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Research Article

A Viscosity Hybrid Steepest Descent Method for Equilibrium Problems, Variational Inequality Problems, and Fixed Point Problems of Infinite Family of Strictly Pseudocontractive Mappings and Nonexpansive Semigroup

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In this paper, modifying the set of variational inequality and extending the nonexpansive mapping of hybrid steepest descent method to nonexpansive semigroups, we introduce a new iterative scheme by using the viscosity hybrid steepest descent method for finding a common element of the set of solutions of a system of equilibrium problems, the set of fixed points of an infinite family of strictly pseudocontractive mappings, the set of solutions of fixed points for nonexpansive semigroups, and the sets of solutions of variational inequality problems with relaxed cocoercive mapping in a real Hilbert space. We prove that the sequence converges strongly to a common element of the above sets under some mild conditions. The results shown in this paper improve and extend the recent ones announced by many others.

1. Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H and let $F : C\times C \rightarrow R$ be a bifunction. We consider the following equilibrium problem (EP) which is to find $x^* \in C$ such that

$$\text{EP: } F(x^*, y) \geq 0, \quad \forall y \in C. \tag{1}$$

The set of solutions of EP is denoted by $\text{EP}(F)$.

Let $\{F_i, i = 1, 2, \dots, N\}$ be a finite family of bifunctions from $C \times C$ into R, where R is the set of real numbers. The system of equilibrium problems for $\{F_1, F_2, \dots, F_N\}$ is to find a common element $x^* \in C$ such that

$$\begin{aligned} &F_1(x^*, y) \geq 0, \quad \forall y \in C,\\ &F_2(x^*, y) \geq 0, \quad \forall y \in C,\\ &\vdots\\ &F_N(x^*, y) \geq 0, \quad \forall y \in C. \end{aligned} \tag{2}$$

We denote the set of solutions of (2) by $\cap_{k=1}^{N} \text{SEP}(F_k)$, where $\text{SEP}(F_k)$ is the set of solutions to the equilibrium problems, that is,

$$F_k(x^*, y) \geq 0, \quad \forall y \in C. \tag{3}$$

If $N = 1$, then the problem (2) is reduced to the equilibrium problems.

If $N = 1$ and $F(x^*, y) = \langle Tx^*, y - x^*\rangle$, then the problem (2) is reduced to the variational inequality problems of finding $x^* \in C$ such that

$$\langle Tx^*, y - x^*\rangle \geq 0, \quad \forall y \in C. \tag{4}$$

The set of solutions of (4) is denoted by $\text{VI}(C, T)$.

The equilibrium problem is very general in the sense that it includes, as special cases, fixed point problems, variational inequality problems, optimization problems, Nash equilibrium problems in noncooperative games, and numerous problems in physics, economics, and others. Some methods have been proposed to solve $\text{VI}(C,T)$, $\text{EP}(F)$, and $\text{SEP}(F_k)$; see, for example, [1–29] and references therein. Formulations

(2) extend this formulism to such problems, covering in particular various forms of feasibility problems [30, 31].

Definition 1. One-parameter family mapping $\Gamma = \{T(t) : t \in R^+\}$ from C into itself is said to be a nonexpansive semigroup on C if it satisfies the following conditions:

(i) $T(0)x = x$ for all $x \in C$,

(ii) $T(s+t) = T(s)T(t)$ for all $s, t \in R^+$,

(iii) for each $x \in C$, the mapping $T(t)x$ is continuous,

(iv) $\|T(t)x - T(t)y\| \leq \|x - y\|$ for all $x, y \in C$ and $t \in R^+$.

Remark 2. We denote by $F(\Gamma)$ the set of all common fixed points of Γ, that is, $F(\Gamma) := \cap_{t\in R^+} F(T(t)) = \{x \in C : T(t)x = x\}$.

Let $B : C \to H$ be a nonlinear mapping. Now, we recall the following definitions.

(1) B is said to be monotone if

$$\langle Bx - By, x - y\rangle \geq 0, \quad \forall x, y \in C. \tag{5}$$

(2) $B : C \to C$ is called ω-Lipschitzian if there exists a positive constant ω such that

$$\|Bx - By\| \leq \omega \|x - y\|, \quad \forall x, y \in C. \tag{6}$$

(3) B is said to be η-strongly monotone if there exists a positive constant η such that

$$\langle Bx - By, x - y\rangle \geq \eta\|x - y\|^2, \quad \forall x, y \in C. \tag{7}$$

(4) B is said to be nonexpansive if

$$\|Bx - By\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{8}$$

And $F(B)$ denotes the set of fixed points of the mapping B, that is, $F(B) = \{x \in C : Bx = x\}$.

(5) $B : C \to C$ is said to be k-strictly pseudocontractive mapping if there exists a constant $0 \leq k < 1$ such that

$$\|Bx - By\|^2 \leq \|x - y\|^2 + k\|(I - B)x - (I - B)y\|^2, \quad \forall x, y \in C. \tag{9}$$

(6) B is said to be α-inverse-strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Bx - By, x - y\rangle \geq \alpha\|Bx - By\|^2, \quad \forall x, y \in C. \tag{10}$$

(7) B is said to be relaxed (u, v)-cocoercive if there exist positive real numbers u, v such that

$$\langle Bx - By, x - y\rangle \geq (-u)\|Bx - By\|^2 + v\|x - y\|^2, \quad \forall x, y \in C. \tag{11}$$

(8) A linear bounded operator B is strong positive if there exists a constant $\overline{\gamma} > 0$ with the property

$$\langle Bx, x\rangle \geq \overline{\gamma}\|x\|^2, \quad \forall x \in C. \tag{12}$$

(9) A set-valued mapping $Q : H \to 2^H$ is called monotone if for all $x, y \in H$, $f \in Qx$ and $g \in Qy$ imply $\langle x - y, f - g\rangle \geq 0$.

(10) A monotone mapping $Q : H \to 2^H$ is called maximal if the graph $G(Q)$ of Q is not properly contained in the graph of any other monotone mapping. It is well known that a monotone mapping Q is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for every $(y, g) \in G(Q)$ implies $f \in Qx$.

Iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems. Convex minimization problems have a great impact and influence on the development of almost all branches of pure and applied sciences. A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\theta(x) = \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle, \quad \forall x \in F(S), \tag{13}$$

where A is a linear bounded operator, $F(S)$ is the fixed point set of a nonexpansive mapping S, and b is a given point in H [16].

For finding a common element of the set of fixed points of nonexpansive mappings and the set of the variational inequalities, in 2006, Marino and Xu [16] introduced the general iterative method and proved that for a given $x_0 \in H$, the sequence $\{x_n\}$ generated by the algorithm

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n B) T x_n, \tag{14}$$

where T is a self-nonexpansive mapping on H, f is an α-contraction of H into itself (i.e., $\|f(x) - f(y)\| \leq \alpha\|x - y\|, \forall x, y \in H$ and $\alpha \in (0, 1)$), $\{\alpha_n\} \subset (0, 1)$ satisfies certain conditions, and B is strongly positive bounded linear operator on H and converges strongly to fixed point x^* of T which is the unique solution to the following variational inequality:

$$\langle(\gamma f - B)x^*, x^* - x\rangle \leq 0, \quad \forall x \in F(T), \tag{15}$$

which is the optimality condition for the minimization problem

$$\min_{x\in F(S)\cap \mathrm{EP}} \frac{1}{2}\langle Bx, x\rangle - h(x), \tag{16}$$

where h is a potential function for rf (i.e., $h'(x) = rf(x)$ for $z \in H$).

Takahashi and Toyoda [32] introduced the following iterative scheme:

$$x_0 \in C, \qquad x_{n+1} = \gamma_n x_n + (1 - \gamma_n)\,\mathrm{SP}_C(x_n - \alpha_n B x_n), \tag{17}$$

where B is a ξ-inverse-strongly monotone mapping, $\{\gamma_n\}$ is a sequence in $(0,1)$, and $\{\alpha_n\}$ is a sequence in $(0,2\xi)$. They showed that if $F(S)\cap \mathrm{VI}(C,B)\neq\emptyset$, then the sequence $\{x_n\}$ generated by (17) converges weakly to some $z\in F(S)\cap \mathrm{VI}(C,B)$.

Yamada [33] introduced the following iterative scheme called the hybrid steepest descent method:

$$x_{n+1}=Sx_n+\alpha_n\mu BSx_n,\quad n\in N, \tag{18}$$

where $x_1=x\in H$, $\{\alpha_n\}\subset(0,1)$, and let $B:H\to H$ be a strongly monotone and Lipschitz continuous mapping and μ is a positive real number. He proved that the sequence $\{x_n\}$ generated by (18) converges strongly to the unique solution of $F(S)\cap\mathrm{VI}(C,B)$.

Let C be a nonempty closed convex subset of H. Given $r>0$ the operators $J_r^F:H\to C$ defined by

$$J_r^F x=\left\{z\in C:F(z,y)+\frac{1}{r}(y-z,z-x)\geq 0,\ \forall y\in C\right\} \tag{19}$$

are called the resolvent of F (see [19]). It is shown in [19] that under suitable hypotheses on F (to be stated precisely in Section 2), $J_r^F:H\to C$ is single-valued and firmly nonexpansive and satisfied $F(J_r^F)=\mathrm{EP}(F),\forall r>0$.

For finding a common element of $\mathrm{EP}(F)\cap F(S)$, S. Takahashi and W. Takahashi [23] introduced an iterative scheme by the viscosity approximation method for finding a common element of the set of solution (1) and the set of fixed points of a nonexpansive mapping in a Hilbert space. Let $S:C\to H$ be a nonexpansive mapping. Starting with arbitrary initial point $x_1\in H$, define sequences $\{x_n\}$ and $\{u_n\}$ recursively by

$$\begin{gathered}F(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\geq 0,\quad \forall y\in C,\\ x_{n+1}=\alpha_n f(x_n)+(1-\alpha_n)Su_n,\quad \forall n\in N.\end{gathered} \tag{20}$$

They proved that under certain appropriate conditions imposed on $\{\alpha_n\}$ and $\{r_n\}$, the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to $z\in F(S)\cap\mathrm{EP}(F)$, where $z=P_{F(S)\cap\mathrm{EP}(F)}f(z)$.

