n+1}=\frac{1}{x_{n-k}}. \tag{33}$$

If $x_{n+1}=x_{n-k}$, then from the first equality in (29) we have that $x_{n-k}=1$, while if $x_{n+1}=1/x_{n-k}$, then from the second equality in (29), we have that $x_{n-k}=1$.

By symmetry (see (30)), we have that if equality holds in the second inequality in (31), then $y_{n-k}=1$. Therefore, equalities in (31) hold if and only if $(x_{n-k},y_{n-k})=(1,1)$. Hence all the conditions of Proposition 2 are fulfilled from which it follows that the positive equilibrium $(1,1)$ is globally asymptotically stable with respect to the set $\mathbb{R}_+^m\times\mathbb{R}_+^m$. □

*Remark 5.* Corollary 4 extends and gives a very short proof of the main result in [23], which is obtained for $k=1$ and $l=0$. Further, it also extends and gives a very short proof of the main result in [25], which is obtained for $k=0$ and $l=1$. Moreover, it confirms the conjecture in [25], which is obtained for $k=0$ and $l\in\mathbb{N}\setminus\{1\}$.

**Corollary 6.** *Let* $k,l\in\mathbb{N}_0$, $k\neq l$. *Consider the system*

$$x_{n+1}=\frac{x_{n-k}+y_{n-l}}{x_{n-k}y_{n-l}+1},\qquad y_{n+1}=\frac{y_{n-k}+x_{n-l}}{y_{n-k}x_{n-l}+1},\quad n\in\mathbb{N}_0. \tag{34}$$

*Then the positive equilibrium point* $(\overline{x},\overline{y})=(1,1)$ *of system* (34) *is globally asymptotically stable with respect to the set* $\mathbb{R}_+^m\times\mathbb{R}_+^m$, *where* $m=\max\{k,l\}$.

*Proof.* We may assume that $m=k$. From system (34), we have that

$$\begin{aligned} x_{n+1}-x_{n-k}&=\frac{y_{n-l}\left(1-x_{n-k}^2\right)}{x_{n-k}y_{n-l}+1},\\ x_{n+1}-\frac{1}{x_{n-k}}&=\frac{x_{n-k}^2-1}{x_{n-k}\left(x_{n-k}y_{n-l}+1\right)},\\ y_{n+1}-y_{n-k}&=\frac{x_{n-l}\left(1-y_{n-k}^2\right)}{x_{n-l}y_{n-k}+1},\\ y_{n+1}-\frac{1}{y_{n-k}}&=\frac{y_{n-k}^2-1}{y_{n-k}\left(x_{n-l}y_{n-k}+1\right)} \end{aligned} \tag{35}$$

from which it follows that

$$\left(x_{n+1}-x_{n-k}\right)\left(x_{n+1}-\frac{1}{x_{n-k}}\right)\le 0,\qquad \left(y_{n+1}-y_{n-k}\right)\left(y_{n+1}-\frac{1}{y_{n-k}}\right)\le 0. \tag{36}$$

Hence condition (12) in Proposition 2 is fulfilled with $r=k+1$. On the other hand, similarly as in the proof of Corollary 4 it is proved that equalities in (36) hold if and only if $(x_{n-k},y_{n-k})=(1,1)$. Hence all the conditions of Proposition 2 are fulfilled from which it follows that the positive equilibrium $(1,1)$ is globally asymptotically stable with respect to the set $\mathbb{R}_+^m\times\mathbb{R}_+^m$. □

*Remark 7.* Corollary 6 extends and gives a very short proof of the main result in [24], which is obtained for $k = 1$ and $l = 0$.

*Remark 8.* Corollary 6 is also a consequence of Corollary 4. Namely, by using the change of variables $(x_n, y_n) = (1/u_n, 1/v_n)$, system (34) is transformed into the system

$$v_{n+1} = \frac{v_{n-k}u_{n-l} + 1}{v_{n-k} + u_{n-l}},$$
$$u_{n+1} = \frac{u_{n-k}v_{n-l} + 1}{u_{n-k} + v_{n-l}}, \quad n \in \mathbb{N}_0, \tag{37}$$

which is system (28). In particular, this shows that systems (1) and (2), for the case $a = 1$, are equivalent and consequently the results in [23, 24].

*Remark 9.* Similar type of issues appear in some literature on scalar difference equations (see, e.g., related results in papers [1, 5, 11, 13]).

It is of some interest to extend results in Corollaries 4 and 6 by using Proposition 2. The next result is of this kind and it extends a result in [5].