In 2012, Chamnarnpan and Kumam [34] introduced the following explicit viscosity scheme with respect to W-mappings for an infinite family of nonexpansive mappings

$$x_{n+1}=\varepsilon_n rf(x_n)+\beta_n x_n+((1-\beta_n)I-\varepsilon_n A)W_nJ_r^F x_n. \tag{21}$$

They prove that sequence $\{x_n\}$ and $J_{r_n}^F$ converge strongly to $z\in(\cap_{n=1}^{\infty}F(T_n))\cap\mathrm{EP}(F)$, where z is an equilibrium point for F and is the unique solution of the variational inequality

$$\langle(rf-A)z,x-z\rangle\leq 0,\quad \forall x\in\bigcap_{n=1}^{\infty}F(T_n)\cap\mathrm{EP}(F). \tag{22}$$

In 2012, Kangtunyakarn [35] modify the set of variational inequality to construct a new iterative scheme for finding a common element of the set of fixed point problems of infinite family of k_i pseudocontractive mappings and the set of equilibrium problem and two sets of variational inequality problems. Let

$$\begin{aligned}F:=&\left(\bigcap_{i=1}^{\infty}\{F(T_i)\}\right)\cap\left(\bigcap_{k=1}^{M}\mathrm{SEP}(F_k)\right)\\ &\cap\mathrm{VI}(C,A)\cap\mathrm{VI}(C,B).\end{aligned} \tag{23}$$

Starting with arbitrary initial point $x_1\in C$, define sequences $\{x_n\}$ and $\{u_n\}$ recursively by

$$\begin{gathered}F(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\geq 0,\quad\forall y\in C,\\ x_{n+1}=\alpha_n u+(1-\alpha_n)S_nP_C(I-\gamma(aA+(1-a)B)u_n),\\ \forall n\in N,\end{gathered} \tag{24}$$

where $\{S_n:C\to C\}$ is the sequence defined by (37), A,B is α and β-inverse-strongly monotone mapping, respectively, $a\in(0,1)$, $0<r<\min\{2\alpha,2\beta\}$ and $\{r_n\}\subset[a,b]\subset(0,\min\{2\alpha,2\beta\})$. Under certain appropriate conditions they proved that the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to $z\in F$, where $z=P_Fu$.

Let $A_i:C\to H$ be a mapping, for $i=1,2,\ldots,N$. By modification of (4), for $\delta_i\in(0,1)$, we have

$$\begin{aligned}\mathrm{VI}\left(C,\sum_{i=1}^{N}\delta_iA_i\right)=&\left\{x^*\in C:\left\langle y-x^*,\sum_{i=1}^{N}\delta_iA_ix^*\right\rangle\geq 0,\right.\\ &\left.\forall y\in C,\sum_{i=1}^{N}\delta_i=1\right\}.\end{aligned} \tag{25}$$

In this paper, motivated by the above results, we extend the nonexpansive mapping of hybrid steepest descent method (18) to nonexpansive semigroups and introduce a new iterative scheme for finding a common element of the set of solutions of a system of equilibrium problems, the set of fixed points of an infinite family of strictly pseudocontractive mappings, the set of solutions of fixed points for nonexpansive semigroups, and the set of solutions of variational inequality problems for relaxed cocoercive mapping in a real Hilbert space by the hybrid steepest descent method. The results shown in this paper improve and extend the recent ones announced by many others.

2. Preliminaries

Throughout this paper, we always assume that C is a nonempty closed convex subset of a Hilbert space H. We write $x_n\rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n\to x$ implies that $\{x_n\}$ converges strongly to x. We denote by N and R the sets of positive integers and real numbers, respectively. For any $x\in H$, there exists a unique nearest point in C, denoted by P_Cx, such that

$$\|x-P_Cx\|\leq\|x-y\|,\quad\forall y\in C. \tag{26}$$

Such a P_C is called the metric projection of H onto C. It is known that P_C is nonexpansive. Furthermore, for $x \in H$ and $u \in C$,

$$u = P_C x \Longleftrightarrow \langle x - u, u - y \rangle \geq 0, \quad \forall y \in C. \quad (27)$$

It is widely known that H satisfies Opial's condition [8], that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\lim_{n\to\infty} \inf \|x_n - x\| < \lim_{n\to\infty} \inf \|x_n - y\| \quad (28)$$

holds for every $y \in H$ with $y \neq x$.

In order to solve the equilibrium problem for a bifunction $F : C \times C \to R$, we assume that F satisfies the following conditions:

(A1) $F(x, x) = 0, \forall x \in C$,

(A2) F is monotone, that is, $F(x, y) + F(y, x) \leq 0, \forall x, y \in C$,

(A3) $\lim_{t\downarrow 0} F(tz + (1 - t)x, y) \leq F(x, y), \forall x, y, z \in C$,

(A4) For each $x \in C, y \mapsto F(x, y)$ is convex and lower semicontinuous.

Let us recall the following lemmas which will be useful for our paper.

Lemma 3 (see [19]). *Let F be a bifunction from $C \times C$ into R satisfying (A1), (A2), (A3), and (A4). Then, for any $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}(y - z, z - x) \geq 0, \quad \forall y \in C. \quad (29)$$

Furthermore, if $J_r^F x = \{z \in C : F(z, y) + (1/r)(y - z, z - x) \geq 0, \forall y \in C\}$, then the following hold:

(1) *J_r^F is single-valued,*

(2) *J_r^F is firmly nonexpansive, that is,*

$$\left\|J_r^F x - J_r^F y\right\|^2 \leq \left\langle J_r^F x - J_r^F y, x - y \right\rangle, \quad \forall x, y \in H, \quad (30)$$

(3) $F(J_r^F) = \mathrm{EP}(F)$,

(4) *$\mathrm{EP}(F)$ is closed and convex.*

Lemma 4 (see [12]). *Let C be a nonempty bounded closed and convex subset of a real Hilbert space H. Let $\Gamma = \{T(s) : s \in R^+\}$ from C be a nonexpansive semigroup on C, then for all $h > 0$,*

$$\lim_{t\to\infty} \sup_{x\in C} \left\| \frac{1}{t}\int_0^t T(s)x\,ds - T(h)\left(\frac{1}{t}\int_0^t T(s)x\,ds\right) \right\| = 0. \quad (31)$$

Lemma 5 (see [13]). *Let C be a nonempty bounded closed and convex subset of a real Hilbert space H, let $\{x_n\}$ be a sequence, and let $\Gamma = \{T(s) : s \in R^+\}$ from C be a nonexpansive semigroup on C, if the following conditions are satisfied:*

(i) $x_n \rightharpoonup z$,

(ii) $\limsup_{s\to\infty} \limsup_{n\to\infty} \|T(s)x_n - x_n\| = 0$,

then, $z \in F(\Gamma)$.

Lemma 6 (see [36]). *In a Hilbert space H, there holds the inequality*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle, \quad \forall x, y \in H. \quad (32)$$

Lemma 7 (see [16]). *Assume A be a strongly positive linear bounded operator on H with coefficient $\overline{\gamma} > 0$ and $0 \leq \rho \geq \|A\|^{-1}$, then $\|I - \rho A\| \leq 1 - \rho\overline{\gamma}$.*

Lemma 8 (see [37]). *Let B be a monotone mapping of C into H and let $N_C\omega_1$ be the normal cone to C at $\omega_1 \in C$, that is,*

$$N_C\omega_1 = \{\omega \in H : \langle \omega_1 - \omega_2, \omega \rangle \geq 0, \forall \omega_2 \in C\}, \quad (33)$$

and define a mapping Q on C by

$$Q\omega_1 = \begin{cases} B\omega_1 + N_C\omega_1, & \omega_1 \in C, \\ \emptyset, & \omega_1 \notin C. \end{cases} \quad (34)$$

Then Q is maximal monotone and $0 \in Q\omega_1$ if and only if, $\langle B\omega_1, \omega_1 - \omega_2 \rangle \geq 0$ for all $\omega_2 \in C$.

Lemma 9 (see [27]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and $\{\gamma_n\}$ be a sequence in $[0, 1]$ satisfying the following condition:*

$$0 < \lim_{n\to\infty} \inf \gamma_n \leq \lim_{n\to\infty} \sup \gamma_n < 1. \quad (35)$$

Suppose that $x_{n+1} = \gamma_n x_n + (1 - \gamma_n) y_n, n \geq 0$ and $\lim_{n\to\infty} \sup(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 10 (see [28]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - b_n) a_n + c_n, \quad n \geq 0, \quad (36)$$

where $\{b_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in R, such that

(i) $\sum_{i=1}^{\infty} b_i = \infty$,

(ii) $\lim_{n\to\infty} \sup(c_n/b_n) \leq 0$ *or* $\sum_{i=1}^{\infty} |c_n| < \infty$.

Then, $\lim_{n\to\infty} a_n = 0$.

Let C be a nonempty closed convex subset of a Hilbert space H. Let $\{T_i\}_{i=1}^{\infty}$ be mapping of C into self. For all $j = 1, 2, \ldots$, let $\rho_j = (\alpha_1^j, \alpha_2^j, \alpha_2^j) \in I \times I \times I$ where $I = [0, 1]$ and

$\alpha_1^j + \alpha_2^j + \alpha_2^j = 1$. For every $n \in N$, we define the mapping $S_n : C \to C$ as follows:

$$\begin{aligned} U_{n,n+1} &:= I, \\ U_{n,n} &:= \alpha_1^n T_n U_{n,n+1} + \alpha_2^n U_{n,n+1} + \alpha_3^n I, \\ &\vdots \\ U_{n,k} &:= \alpha_1^k T_k U_{n,k+1} + \alpha_2^k U_{n,k+1} + \alpha_3^k I, \\ &\vdots \\ U_{n,2} &:= \alpha_1^2 T_2 U_{n,3} + \alpha_2^2 U_{n,3} + \alpha_3^2 I, \\ S_n = U_{n,1} &:= \alpha_1^1 T_1 U_{n,2} + \alpha_2^1 U_{n,2} + \alpha_3^1 I. \end{aligned} \tag{37}$$

This mapping is called S-mapping generated by $T_1, \ldots, T_n$ and $\rho_1, \ldots, \rho_n$.

Lemma 11 (see [38]). *Let C be a nonempty closed convex subset of a Hilbert space H. Let $\{T_i\}_{i=1}^{\infty}$ be a k_i-strict pseudo-contractive mapping of C into self with $\kappa = \sup_i k_i$ and let $\rho_j = (\alpha_1^j, \alpha_2^j, \alpha_2^j) \in I \times I \times I$ where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_2^j = 1$, $\alpha_1^j + \alpha_2^j \leq b < 1$, and $\alpha_1^j, \alpha_2^j, \alpha_2^j \in (\kappa, 1)$ for all $j = 1, 2, \ldots$. For every $n \in N$, let S_n and S-mapping generated by $T_1, T_2, \ldots, T_n$ and $\rho_1, \rho_2, \ldots, \rho_n$ and $T_1, T_2, \ldots$, and $\rho_1, \rho_2, \ldots$, respectively. Then, for every $x \in C$ and $k \in N$, the limit $\lim_{n\to\infty} U_{n,k}x$ exists.*

In view of the previous lemma, we will define the mapping $S : C \to C$ as follows:

$$Sx := \lim_{n\to\infty} S_n x = \lim_{n\to\infty} U_{n,1} x, \quad x \in C. \tag{38}$$

Remark 12 (see [38]). For each $n \in N$, S_n is nonexpansive and $\lim_{n\to\infty} \sup_{x\in D} \|S_n x - Sx\| = 0$ for every bounded subset D of C.

Lemma 13 (see [38]). *Let C be a nonempty closed convex subset of a Hilbert space H. Let $\{T_i\}_{i=1}^{\infty}$ be a k_i-strict pseudo-contractive mapping of C into self such that $\cap_{i=1}^{\infty} F(T_i) \neq \emptyset$ with $\kappa = \sup_i k_i$ and let $\rho_j = (\alpha_1^j, \alpha_2^j, \alpha_2^j) \in I \times I \times I$ where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_2^j = 1$, $\alpha_1^j + \alpha_2^j \leq b < 1$, and $\alpha_1^j, \alpha_2^j, \alpha_2^j \in (\kappa, 1)$ for all $j = 1, 2, \ldots$. For every $n \in N$, let S_n and S-mapping generated by $T_1, T_2, \ldots, T_n$ and $\rho_1, \rho_2, \ldots, \rho_n$, respectively. Then, $F(S) = \cap_{i=1}^{\infty} F(T_i) \neq \emptyset$.*

3. Main Results

In this section, we will present our main results. To establish our results, we need the following technical lemmas.

Lemma 14. *Let C be a nonempty closed convex subset of a Hilbert space H and let $A_i : C \to H$ be ω_i-Lipschitz continuous and relaxed (u_i, v_i)-cocoercive mappings with $v_i - u_i\omega_i^2 > 0$, for $i = 1, 2, \ldots, N$. If $\cap_{i=1}^{N} \mathrm{VI}(C, A_i) \neq \emptyset$, then, for $\delta_i \in (0,1)$ and $\sum_{i=1}^{N} \delta_i = 1$,*

$$\bigcap_{i=1}^{N} \mathrm{VI}(C, A_i) = \mathrm{VI}\left(C, \sum_{i=1}^{N} \delta_i A_i\right). \tag{39}$$

Proof. The proof is by induction. This holds for $N = 2$. In fact, for $a \in (0,1)$, it is obvious that

$$\mathrm{VI}(C, A_1) \cap \mathrm{VI}(C, A_2) \subseteq \mathrm{VI}\left(C, \sum_{i=1}^{N} \delta_i A_i\right). \tag{40}$$

Next, we will show that $\mathrm{VI}(C, \sum_{i=1}^{N} \delta_i A_i) \subseteq \mathrm{VI}(C, A_1) \cap \mathrm{VI}(C, A_2)$.

Let

$$\begin{gathered} x_0 \in \mathrm{VI}\left(C, \sum_{i=1}^{N} \delta_i A_i\right), \\ x^* \in \mathrm{VI}(C, A_1) \cap \mathrm{VI}(C, A_2). \end{gathered} \tag{41}$$

It follows that

$$\langle y - x^*, A_1 x^* \rangle \geq 0, \quad \forall y \in C, \tag{42}$$

$$\langle y - x^*, A_2 x^* \rangle \geq 0, \quad \forall y \in C. \tag{43}$$

Then, for every $a \in (0,1)$, one has

$$\begin{gathered} \langle y - x^*, aA_1 x^* \rangle \geq 0, \quad \forall y \in C, \\ \langle y - x^*, (1-a) A_2 x^* \rangle \geq 0, \quad \forall y \in C. \end{gathered} \tag{44}$$

From $x_0 \in \mathrm{VI}(C, \sum_{i=1}^{N} \delta_i A_i)$ and (43), one has

$$\begin{aligned} \langle x^* - x_0, aA_1 x_0 \rangle &= \left\langle x^* - x_0, \left(\sum_{i=1}^{N} \delta_i A_i\right) x_0 \right\rangle \\ &\quad - \langle x^* - x_0, (1-a) A_2 x_0 \rangle \\ &\geq (1-a) \langle x_0 - x^*, A_2 x_0 \rangle \\ &= (1-a) \langle x_0 - x^*, A_2 x_0 - A_2 x^* \rangle \\ &\quad + (1-a) \langle x_0 - x^*, A_2 x^* \rangle \\ &\geq (1-a) \left(v_2 - u_2 \omega_2^2\right) \|x_0 - x^*\|^2 \\ &\geq 0, \end{aligned} \tag{45}$$

which means

$$\langle x^* - x_0, A_1 x_0 \rangle \geq 0. \tag{46}$$

On the other hand, from $x^* \in \mathrm{VI}(C, A_1)$, we have

$$\begin{aligned}\langle x^* - x_0, A_1x_0\rangle &= \langle x^* - x_0, A_1x_0 - A_1x^*\rangle \\ &\quad + \langle x^* - x_0, A_1x^*\rangle \\ &\leq \langle x^* - x_0, A_1x_0 - A_1x^*\rangle \\ &\leq u_1\|A_1x_0 - A_1x^*\|^2 - v_1\|x_0 - x^*\|^2 \\ &\leq u_1\omega_1^2\|x_0 - x^*\|^2 - v_1\|x_0 - x^*\|^2 \\ &= \left(u_1\omega_1^2 - v_1\right)\|x_0 - x^*\|^2 \\ &\leq 0.\end{aligned} \tag{47}$$

This together with (46) leads to

$$A_1x^* = A_1x_0. \tag{48}$$

Furthermore, for every $y \in C$, from (46) and (48), we obtain

$$\begin{aligned}\langle y - x_0, A_1x_0\rangle &= \langle y - x^*, A_1x_0\rangle + \langle x^* - x_0, A_1x_0\rangle \\ &\geq \langle y - x^*, A_1x_0\rangle \\ &= \langle y - x^*, A_1x^*\rangle \\ &\geq 0,\end{aligned} \tag{49}$$

which implies

$$x_0 \in \mathrm{VI}(C, A_1). \tag{50}$$

It follows from (45) and (42) that

$$\begin{aligned}\langle x^* - x_0, (1 - a)A_2x_0\rangle &\geq \langle x_0 - x^*, aA_1x_0\rangle \\ &= a\langle x_0 - x^*, A_1(x_0 - x^*)\rangle \\ &\quad + a\langle x_0 - x^*, A_1x^*\rangle \\ &\geq a\langle x_0 - x^*, A_1(x_0 - x^*)\rangle \\ &\geq a\left(v_1 - u_1\omega_1^2\right)\|x_0 - x^*\|^2 \\ &\geq 0.\end{aligned} \tag{51}$$

It yields that

$$\langle x^* - x_0, A_2x_0\rangle \geq 0. \tag{52}$$

From $x^* \in \mathrm{VI}(C, A_2)$ and (52), one has

$$\begin{aligned}0 &\leq \langle x^* - x_0, A_2x_0\rangle \\ &= \langle x^* - x_0, A_2x_0 - A_2x^*\rangle \\ &\quad + \langle x^* - x_0, A_2x^*\rangle \\ &\leq \langle x^* - x_0, A_2x_0 - A_2x^*\rangle \\ &\leq u_2\|A_2x_0 - A_2x^*\|^2 - v_2\|x_0 - x^*\|^2 \\ &\leq u_2\omega_2^2\|x_0 - x^*\|^2 - v_1\|x_0 - x^*\|^2 \\ &= \left(u_2\omega_2^2 - v_2\right)\|x_0 - x^*\|^2 \\ &\leq 0.\end{aligned} \tag{53}$$

That is,

$$A_2x^* = A_2x_0. \tag{54}$$

Therefore, for every $y \in C$, from (52) and (54), we obtain

$$\begin{aligned}\langle y - x_0, A_2x_0\rangle &= \langle y - x^*, A_2x_0\rangle + \langle x^* - x_0, A_2x_0\rangle \\ &\geq \langle y - x^*, A_2x_0\rangle \\ &= \langle y - x^*, A_2x^*\rangle \\ &\geq 0,\end{aligned} \tag{55}$$

which means

$$x_0 \in \mathrm{VI}(C, A_2). \tag{56}$$

And hence,

$$x_0 \in \mathrm{VI}(C, A_1) \cap \mathrm{VI}(C, A_2). \tag{57}$$

Thus, we have

$$\mathrm{VI}\left(C, \sum_{i=1}^{N}\delta_iA_i\right) \subseteq \mathrm{VI}(C, A_1) \cap \mathrm{VI}(C, A_2). \tag{58}$$

Thus,

$$\mathrm{VI}\left(C, \sum_{i=1}^{N}\delta_iA_i\right) = \mathrm{VI}(C, A_1) \cap \mathrm{VI}(C, A_2). \tag{59}$$

Assume now that $\cap_{i=1}^{k} \mathrm{VI}(C, A_i) = \mathrm{VI}(C, \sum_{i=1}^{k} \delta_i A_i)$ is true for some k, and we show that it continues to hold for $k+1$. For $\delta_i \in (0,1)$ and $\sum_{i=1}^{k+1} \delta_i = 1$, we have

$$\begin{aligned}
\mathrm{VI}\left(C, \sum_{i=1}^{k+1} \delta_i A_i\right)
&= \mathrm{VI}\left(C, \delta_1 A_1 + \sum_{i=2}^{k+1} \delta_i A_i\right) \\
&= \mathrm{VI}\left(C, \delta_1 A_1 + (1-\delta_1) \sum_{i=2}^{k+1} \frac{\delta_i}{1-\delta_1} A_i\right) \\
&= \mathrm{VI}(C, \delta_1 A_1) \cap \mathrm{VI}\left(C, (1-\delta_1) \sum_{i=2}^{k+1} \frac{\delta_i}{1-\delta_1} A_i\right) \\
&= \mathrm{VI}(C, A_1) \cap \mathrm{VI}\left(C, \sum_{i=2}^{k+1} \frac{\delta_i}{1-\delta_1} A_i\right) \\
&= \mathrm{VI}(C, A_1) \cap \left(\bigcap_{i=2}^{k+1} \mathrm{VI}(C, A_i)\right) \\
&= \bigcap_{i=1}^{k+1} \mathrm{VI}(C, A_i).
\end{aligned} \tag{60}$$

By induction, $\cap_{i=1}^{k} \mathrm{VI}(C, A_i) = \mathrm{VI}(C, \sum_{i=1}^{k} \delta_i A_i)$ holds for $k = 1, 2, \ldots, N$ and this completes the proof. □

Lemma 15. *Let C be a nonempty closed convex subset of a Hilbert space H, let $\Gamma = \{T(s) : s \in R^+\}$ from C be a nonexpansive semigroup on C, and let $A_i : C \to H$ be ω_i-Lipschitz continuous and relaxed (μ_i, ν_i)-cocoercive mappings with $\nu_i - \mu_i \omega_i^2 > 0$, for $i = 1, 2, \ldots, N$. Assume that $D = \sum_{i=1}^{N} \delta_i A_i$, for $\delta_i \in (0,1)$ and $\sum_{i=1}^{N} \delta_i = 1$. If $K_n(x) = (1/t_n) \int_0^{t_n} T(s) S_n x\, ds$, where $\{S_n : C \to C\}$ is the sequence defined by* (37) *with $0 \le \alpha_n \le (2 \sum_{i=1}^{N} \delta_i (\nu_i - \mu_i \omega_i^2)) / (\sum_{i=1}^{N} \delta_i \omega_i^2)$, then $K_n - \alpha_n D K_n$ is a nonexpansive mapping in H. Furthermore, $I - \alpha_n D$ is a nonexpansive mapping in H.*