**Corollary 10.** *Let $f \in C(\mathbb{R}_+^k, \mathbb{R}_+)$ and $g \in C(\mathbb{R}_+^l, \mathbb{R}_+)$ with $k, l \in \mathbb{N}$, $0 \le r_1 < \cdots < r_k$ and $0 \le m_1 < \cdots < m_l \le r_k$ and satisfy the following two conditions:*

(*H1*) $[f(u_1, u_2, \ldots, u_k)]^* = f(u_1^*, u_2^*, \ldots, u_k^*)$,

(*H2*) $f(u_1^*, u_2^*, \ldots, u_k^*) \le u_k^*$,

*where $a^* := \max\{a, 1/a\}$.*

*Then $(\overline{x}, \overline{y}) = (1, 1)$ is the unique positive equilibrium of the system of difference equations*

$$x_n = \frac{f\left(x_{n-r_1-1}, \ldots, x_{n-r_k-1}\right) g\left(y_{n-m_1-1}, \ldots, y_{n-m_l-1}\right) + 1}{f\left(x_{n-r_1-1}, \ldots, x_{n-r_k-1}\right) + g\left(y_{n-m_1-1}, \ldots, y_{n-m_l-1}\right)}, \quad n \in \mathbb{N},$$
$$y_n = \frac{f\left(y_{n-r_1-1}, \ldots, y_{n-r_k-1}\right) g\left(x_{n-m_1-1}, \ldots, x_{n-m_l-1}\right) + 1}{f\left(y_{n-r_1-1}, \ldots, y_{n-r_k-1}\right) + g\left(x_{n-m_1-1}, \ldots, x_{n-m_l-1}\right)}, \quad n \in \mathbb{N}, \tag{38}$$

*and it is globally asymptotically stable.*

*Proof.* Let

$$f_n = f\left(x_{n-r_1-1}, \ldots, x_{n-r_k-1}\right),$$
$$g_n = g\left(y_{n-m_1-1}, \ldots, y_{n-m_l-1}\right). \tag{39}$$

We should determine the sign of the product of the following expressions:

$$P_n := \frac{f_n g_n + 1}{f_n + g_n} - x_{n-r_k-1} = \frac{1}{f_n + g_n}\left(f_n g_n\left(1 - \frac{x_{n-r_k-1}}{f_n}\right) + 1 - x_{n-r_k-1}f_n\right), \tag{40}$$

$$Q_n := \frac{f_n g_n + 1}{f_n + g_n} - \frac{1}{x_{n-r_k-1}} = \frac{1}{x_{n-r_k-1}\left(f_n + g_n\right)}\left(g_n\left(x_{n-r_k-1}f_n - 1\right) + f_n\left(\frac{x_{n-r_k-1}}{f_n} - 1\right)\right). \tag{41}$$

From (40) and (41), we see if we show that $x_{n-r_k-1}f_n - 1$ and $(x_{n-r_k-1}/f_n) - 1$ have the same sign for $n \in \mathbb{N}$, then $P_nQ_n$ will be nonpositive.

We consider four cases.

*Case 1.* $x_{n-r_k-1} \ge 1$, $f_n \ge 1$. Clearly in this case $x_{n-r_k-1}f_n - 1 \ge 0$. By (*H1*) and (*H2*), we have that

$$1 \le f_n = \left(f_n\right)^* = f\left(x_{n-r_1-1}^*, \ldots, x_{n-r_k-1}^*\right) \le x_{n-r_k-1}^* = x_{n-r_k-1}. \tag{42}$$

Hence $(x_{n-r_k-1}/f_n) - 1 \ge 0$ and consequently

$$\left(x_{n-r_k-1}f_n - 1\right)\left(\frac{x_{n-r_k-1}}{f_n} - 1\right) \ge 0. \tag{43}$$

*Case 2.* $x_{n-r_k-1} \ge 1$, $f_n \le 1$. Since $1/f_n \ge 1$, we obtain $(x_{n-r_k-1}/f_n) - 1 \ge 0$. On the other hand, by (*H1*) and (*H2*), we have

$$\frac{1}{f_n} = \left(f_n\right)^* = f\left(x_{n-r_1-1}^*, \ldots, x_{n-r_k-1}^*\right) \le x_{n-r_k-1}^* = x_{n-r_k-1}, \tag{44}$$

so that $x_{n-r_k-1}f_n - 1 \ge 0$. Hence (43) follows in this case.

*Case 3.* Case $x_{n-r_k-1} \le 1$, $f_n \ge 1$. Then we have that $1/f_n \le 1$ and consequently $(x_{n-r_k-1}/f_n) - 1 \le 0$. On the other hand, we have

$$f_n = \left(f_n\right)^* = f\left(x_{n-r_1-1}^*, \ldots, x_{n-r_k-1}^*\right) \le x_{n-r_k-1}^* = \frac{1}{x_{n-r_k-1}}, \tag{45}$$

so that $x_{n-r_k-1}f_n - 1 \le 0$. Hence (43) follows in this case too.