Proof. Since $0 \le \alpha_n \le (2 \sum_{i=1}^{N} \delta_i (\nu_i - \mu_i \omega_i^2)) / (\sum_{i=1}^{N} \delta_i \omega_i^2)$, for every $x, y \in C$, we have

$$\begin{aligned}
&\|(K_n - \alpha_n D K_n) x - (K_n - \alpha_n D K_n) y\|^2 \\
&\quad = \|(K_n x - K_n y) - \alpha_n (D K_n x - D K_n y)\|^2 \\
&\quad = \|K_n x - K_n y\|^2 - 2\alpha_n \langle K_n x - K_n y, D K_n x - D K_n y \rangle \\
&\qquad + \alpha_n^2 \|D K_n x - D K_n y\|^2 \\
&\quad \le \|K_n x - K_n y\|^2 - 2\alpha_n \sum_{i=1}^{N} \delta_i (\nu_i - \mu_i \omega_i^2) \\
&\qquad \times \|K_n x - K_n y\|^2 + \alpha_n^2 \sum_{i=1}^{N} \delta_i \omega_i^2 \|K_n x - K_n y\|^2 \\
&\quad = \left(1 - 2\alpha_n \sum_{i=1}^{N} \delta_i (\nu_i - \mu_i \omega_i^2) + \alpha_n^2 \sum_{i=1}^{N} \delta_i \omega_i^2\right) \\
&\qquad \times \|K_n x - K_n y\|^2 \\
&\quad \le \|x - y\|^2.
\end{aligned} \tag{61}$$

Thus, we obtain that $K_n - \alpha_n D K_n$ is a nonexpansive mapping.

Similarly, we can obtain that $I - \alpha_n D$ is a nonexpansive mapping in H and this completes the proof. □

The following main results follow from Lemmas 14 and 15.

Theorem 16. *Let C be a nonempty closed convex subset of a real Hilbert space H, and let F_k, $k \in \{1, 2, \ldots, M\}$ be a bifunction from $C \times C \to R$ satisfying (A1)–(A4). Let $\Gamma = \{T(s) : 0 \le s < \infty\}$ be a nonexpansive semigroup on C and let t_n be a positive real divergent sequence. Let $\{T_i\}_{i=1}^{\infty}$ be k_i-strict pseudo-contractive mappings of C into self with $\kappa = \sup_i k_i$ and let $\rho_j = (\alpha_1^j, \alpha_2^j, \alpha_2^j) \in I \times I \times I$, where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_2^j = 1$, $\alpha_1^j + \alpha_2^j \le b < 1$, and $\alpha_1^j, \alpha_2^j, \alpha_2^j \in (\kappa, 1)$ for all $j = 1, 2, \ldots$. For every $n \in N$, let S_n and S-mapping generated by $T_1, T_2, \ldots, T_n$ and $\rho_1, \rho_2, \ldots, \rho_n$ with $T_i(F(\Gamma)) \subset F(\Gamma)$. Let $A_i : C \to H$ be ω_i-Lipschitz continuous and relaxed (μ_i, ν_i)-cocoercive mappings with $\nu_i - \mu_i \omega_i^2 > 0$, for $i = 1, 2, \ldots, N$, let f be a contraction of H into itself with $\eta \in (0,1)$, and let A be is a strongly positive linear bounded self-adjoint operator with the coefficients $\overline{\gamma} > 0$ and $0 < r < \overline{\gamma}/\eta$. Assume that*

$$\begin{aligned}
\Theta := F(\Gamma) \cap \left(\bigcap_{i=1}^{\infty} \{F(T_i)\}\right) \cap \left(\bigcap_{k=1}^{M} \mathrm{SEP}(F_k)\right) \\
\cap \left(\bigcap_{i=1}^{N} \mathrm{VI}(C, A_i)\right).
\end{aligned} \tag{62}$$

Let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and

$$\begin{aligned}
u_n &= J_{r_{M,n}}^{F_M} J_{r_{M-1,n}}^{F_{M-1}} \cdots J_{r_{2,n}}^{F_2} J_{r_{1,n}}^{F_1} x_n, \\
z_n &= P_C\left(\frac{1}{t_n} \int_0^{t_n} T(s) S_n u_n ds - \alpha_n \left(\sum_{i=1}^{N} \delta_i A_i\right) \right. \\
&\qquad \left. \times \frac{1}{t_n} \int_0^{t_n} T(s) S_n u_n ds\right), \\
y_n &= \varepsilon_n r f(x_n) + \beta_n x_n + ((1-\beta_n) I - \varepsilon_n A) z_n, \\
x_{n+1} &= \gamma_n x_n + (1-\gamma_n) y_n, \quad \forall n \in N,
\end{aligned} \tag{63}$$

where $\{S_n : C \to C\}$ is the sequence defined by (37) *and $\delta_i \in (0,1)$, $\sum_{i=1}^{N} \delta_i = 1$. If $\{\varepsilon_n\}, \{\beta_n\}$ are two sequences in $(0,1)$ and*

$\{\gamma_n\} \subset [c_1, c_2] \subset (0,1)$ *and* $\{r_{k,n}\}$, *for* $k \in \{1,2,\ldots,M\}$ *is a real sequence in* $(0,\infty)$ *satisfy the following conditions:*

(i) $\lim_{n\to\infty}\varepsilon_n = 0$, $\sum_{i=1}^{\infty}\varepsilon_n = \infty$,

(ii) $0 < \lim_{n\to\infty}\inf\beta_n \le \lim_{n\to\infty}\sup\beta_n < 1$ *and* $\lim_{n\to\infty}|\beta_{n+1}-\beta_n| = 0$,

(iii) $\lim_{n\to\infty}\inf r_{k,n} > 0$ *and* $\lim_{n\to\infty}|r_{k,n+1}-r_{k,n}| = 0$, *for* $k \in \{1,2,\ldots,M\}$,

(iv) $\{\alpha_n\} \subset [g_1, g_2] \subset (0, (2\sum_{i=1}^{N}\delta_i(\nu_i - \mu_i\omega_i^2))/(\sum_{i=1}^{N}\delta_i\omega_i^2))$ *and* $\lim_{n\to\infty}|\alpha_{n+1}-\alpha_n| = 0$,

(v) $\lim_{n\to\infty}|t_n - t_{n-1}|/t_n = 0$.

Then $\{x_n\}$ *converges strongly to* $z \in \Theta$, *where* z *is the unique solution of variational inequality*

$$\lim_{n\to\infty}\sup\langle (rf - A)z, p - z\rangle \le 0, \quad \forall p \in \Theta, \tag{64}$$

which is the optimality condition for the minimization problem

$$\min_{z\in\Theta}\frac{1}{2}\langle Az, z\rangle - h(z), \tag{65}$$

where h *is a potential function for* rf *(i.e.,* $h'(z) = rf(z)$ *for* $z \in H$*).*

Proof. From the restrictions on control sequences, we may assume, without loss of generality, that $\varepsilon_n \le (1-\beta_n)\|A\|^{-1}$ for all $n \ge 1$. From Lemma 7, we know that if $0 \le \rho \le \|A\|^{-1}$, then $\|I - \rho A\| \le 1 - \rho\overline{\gamma}$. We will assume that $\|I - A\| \le 1 - \overline{\gamma}$. Since A is a strongly positive linear bounded self-adjoint operator on H, we have

$$\|A\| = \sup\{|\langle Ax, x\rangle| : x \in H, \|x\| = 1\}. \tag{66}$$

Note that

$$\begin{aligned}\langle ((1-\beta_n)I - \varepsilon_n A)x, x\rangle &= 1 - \beta_n - \varepsilon_n\langle Ax, x\rangle \\ &\ge 1 - \beta_n - \varepsilon_n\|A\| \\ &\ge 0.\end{aligned} \tag{67}$$

That is, $(1-\beta_n)I - \varepsilon_n A$ is positive. Furthermore,

$$\begin{aligned}&\|(1-\beta_n)I - \varepsilon_n A\| \\ &\quad = \sup\{|\langle ((1-\beta_n)I - \varepsilon_n A)x, x\rangle| : x \in H, \|x\| = 1\} \\ &\quad = \sup\{1 - \beta_n - \varepsilon_n\langle Ax, x\rangle : x \in H, \|x\| = 1\} \\ &\quad \le 1 - \beta_n - \varepsilon_n\overline{\gamma}.\end{aligned} \tag{68}$$

Next, We divide the proof of Theorem into five steps.

Step 1. We show that $\{x_n\}$ is bounded.

Take $p \in \Theta$. Let $\mathfrak{T}_n^k = J_{r_{k,n}}^{F_k} J_{r_{k-1,n}}^{F_{k-1}} \cdots J_{r_{2,n}}^{F_2} J_{r_{1,n}}^{F_1}$, for $k \in \{1,2,\ldots M\}$ and $\mathfrak{T}_n^0 = I$, for any $n \in N$. Since $J_{r_{k,n}}^{F_k}$ is nonexpansive for each $k = 1,2,\ldots M$ and $u_n = \mathfrak{T}_n^M x_n$, we have

$$\|u_n - p\| = \left\|\mathfrak{T}_n^M x_n - \mathfrak{T}_n^M p\right\| \le \|x_n - p\|. \tag{69}$$

From Lemma 15 and (69), one has

$$\begin{aligned}&\|z_n - p\| \\ &= \left\| P_C\left(\frac{1}{t_n}\int_0^{t_n} T(s) S_n u_n ds - \alpha_n\left(\sum_{i=1}^{N}\delta_i A_i\right) \times \frac{1}{t_n}\int_0^{t_n} T(s) S_n u_n ds\right) - p\right\| \\ &= \left\| P_C\left(\frac{1}{t_n}\int_0^{t_n} T(s) S_n u_n ds - \alpha_n\left(\sum_{i=1}^{N}\delta_i A_i\right) \times \frac{1}{t_n}\int_0^{t_n} T(s) S_n u_n ds\right)\right. \\ &\qquad \left. - P_C\left(\frac{1}{t_n}\int_0^{t_n} T(s) S_n p\, ds - \alpha_n\left(\sum_{i=1}^{N}\delta_i A_i\right) \times \frac{1}{t_n}\int_0^{t_n} T(s) S_n p\, ds\right)\right\| \\ &\le \left\|\left(\frac{1}{t_n}\int_0^{t_n} T(s) S_n u_n ds - \alpha_n\left(\sum_{i=1}^{N}\delta_i A_i\right) \times \frac{1}{t_n}\int_0^{t_n} T(s) S_n u_n ds\right)\right. \\ &\qquad \left. - \left(\frac{1}{t_n}\int_0^{t_n} T(s) S_n p\, ds - \alpha_n\left(\sum_{i=1}^{N}\delta_i A_i\right) \times \frac{1}{t_n}\int_0^{t_n} T(s) S_n p\, ds\right)\right\| \\ &\le \|u_n - p\| \\ &\le \|x_n - p\|.\end{aligned} \tag{70}$$

It follows that

$$\begin{aligned}&\|y_n - p\| \\ &\quad = \|\varepsilon_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \varepsilon_n A)z_n - p\| \\ &\quad = \|\varepsilon_n(\gamma f(x_n) - Ap) + \beta_n(x_n - p) + ((1-\beta_n)I - \varepsilon_n A)(z_n - p)\| \\ &\quad \le \varepsilon_n\|\gamma(f(x_n) - f(p)) + \gamma f(p) - Ap\| + \beta_n\|x_n - p\| + (1 - \beta_n - \varepsilon_n\overline{\gamma})\|z_n - p\| \\ &\quad \le \varepsilon_n\gamma\eta\|x_n - p\| + \varepsilon_n\|\gamma f(p) - Ap\| + (1 - \varepsilon_n\overline{\gamma})\|z_n - p\|\end{aligned}$$

$$\leq (1-\varepsilon_n(\overline{\gamma}-\gamma\eta))\|x_n-p\| + \varepsilon_n\|\gamma f(p)-Ap\|. \tag{71}$$

Furthermore,

$$\begin{aligned}\|x_{n+1}-p\| &= \|\gamma_n x_n+(1-\gamma_n)y_n-p\| \\ &\leq \gamma_n\|x_n-p\|+(1-\gamma_n)\|y_n-p\| \\ &\leq (1-\varepsilon_n(1-\gamma_n)(\overline{\gamma}-\gamma\eta))\|x_n-p\| \\ &\quad +(1-\gamma_n)\varepsilon_n\|\gamma f(p)-Ap\| \\ &\leq \max\left\{\|x_n-p\|,\frac{\|\gamma f(p)-Ap\|}{\overline{\gamma}-\gamma\eta}\right\}.\end{aligned} \tag{72}$$

By induction, we have

$$\|x_n-p\|\leq \max\left\{\|x_1-p\|,\frac{\|\gamma f(p)-Ap\|}{\overline{\gamma}-\gamma\eta}\right\},\quad n\geq 1. \tag{73}$$

Hence, $\{x_n\}$ is bounded and we also obtain that $\{u_n\},\{z_n\}$, $\{y_n\},\{(\sum_{i=1}^N\delta_iA_i)(1/t_n)\int_0^{t_n}T(s)S_nu_nds\}$ and $\{f(x_n)\}$ are all bounded.

Step 2. We claim that $\lim_{n\to\infty}\|x_n-x_{n+1}\|=0$.

From the definition of z_n and Lemma 15, for $p\in\Theta$, we have

$$\begin{aligned}
&\|z_{n+1}-z_n\| \\
&= \Bigg\|P_C\left(\frac{1}{t_{n+1}}\int_0^{t_{n+1}}T(s)S_{n+1}u_{n+1}ds-\alpha_{n+1}\left(\sum_{i=1}^N\delta_iA_i\right)\times\frac{1}{t_{n+1}}\int_0^{t_{n+1}}T(s)S_{n+1}u_{n+1}ds\right) \\
&\quad -P_C\left(\frac{1}{t_n}\int_0^{t_n}T(s)S_nu_nds-\alpha_n\left(\sum_{i=1}^N\delta_iA_i\right)\times\frac{1}{t_n}\int_0^{t_n}T(s)S_nu_nds\right)\Bigg\| \\
&\leq \Bigg\|\left(\frac{1}{t_{n+1}}\int_0^{t_{n+1}}T(s)S_{n+1}u_{n+1}ds-\alpha_{n+1}\left(\sum_{i=1}^N\delta_iA_i\right)\times\frac{1}{t_{n+1}}\int_0^{t_{n+1}}T(s)S_{n+1}u_{n+1}ds\right) \\
&\quad -\left(\frac{1}{t_n}\int_0^{t_n}T(s)S_nu_nds-\alpha_n\left(\sum_{i=1}^N\delta_iA_i\right)\times\frac{1}{t_n}\int_0^{t_n}T(s)S_nu_nds\right)\Bigg\| \\
&\leq \Bigg\|\left(I-\alpha_{n+1}\left(\sum_{i=1}^N\delta_iA_i\right)\right)\times\left(\frac{1}{t_{n+1}}\int_0^{t_{n+1}}T(s)S_{n+1}u_{n+1}ds-\frac{1}{t_n}\int_0^{t_n}T(s)S_nu_nds\right)\Bigg\| \\
&\quad +\left\|(\alpha_n-\alpha_{n+1})\left(\sum_{i=1}^N\delta_iA_i\right)\frac{1}{t_n}\int_0^{t_n}T(s)S_nu_nds\right\| \\
&\leq \left\|\frac{1}{t_{n+1}}\int_0^{t_{n+1}}T(s)S_{n+1}u_{n+1}ds-\frac{1}{t_n}\int_0^{t_n}T(s)S_nu_nds\right\| \\
&\quad +|\alpha_n-\alpha_{n+1}|\left\|\left(\sum_{i=1}^N\delta_iA_i\right)\frac{1}{t_n}\int_0^{t_n}T(s)S_nu_nds\right\| \\
&\leq \left\|\frac{1}{t_{n+1}}\int_0^{t_{n+1}}T(s)(S_{n+1}u_{n+1}-S_nu_n)ds\right\| \\
&\quad +\left\|\left(\frac{1}{t_{n+1}}-\frac{1}{t_n}\right)\int_0^{t_n}T(s)S_nu_nds+\frac{1}{t_{n+1}}\int_{t_n}^{t_{n+1}}T(s)S_nu_nds\right\| \\
&\quad +|\alpha_n-\alpha_{n+1}|\left\|\left(\sum_{i=1}^N\delta_iA_i\right)\frac{1}{t_n}\int_0^{t_n}T(s)S_nu_nds\right\| \\
&\leq \left\|\frac{1}{t_{n+1}}\int_0^{t_{n+1}}T(s)(S_{n+1}u_{n+1}-S_nu_n)ds\right\| \\
&\quad +\left\|\left(\frac{1}{t_{n+1}}-\frac{1}{t_n}\right)\int_0^{t_n}T(s)(S_nu_n-S_np)ds+\frac{1}{t_{n+1}}\int_{t_n}^{t_{n+1}}T(s)(S_nu_n-S_np)ds\right\| \\
&\quad +|\alpha_n-\alpha_{n+1}|\left\|\left(\sum_{i=1}^N\delta_iA_i\right)\frac{1}{t_n}\int_0^{t_n}T(s)S_nu_nds\right\| \\
&\leq \|S_{n+1}u_{n+1}-S_nu_n\|+\frac{2|t_{n+1}-t_n|}{t_{n+1}}\|u_n-p\| \\
&\quad +|\alpha_n-\alpha_{n+1}|\left\|\left(\sum_{i=1}^N\delta_iA_i\right)\frac{1}{t_n}\int_0^{t_n}T(s)S_nu_nds\right\| \\
&\leq \|u_{n+1}-u_n\|+\|S_{n+1}u_n-S_nu_n\|+\frac{2|t_{n+1}-t_n|}{t_{n+1}}\|u_n-p\| \\
&\quad +|\alpha_n-\alpha_{n+1}|\left\|\left(\sum_{i=1}^N\delta_iA_i\right)\frac{1}{t_n}\int_0^{t_n}T(s)S_nu_nds\right\|.
\end{aligned} \tag{74}$$

First, we will show that if $\{x_n\}$ is bounded, then

$$\lim_{n\to\infty}\left\|\mathfrak{T}_n^kx_n-\mathfrak{T}_{n+1}^kx_n\right\|=0 \tag{75}$$

for $k\in\{1,2,\ldots,M\}$.

From Step 2 of the proof in [4], we have for $k \in \{1, 2, \ldots, M\}$

$$\lim_{n\to\infty} \left\| J^{F_k}_{r_{k,n+1}} x_n - J^{F_k}_{r_{k,n}} x_n \right\| = 0. \tag{76}$$

For $k \in \{1, 2, \ldots, M\}$, notice that

$$\mathfrak{F}^k_n = J^{F_k}_{r_{k,n}} J^{F_{k-1}}_{r_{k-1,n}} \cdots J^{F_2}_{r_{2,n}} J^{F_1}_{r_{1,n}} = J^{F_k}_{r_{k,n}} \mathfrak{F}^{k-1}_n. \tag{77}$$

It follows that

$$\begin{aligned}
&\left\| \mathfrak{F}^k_n x_n - \mathfrak{F}^k_{n+1} x_n \right\| \\
&\quad = \left\| J^{F_k}_{r_{k,n}} \mathfrak{F}^{k-1}_n x_n - J^{F_k}_{r_{k,n+1}} \mathfrak{F}^{k-1}_{n+1} x_n \right\| \\
&\quad \le \left\| J^{F_k}_{r_{k,n}} \mathfrak{F}^{k-1}_n x_n - J^{F_k}_{r_{k,n+1}} \mathfrak{F}^{k-1}_n x_n \right\| \\
&\qquad + \left\| J^{F_k}_{r_{k,n+1}} \mathfrak{F}^{k-1}_n x_n - J^{F_k}_{r_{k,n+1}} \mathfrak{F}^{k-1}_{n+1} x_n \right\| \\
&\quad \le \left\| J^{F_k}_{r_{k,n}} \mathfrak{F}^{k-1}_n x_n - J^{F_k}_{r_{k,n+1}} \mathfrak{F}^{k-1}_n x_n \right\| \\
&\qquad + \left\| J^{F_{k-1}}_{r_{k-1,n}} \mathfrak{F}^{k-2}_n x_n - J^{F_{k-1}}_{r_{k-1,n+1}} \mathfrak{F}^{k-2}_n x_n \right\| \\
&\qquad + \left\| \mathfrak{F}^{k-2}_n x_n - \mathfrak{F}^{k-2}_{n+1} x_n \right\| \\
&\quad \le \left\| J^{F_k}_{r_{k,n}} \mathfrak{F}^{k-1}_n x_n - J^{F_k}_{r_{k,n+1}} \mathfrak{F}^{k-1}_n x_n \right\| \\
&\qquad + \left\| J^{F_{k-1}}_{r_{k-1,n}} \mathfrak{F}^{k-2}_n x_n - J^{F_{k-1}}_{r_{k-1,n+1}} \mathfrak{F}^{k-2}_n x_n \right\| \\
&\qquad + \cdots + \left\| J^{F_2}_{r_{2,n}} \mathfrak{F}^1_n x_n - J^{F_2}_{r_{2,n+1}} \mathfrak{F}^1_n x_n \right\| \\
&\qquad + \left\| J^{F_1}_{r_{1,n}} x_n - J^{F_1}_{r_{1,n+1}} x_n \right\|.
\end{aligned} \tag{78}$$

Therefore, from (76), we conclude (75).