*Case 4.* Case $x_{n-r_k-1} \le 1$, $f_n \le 1$. Then $x_{n-r_k-1}f_n - 1 \le 0$. On the other hand, we have

$$\frac{1}{f_n} = \left(f_n\right)^* = f\left(x_{n-r_1-1}^*, \ldots, x_{n-r_k-1}^*\right) \le x_{n-r_k-1}^* = \frac{1}{x_{n-r_k-1}}, \tag{46}$$

so that $(x_{n-r_k-1}/f_n)-1 \leq 0$. Hence (43) also holds in this case. Thus $P_nQ_n \leq 0$, for every $n \in \mathbb{N}$.

Assume that $P_nQ_n = 0$, then $P_n = 0$ or $Q_n = 0$. Using (40) or (41) along with (43) in any of these two cases, we have that

$$f_n = \frac{1}{x_{n-r_k-1}} = x_{n-r_k-1}, \quad n \in \mathbb{N}. \tag{47}$$

Hence $x_{n-r_k-1} = 1$, $n \in \mathbb{N}$.

Let

$$\widehat{f}_n = f\left(y_{n-r_1-1}, \ldots, y_{n-r_k-1}\right), \\ \widehat{g}_n = g\left(x_{n-m_1-1}, \ldots, x_{n-m_l-1}\right). \tag{48}$$

Using the following expressions:

$$\begin{aligned} \widehat{P}_n &:= \frac{\widehat{f}_n\widehat{g}_n + 1}{\widehat{f}_n + \widehat{g}_n} - y_{n-r_k-1} \\ &= \frac{1}{\widehat{f}_n + \widehat{g}_n}\left(\widehat{f}_n\widehat{g}_n\left(1 - \frac{y_{n-r_k-1}}{\widehat{f}_n}\right) + 1 - y_{n-r_k-1}\widehat{f}_n\right), \\ \widehat{Q}_n &:= \frac{\widehat{f}_n\widehat{g}_n + 1}{\widehat{f}_n + \widehat{g}_n} - \frac{1}{y_{n-r_k-1}} \\ &= \frac{1}{y_{n-r_k-1}\left(\widehat{f}_n + \widehat{g}_n\right)}\left(\widehat{g}_n\left(y_{n-r_k-1}\widehat{f}_n - 1\right) + \widehat{f}_n\left(\frac{y_{n-r_k-1}}{\widehat{f}_n} - 1\right)\right), \end{aligned} \tag{49}$$

it can be proved similarly that $\widehat{P}_n\widehat{Q}_n \leq 0$, for every $n \in \mathbb{N}$, and that $\widehat{P}_n\widehat{Q}_n = 0$, if and only if $y_{n-r_k-1} = 1$, $n \in \mathbb{N}$.

Finally, let $(x^*, y^*)$ be a solution of the system

$$x^* = \frac{f\left(\vec{x}_k^*\right)g\left(\vec{y}_l^*\right) + 1}{f\left(\vec{x}_k^*\right) + g\left(\vec{y}_l^*\right)}, \qquad y^* = \frac{f\left(\vec{y}_k^*\right)g\left(\vec{x}_l^*\right) + 1}{f\left(\vec{y}_k^*\right) + g\left(\vec{x}_l^*\right)}. \tag{50}$$

Then we have that

$$\begin{aligned} 0 &= \frac{f\left(\vec{x}_k^*\right)g\left(\vec{y}_l^*\right) + 1}{f\left(\vec{x}_k^*\right) + g\left(\vec{y}_l^*\right)} - x^* \\ &= \frac{1}{f\left(\vec{x}_k^*\right) + g\left(\vec{y}_l^*\right)}\left(f\left(\vec{x}_k^*\right)g\left(\vec{y}_l^*\right)\left(1 - \frac{x^*}{f\left(\vec{x}_k^*\right)}\right) + 1 - x^* f\left(\vec{x}_k^*\right)\right), \end{aligned} \tag{51}$$

where $\vec{z}_j^* = (z^*, \ldots, z^*)$ denotes the vector consisting of $j$ copies of $z^*$. Then similar to the considerations in Cases (i)–(iv), it follows that $f(\vec{x}_k^*) = x^* = 1/x^*$, so that $x^* = 1$, and similarly it is obtained that $y^* = 1$. Hence $(x^*, y^*) = (1, 1)$ is a unique positive equilibrium of system (26).

From all above mentioned and by Proposition 2, we get the result. □

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