Second, we estimate $\|u_{n+1} - u_n\|$. From $u_{n+1} = \mathfrak{F}^M_{n+1} x_{n+1}$ and $u_n = \mathfrak{F}^M_n x_n = J^{F_M}_{r_{M,n}} \mathfrak{F}^{M-1}_n x_n$, we obtain

$$\begin{aligned}
&F_M(u_{n+1}, y) + \frac{1}{r_{M,n+1}} \left\langle y - u_{n+1}, u_{n+1} - \mathfrak{F}^{M-1}_{n+1} x_{n+1} \right\rangle \\
&\quad \ge 0, \quad \forall y \in C,
\end{aligned} \tag{79}$$

$$F_M(u_n, y) + \frac{1}{r_{M,n}} \left\langle y - u_n, u_n - \mathfrak{F}^{M-1}_n x_n \right\rangle \ge 0, \quad \forall y \in C. \tag{80}$$

Taking $y = u_n$ in (79) and $y = u_{n+1}$ in (80), we have

$$\begin{aligned}
&F_M(u_{n+1}, u_n) + \frac{1}{r_{M,n+1}} \left\langle u_n - u_{n+1}, u_{n+1} - \mathfrak{F}^{M-1}_{n+1} x_{n+1} \right\rangle \ge 0, \\
&F_M(u_n, u_{n+1}) + \frac{1}{r_{M,n}} \left\langle u_{n+1} - u_n, u_n - \mathfrak{F}^{M-1}_n x_n \right\rangle \ge 0.
\end{aligned} \tag{81}$$

So, from (A2), one has

$$\left\langle u_{n+1} - u_n, \frac{u_n - \mathfrak{F}^{M-1}_n x_n}{r_{M,n}} - \frac{u_{n+1} - \mathfrak{F}^{M-1}_{n+1} x_{n+1}}{r_{M,n+1}} \right\rangle \ge 0. \tag{82}$$

Furthermore,

$$\begin{aligned}
&\Big\langle u_{n+1} - u_n, u_n - \mathfrak{F}^{M-1}_n x_n - \left(u_{n+1} - \mathfrak{F}^{M-1}_{n+1} x_{n+1}\right) \\
&\quad + \left(1 - \frac{r_{M,n}}{r_{M,n+1}}\right)\left(u_{n+1} - \mathfrak{F}^{M-1}_{n+1} x_{n+1}\right) \Big\rangle \ge 0.
\end{aligned} \tag{83}$$

Since $\lim_{n\to\infty} r_{k,n} > 0$, we assume that there exists a real number such that $r_{k,n} > a > 0$ for all $n \in N$. Thus, we obtain

$$\begin{aligned}
&\|u_{n+1} - u_n\| \\
&\quad \le \left\| \mathfrak{F}^{M-1}_n x_n - \mathfrak{F}^{M-1}_{n+1} x_{n+1} \right\| \\
&\qquad + \left| 1 - \frac{r_{M,n}}{r_{M,n+1}} \right| \left\| u_{n+1} - \mathfrak{F}^{M-1}_{n+1} x_{n+1} \right\| \\
&\quad \le \left\| \mathfrak{F}^{M-1}_n x_n - \mathfrak{F}^{M-1}_{n+1} x_n \right\| + \left\| \mathfrak{F}^{M-1}_{n+1} x_n - \mathfrak{F}^{M-1}_{n+1} x_{n+1} \right\| \\
&\qquad + \frac{1}{a} \left| r_{M,n+1} - r_{M,n} \right| \left\| u_{n+1} - \mathfrak{F}^{M-1}_{n+1} x_{n+1} \right\| \\
&\quad \le \left\| \mathfrak{F}^{M-1}_n x_n - \mathfrak{F}^{M-1}_{n+1} x_n \right\| + \left\| x_n - x_{n+1} \right\| \\
&\qquad + \frac{1}{a} \left| r_{M,n+1} - r_{M,n} \right| \left\| u_{n+1} - \mathfrak{F}^{M-1}_{n+1} x_{n+1} \right\|.
\end{aligned} \tag{84}$$

Third, we estimate $\|S_{n+1} u_n - S_n u_n\|$. It follows from (37) that

$$\begin{aligned}
&\left\| S_{n+1} u_n - S_n u_n \right\|^2 \\
&\quad = \left\| U_{n+1,1} u_n - U_{n,1} u_n \right\|^2 \\
&\quad = \Big\| \alpha^1_1 T_1 U_{n+1,2} u_n + \alpha^1_2 U_{n+1,2} u_n + \alpha^1_3 u_n \\
&\qquad\quad - \left(\alpha^1_1 T_1 U_{n,2} u_n + \alpha^1_2 U_{n,2} u_n + \alpha^1_3 u_n \right) \Big\|^2 \\
&\quad = \Big\| \alpha^1_1 \left(T_1 U_{n+1,2} - T_1 U_{n,2} \right) u_n \\
&\qquad\quad + \alpha^1_2 \left(U_{n+1,2} - U_{n,2} \right) u_n \Big\|^2 \\
&\quad \le \alpha^1_1 \left\| \left(T_1 U_{n+1,2} - T_1 U_{n,2} \right) u_n \right\|^2 \\
&\qquad + \alpha^1_2 \left\| \left(U_{n+1,2} - U_{n,2} \right) u_n \right\|^2 \\
&\qquad - \alpha^1_1 \alpha^1_2 \Big\| \left(T_1 U_{n+1,2} - T_1 U_{n,2} \right) u_n \\
&\qquad\qquad - \left(U_{n+1,2} - U_{n,2} \right) u_n \Big\|^2 \\
&\quad \le \alpha^1_1 \Big(\left\| U_{n+1,2} u_n - U_{n,2} u_n \right\|^2 \\
&\qquad\qquad + \kappa \left\| \left(I - T_1\right) U_{n+1,2} u_n - \left(I - T_1\right) U_{n,2} u_n \right\|^2 \Big) \\
&\qquad + \alpha^2_1 \left\| U_{n+1,2} u_n - U_{n,2} u_n \right\|^2 \\
&\qquad - \alpha^1_1 \alpha^1_2 \left\| \left(I - T_1\right) U_{n+1,2} u_n - \left(I - T_1\right) U_{n,2} u_n \right\|^2
\end{aligned}$$

$$\le \left(1-\alpha_1^3\right)\left\|U_{n+1,2}u_n - U_{n,2}u_n\right\|^2$$
$$\vdots$$
$$\le \prod_{i=1}^{n}\left(1-\alpha_3^i\right)\left\|U_{n+1,n+1}u_n - U_{n,n+1}u_n\right\|^2, \tag{85}$$

which means that

$$\left\|S_{n+1}u_n - S_nu_n\right\| \le L_1\prod_{i=1}^{n}\left(1-\alpha_3^i\right), \tag{86}$$

where $L_1 \ge 0$ is a constant such that $\|U_{n+1,n+1}u_n - U_{n,n+1}u_n\| \le L_1$, for all $n \in N$.

Next, we estimate $\|y_{n+1}-y_n\|$. Substituting (84) and (86) into (74), one has

$$\begin{aligned}\|z_{n+1}-z_n\| &\le \|x_n - x_{n+1}\| + \left\|\mathfrak{I}_n^{M-1}x_n - \mathfrak{I}_{n+1}^{M-1}x_n\right\| \\ &\quad + \frac{1}{a}\left|r_{M,n+1}-r_{M,n}\right|\left\|u_{n+1}-\mathfrak{I}_{n+1}^{M-1}x_{n+1}\right\| \\ &\quad + L_1\prod_{i=1}^{n}\left(1-\alpha_3^i\right) \\ &\quad + \left|\alpha_n-\alpha_{n+1}\right|\left\|\left(\sum_{i=1}^{N}\delta_iA_i\right)\frac{1}{t_n}\int_0^{t_n}T(s)S_nu_nds\right\| \\ &\quad + \frac{2\left|t_{n+1}-t_n\right|}{t_{n+1}}\left\|u_n-p\right\|.\end{aligned} \tag{87}$$

From (61), we have

$$\begin{aligned}\|y_{n+1}-y_n\| &= \|\varepsilon_{n+1}\gamma\left(f(x_{n+1})-f(x_n)\right) \\ &\qquad + \left(\varepsilon_{n+1}-\varepsilon_n\right)\left(\gamma f(x_n)-Az_n\right) \\ &\qquad + \beta_{n+1}\left(x_{n+1}-x_n\right)+\left(\beta_{n+1}-\beta_n\right)\left(x_n-z_n\right) \\ &\qquad + \left(\left(1-\beta_{n+1}\right)I-\varepsilon_{n+1}A\right)\left(z_{n+1}-z_n\right)\| \\ &\le \varepsilon_{n+1}\gamma\eta\left\|x_{n+1}-x_n\right\| + \left|\varepsilon_{n+1}-\varepsilon_n\right| \\ &\quad \times\left\|\gamma f(x_n)-Az_n\right\| \\ &\quad + \beta_{n+1}\left\|x_{n+1}-x_n\right\| + \left|\beta_{n+1}-\beta_n\right|\left\|x_n-z_n\right\| \\ &\quad + \left(1-\beta_{n+1}-\varepsilon_{n+1}\overline{\gamma}\right)\left\|z_{n+1}-z_n\right\|.\end{aligned} \tag{88}$$

Substitution (87) into (88) yields that

$$\begin{aligned}\|y_{n+1}-y_n\| &\le \left(1-\varepsilon_{n+1}\left(\overline{\gamma}-\gamma\eta\right)\right)\left\|x_{n+1}-x_n\right\| \\ &\quad + L_2\Bigg(\left|\varepsilon_{n+1}-\varepsilon_n\right| + \left|\beta_{n+1}-\beta_n\right| + \frac{2\left|t_{n+1}-t_n\right|}{t_{n+1}} \\ &\qquad + \left|r_{M,n+1}-r_{M,n}\right| + \left|\alpha_n-\alpha_{n+1}\right|\Bigg) \\ &\quad + \left\|\mathfrak{I}_n^{M-1}x_n-\mathfrak{I}_{n+1}^{M-1}x_n\right\| + L_1\prod_{i=1}^{n}\left(1-\alpha_3^i\right),\end{aligned} \tag{89}$$

where L_2 is an appropriate constant such that

$$\begin{aligned}L_2 = \max\Bigg\{&\sup_{n\ge1}\left\{\left\|\left(\sum_{i=1}^{N}\delta_iA_i\right)\frac{1}{t_n}\int_0^{t_n}T(s)S_nu_nds\right\|\right\}, \\ &\sup_{n\ge1}\left\{\frac{1}{a}\left\|u_{n+1}-\mathfrak{I}_{n+1}^{M-1}x_{n+1}\right\|\right\}, \\ &\sup_{n\ge1}\left\{\left\|\gamma f(x_n)-Az_n\right\|\right\}, \sup_{n\ge1}\left\{\left\|x_n-z_n\right\|\right\}, \\ &\sup_{n\ge1}\left\{\left\|u_n-p\right\|\right\}\Bigg\}.\end{aligned} \tag{90}$$

It follows from (89) that

$$\begin{aligned}\|y_{n+1}-y_n\| - \|x_{n+1}-x_n\| &\le L_2\Bigg(\left|\varepsilon_{n+1}-\varepsilon_n\right| + \left|\beta_{n+1}-\beta_n\right| + \frac{2\left|t_{n+1}-t_n\right|}{t_{n+1}} \\ &\qquad + \left|r_{M,n+1}-r_{M,n}\right| + \left|\alpha_n-\alpha_{n+1}\right|\Bigg) \\ &\quad + \left\|\mathfrak{I}_n^{M-1}x_n-\mathfrak{I}_{n+1}^{M-1}x_n\right\| + L_1\prod_{i=1}^{n}\left(1-\alpha_3^i\right).\end{aligned} \tag{91}$$

Consequently, from (75) and the conditions in Theorem 16, we obtain

$$\lim_{n\to\infty}\sup\left(\|y_{n+1}-y_n\| - \|x_{n+1}-x_n\|\right) \le 0. \tag{92}$$

Hence, by Lemma 9, one has

$$\lim_{n\to\infty}\|y_n-x_n\| = 0. \tag{93}$$

Since $x_{n+1} = \gamma_nx_n + (1-\gamma_n)y_n$, this shows that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim_{n\to\infty}\left(1-\gamma_n\right)\|y_n-x_n\| = 0. \tag{94}$$

Step 3. We claim that $\lim_{n\to\infty}\|(1/t_n)\int_0^{t_n}T(s)S_nu_nds-u_n\| = 0$.

Observing $y_n = \varepsilon_n\gamma f(x_n) + \beta_nx_n + ((1-\beta_n)I - \varepsilon_nA)z_n$, we obtain

$$\begin{aligned}\|x_n-z_n\| &\le \|x_n-y_n\| + \|y_n-z_n\| \\ &\le \|x_n-y_n\| + \varepsilon_n\left\|\gamma f(x_n)-Az_n\right\| \\ &\quad + \beta_n\left\|x_n-z_n\right\|,\end{aligned} \tag{95}$$

which means that

$$\|x_n - z_n\| \le \frac{1}{1-\beta_n}\|x_n - y_n\| + \frac{\varepsilon_n}{1-\beta_n}\|\gamma f(x_n) - Az_n\|. \tag{96}$$

This together with the conditions (i) and (ii) imply that

$$\lim_{n\to\infty}\|x_n - z_n\| = 0. \tag{97}$$

From (93) and (97), one has

$$\lim_{n\to\infty}\|y_n - z_n\| \le \lim_{n\to\infty}(\|y_n - x_n\| + \|x_n - z_n\|) = 0. \tag{98}$$

For $p \in \Theta$, we see that

$$\begin{aligned}
&\|z_n - p\|^2 \\
&= \Bigg\|P_C\Bigg(\frac{1}{t_n}\int_0^{t_n} T(s)S_n u_n ds - \alpha_n\left(\sum_{i=1}^N \delta_i A_i\right) \times \frac{1}{t_n}\int_0^{t_n} T(s)S_n u_n ds\Bigg) - p\Bigg\|^2 \\
&= \Bigg\|P_C\Bigg(\frac{1}{t_n}\int_0^{t_n} T(s)S_n u_n ds - \alpha_n\left(\sum_{i=1}^N \delta_i A_i\right) \times \frac{1}{t_n}\int_0^{t_n} T(s)S_n u_n ds\Bigg) \\
&\quad - P_C\Bigg(\frac{1}{t_n}\int_0^{t_n} T(s)S_n p\, ds - \alpha_n\left(\sum_{i=1}^N \delta_i A_i\right) \times \frac{1}{t_n}\int_0^{t_n} T(s)S_n p\, ds\Bigg)\Bigg\|^2 \\
&\le \Bigg\|\left(\frac{1}{t_n}\int_0^{t_n} T(s)S_n u_n ds - \frac{1}{t_n}\int_0^{t_n} T(s)S_n p\, ds\right) - \alpha_n\left(\sum_{i=1}^N \delta_i A_i\right) \\
&\quad \times \left(\frac{1}{t_n}\int_0^{t_n} T(s)S_n u_n ds - \frac{1}{t_n}\int_0^{t_n} T(s)S_n p\, ds\right)\Bigg\|^2 \\
&= \left\|\left(\frac{1}{t_n}\int_0^{t_n} T(s)S_n u_n ds - \frac{1}{t_n}\int_0^{t_n} T(s)S_n p\, ds\right)\right\|^2 \\
&\quad - 2\alpha_n \Bigg\langle\left(\frac{1}{t_n}\int_0^{t_n} T(s)S_n u_n ds - \frac{1}{t_n}\int_0^{t_n} T(s)S_n p\, ds\right), \\
&\quad \left(\sum_{i=1}^N \delta_i A_i\right)\left(\frac{1}{t_n}\int_0^{t_n} T(s)S_n u_n ds - \frac{1}{t_n}\int_0^{t_n} T(s)S_n p\, ds\right)\Bigg\rangle \\
&\quad + \alpha_n^2\Bigg\|\left(\sum_{i=1}^N \delta_i A_i\right) \times \left(\left(\frac{1}{t_n}\int_0^{t_n} T(s)S_n u_n ds - \frac{1}{t_n}\int_0^{t_n} T(s)S_n p\, ds\right)\right)\Bigg\|^2 \\
&\le \|x_n - p\|^2 - \alpha_n\sum_{i=1}^N \delta_i\left(\frac{2\nu_i}{\omega_i^2} - 2\mu_i - \alpha_n\right) \\
&\quad \times \left\|A_i\left(\frac{1}{t_n}\int_0^{t_n} T(s)S_n u_n ds - \frac{1}{t_n}\int_0^{t_n} T(s)S_n p\, ds\right)\right\|^2.
\end{aligned} \tag{99}$$

It follows from (42) that

$$\begin{aligned}
&\|y_n - p\|^2 \\
&= \|\varepsilon_n r f(x_n) + \beta_n x_n + ((1-\beta_n)I - \varepsilon_n A)z_n - p\|^2 \\
&= \|((1-\beta_n)I - \varepsilon_n A)(z_n - p) + \beta_n(x_n - p) + \varepsilon_n(rf(x_n) - Ap)\|^2 \\
&= \|((1-\beta_n)I - \varepsilon_n A)(z_n - p) + \beta_n(x_n - p)\|^2 \\
&\quad + \varepsilon_n^2\|rf(x_n) - Ap\|^2 \\
&\quad + 2\varepsilon_n\langle((1-\beta_n)I - \varepsilon_n A)(z_n - p), rf(x_n) - Ap\rangle \\
&\quad + 2\beta_n\varepsilon_n\langle x_n - p, rf(x_n) - Ap\rangle \\
&\le ((1-\beta_n - \varepsilon_n\overline{\gamma})\|z_n - p\| + \beta_n\|x_n - p\|)^2 \\
&\quad + \varepsilon_n^2\|rf(x_n) - Ap\|^2 \\
&\quad + 2\varepsilon_n\langle((1-\beta_n)I - \varepsilon_n A)(z_n - p), rf(x_n) - Ap\rangle \\
&\quad + 2\beta_n\varepsilon_n\langle x_n - p, rf(x_n) - Ap\rangle \\
&= (1-\beta_n - \varepsilon_n\overline{\gamma})^2\|z_n - p\|^2 \\
&\quad + \beta_n^2\|x_n - p\|^2 + 2\beta_n(1-\beta_n - \varepsilon_n\overline{\gamma})
\end{aligned}$$

$$\begin{aligned}
&\times \|z_n - p\| \|x_n - p\| \\
&+ \varepsilon_n^2 \|rf(x_n) - Ap\|^2 + 2\varepsilon_n \\
&\times \langle ((1-\beta_n) I - \varepsilon_n A)(z_n - p), \\
&\quad rf(x_n) - Ap \rangle \\
&+ 2\beta_n \varepsilon_n \langle x_n - p, rf(x_n) - Ap \rangle \\
\le{}& (1-\beta_n-\varepsilon_n\overline{\gamma})^2 \|z_n - p\|^2 + \beta_n^2 \|x_n - p\|^2 \\
&+ \beta_n (1-\beta_n-\varepsilon_n\overline{\gamma}) \\
&\times \left(\|z_n - p\|^2 + \|x_n - p\|^2\right) \\
&+ \varepsilon_n^2 \|rf(x_n) - Ap\|^2 + 2\varepsilon_n \\
&\times \langle ((1-\beta_n) I - \varepsilon_n A)(z_n - p), \\
&\quad rf(x_n) - Ap \rangle \\
&+ 2\beta_n \varepsilon_n \langle x_n - p, rf(x_n) - Ap \rangle \\
={}& (1-\varepsilon_n\overline{\gamma})(1-\beta_n-\varepsilon_n\overline{\gamma}) \|z_n - p\|^2 \\
&+ (1-\varepsilon_n\overline{\gamma}) \beta_n \|x_n - p\|^2 \\
&+ \varepsilon_n^2 \|rf(x_n) - Ap\|^2 + 2\varepsilon_n \\
&\times \langle ((1-\beta_n) I - \varepsilon_n A)(z_n - p), \\
&\quad rf(x_n) - Ap \rangle \\
&+ 2\beta_n \varepsilon_n \langle x_n - p, rf(x_n) - Ap \rangle .
\end{aligned} \tag{100}$$

Substituting (99) into (100) yields that

$$\begin{aligned}
&\|y_n - p\|^2 \\
&\le (1-\varepsilon_n\overline{\gamma})(1-\beta_n-\varepsilon_n\overline{\gamma}) \\
&\quad \times \Bigg\{ \|x_n - p\|^2 - a\alpha_n \left(\frac{2\nu_1}{\omega_1^2} - 2\mu_1 + \alpha_n \right) \\
&\qquad \times \left\| A_1 \left(\frac{1}{t_n} \int_0^{t_n} T(s) S_n u_n ds \right.\right. \\
&\qquad\qquad \left.\left. - \frac{1}{t_n} \int_0^{t_n} T(s) S_n p\, ds \right) \right\|^2 \\
&\qquad - (1-a)\alpha_n \left(\frac{2\nu_2}{\omega_2^2} - 2\mu_2 + \alpha_n \right) \\
&\qquad \times \left\| A_2 \left(\frac{1}{t_n} \int_0^{t_n} T(s) S_n u_n ds \right.\right. \\
&\qquad\qquad \left.\left. - \frac{1}{t_n} \int_0^{t_n} T(s) S_n p\, ds \right) \right\|^2 \Bigg\} \\
&\quad + (1-\varepsilon_n\overline{\gamma}) \beta_n \|x_n - p\|^2 \\
&\quad + \varepsilon_n^2 \|rf(x_n) - Ap\|^2 \\
&\quad + 2\varepsilon_n \langle ((1-\beta_n) I - \varepsilon_n A)(z_n - p), \\
&\qquad rf(x_n) - Ap \rangle \\
&\quad + 2\beta_n \varepsilon_n \langle x_n - p, rf(x_n) - Ap \rangle \\
&= (1-\varepsilon_n\overline{\gamma})^2 \|x_n - p\|^2 \\
&\quad + (1-\varepsilon_n\overline{\gamma})(1-\beta_n-\varepsilon_n\overline{\gamma}) \\
&\quad \times \Bigg\{ -\alpha_n \sum_{i=1}^{N} \delta_i \left(\frac{2\nu_i}{\omega_i^2} - 2\mu_i - \alpha_n \right) \\
&\qquad \times \left\| A_i \left(\frac{1}{t_n} \int_0^{t_n} T(s) S_n u_n ds \right.\right. \\
&\qquad\qquad \left.\left. - \frac{1}{t_n} \int_0^{t_n} T(s) S_n p\, ds \right) \right\|^2 \Bigg\} \\
&\quad + 2\varepsilon_n \langle ((1-\beta_n) I - \varepsilon_n A)(z_n - p), \\
&\qquad rf(x_n) - Ap \rangle \\
&\quad + 2\beta_n \varepsilon_n \langle x_n - p, rf(x_n) - Ap \rangle \\
&\quad + \varepsilon_n^2 \|rf(x_n) - Ap\|^2 \\
&\le \|x_n - p\|^2 + (1-\varepsilon_n\overline{\gamma})(1-\beta_n-\varepsilon_n\overline{\gamma}) \\
&\quad \times \Bigg\{ -\alpha_n \sum_{i=1}^{N} \delta_i \left(\frac{2\nu_i}{\omega_i^2} - 2\mu_i - \alpha_n \right) \\
&\qquad \times \left\| A_i \left(\frac{1}{t_n} \int_0^{t_n} T(s) S_n u_n ds \right.\right. \\
&\qquad\qquad \left.\left. - \frac{1}{t_n} \int_0^{t_n} T(s) S_n p\, ds \right) \right\|^2 \Bigg\} \\
&\quad + 2\varepsilon_n \langle ((1-\beta_n) I - \varepsilon_n A)(z_n - p), \\
&\qquad rf(x_n) - Ap \rangle \\
&\quad + 2\beta_n \varepsilon_n \langle x_n - p, rf(x_n) - Ap \rangle \\
&\quad + \varepsilon_n^2 \|rf(x_n) - Ap\|^2.
\end{aligned} \tag{101}$$

Furthermore,

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\quad = \|\gamma_n x_n + (1-\gamma_n) y_n - p\|^2 \\
&\quad \le \gamma_n \|x_n - p\|^2 + (1-\gamma_n) \|y_n - p\|^2 \\
&\quad \le \|x_n - p\|^2 + (1-\gamma_n)(1-\varepsilon_n\overline{\gamma})
\end{aligned}$$

$$\begin{aligned}
&\times\left(1-\beta_n-\varepsilon_n\overline{\gamma}\right)\\
&\times\left\{-\alpha_n\sum_{i=1}^{N}\delta_i\left(\frac{2\nu_i}{\omega_i^2}-2\mu_i-\alpha_n\right)A_i\right.\\
&\qquad\times\left\|\left(\frac{1}{t_n}\int_0^{t_n}T(s)S_nu_nds\right.\right.\\
&\qquad\qquad\left.\left.\left.-\frac{1}{t_n}\int_0^{t_n}T(s)S_np\,ds\right)\right\|^2\right\}\\
&+\left(1-\gamma_n\right)\varepsilon_n^2\left\|rf\left(x_n\right)-Ap\right\|^2\\
&+2\left(1-\gamma_n\right)\varepsilon_n\\
&\times\left\langle\left(\left(1-\beta_n\right)I-\varepsilon_nA\right)\left(z_n-p\right),\right.\\
&\qquad\left.rf\left(x_n\right)-Ap\right\rangle\\
&+2\left(1-\gamma_n\right)\beta_n\varepsilon_n\left\langle x_n-p,rf\left(x_n\right)-Ap\right\rangle.
\end{aligned}\tag{102}$$

It follows that

$$\begin{aligned}
&\left(1-e_1\right)\left(1-\varepsilon_n\overline{\gamma}\right)\left(1-\beta_n-\varepsilon_n\overline{\gamma}\right)\\
&\quad\times\left\{\sum_{i=1}^{N}\delta_i\left(\frac{2g_1\nu_i}{\omega_i^2}-2g_2\mu_i-g_2^2\right)\right.\\
&\qquad\times\left\|A_i\left(\frac{1}{t_n}\int_0^{t_n}T(s)S_nu_nds\right.\right.\\
&\qquad\qquad\left.\left.\left.-\frac{1}{t_n}\int_0^{t_n}T(s)S_np\,ds\right)\right\|^2\right\}\\
&\le\left(1-\gamma_n\right)\left(1-\varepsilon_n\overline{\gamma}\right)\left(1-\beta_n-\varepsilon_n\overline{\gamma}\right)\\
&\quad\times\left\{-\alpha_n\sum_{i=1}^{N}\delta_i\left(\frac{2\nu_i}{\omega_i^2}-2\mu_i-\alpha_n\right)\right.\\
&\qquad\times\left\|A_i\left(\frac{1}{t_n}\int_0^{t_n}T(s)S_nu_nds\right.\right.\\
&\qquad\qquad\left.\left.\left.-\frac{1}{t_n}\int_0^{t_n}T(s)S_np\,ds\right)\right\|^2\right\}\\
&\le\left\|x_n-p\right\|^2-\left\|x_{n+1}-p\right\|^2\\
&\quad+\left(1-\gamma_n\right)\varepsilon_n^2\left\|rf(x_n)-Ap\right\|^2\\
&\quad+2\left(1-\gamma_n\right)\varepsilon_n\\
&\quad\times\left\langle\left(\left(1-\beta_n\right)I-\varepsilon_nA\right)\left(z_n-p\right),rf\left(x_n\right)-Ap\right\rangle\\
&\quad+2\left(1-\gamma_n\right)\beta_n\varepsilon_n\left\langle x_n-p,rf\left(x_n\right)-Ap\right\rangle.
\end{aligned}\tag{103}$$

From (94) and the condition (i), for $i=1,2,\ldots,N$, we have

$$\lim_{n\to\infty}\left\|A_i\left(\frac{1}{t_n}\int_0^{t_n}T(s)S_nu_nds-\frac{1}{t_n}\int_0^{t_n}T(s)S_np\,ds\right)\right\|=0.\tag{104}$$

Then, for $\delta_i\in(0,1)$ and $\sum_{i=1}^{N}\delta_i=1$,

$$\begin{aligned}
\lim_{n\to\infty}&\left\|\left(\sum_{i=1}^{N}\delta_iA_i\right)\left(\frac{1}{t_n}\int_0^{t_n}T(s)S_nu_nds\right.\right.\\
&\qquad\left.\left.-\frac{1}{t_n}\int_0^{t_n}T(s)S_np\,ds\right)\right\|=0.
\end{aligned}\tag{105}$$

On the other hand, one has

$$\begin{aligned}
&\left\|z_n-p\right\|^2\\
&=\left\|P_C\left(\frac{1}{t_n}\int_0^{t_n}T(s)S_nu_nds\right.\right.\\
&\qquad\left.\left.-\alpha_n\left(\sum_{i=1}^{N}\delta_iA_i\right)\frac{1}{t_n}\int_0^{t_n}T(s)S_nu_nds\right)-p\right\|^2\\
&\le\left\langle\frac{1}{t_n}\int_0^{t_n}T(s)S_nu_nds\right.\\
&\qquad-\alpha_n\left(\sum_{i=1}^{N}\delta_iA_i\right)\frac{1}{t_n}\int_0^{t_n}T(s)S_nu_nds\\
&\qquad-\left(\frac{1}{t_n}\int_0^{t_n}T(s)S_np\,ds-\alpha_n\left(\sum_{i=1}^{N}\delta_iA_i\right)\right.\\
&\qquad\qquad\left.\left.\times\frac{1}{t_n}\int_0^{t_n}T(s)S_np\,ds\right),z_n-p\right\rangle\\
&=\frac{1}{2}\left\{\left\|\frac{1}{t_n}\int_0^{t_n}T(s)S_nu_nds\right.\right.\\
&\qquad-\alpha_n\left(\sum_{i=1}^{N}\delta_iA_i\right)\frac{1}{t_n}\int_0^{t_n}T(s)S_nu_nds\\
&\qquad-\left(\frac{1}{t_n}\int_0^{t_n}T(s)S_np\,ds\right.\\
&\qquad\qquad\left.\left.-\alpha_n\left(\sum_{i=1}^{N}\delta_iA_i\right)\frac{1}{t_n}\int_0^{t_n}T(s)S_np\,ds\right)\right\|^2\\
&\quad+\left\|z_n-p\right\|^2\\
&\quad-\left\|\frac{1}{t_n}\int_0^{t_n}T(s)S_nu_nds\right.
\end{aligned}$$

$$
\begin{aligned}
&-\alpha_n\left(\sum_{i=1}^{N}\delta_i A_i\right)\frac{1}{t_n}\int_0^{t_n}T(s)S_nu_n ds\\
&-\left(\frac{1}{t_n}\int_0^{t_n}T(s)S_np\,ds-\alpha_n\left(\sum_{i=1}^{N}\delta_i A_i\right)\right.\\
&\left.\left.\left.\times\frac{1}{t_n}\int_0^{t_n}T(s)S_np\,ds\right)-(z_n-p)\right\|^2\right\}\\
\leq{}&\frac{1}{2}\left\{\|u_n-p\|^2+\|z_n-p\|^2\right.\\
&-\left\|\left(\frac{1}{t_n}\int_0^{t_n}T(s)S_nu_n ds-z_n\right)\right.\\
&-\alpha_n\left(\sum_{i=1}^{N}\delta_i A_i\right)\\
&\times\left(\frac{1}{t_n}\int_0^{t_n}T(s)S_nu_n ds\right.\\
&\left.\left.\left.-\frac{1}{t_n}\int_0^{t_n}T(s)S_np\,ds\right)\right\|^2\right\}\\
\leq{}&\frac{1}{2}\left\{\|x_n-p\|^2+\|z_n-p\|^2\right.\\
&-\left\|\frac{1}{t_n}\int_0^{t_n}T(s)S_nu_n ds-z_n\right\|^2\\
&-\alpha_n^2\left\|\left(\sum_{i=1}^{N}\delta_i A_i\right)\right.\\
&\left.\times\left(\frac{1}{t_n}\int_0^{t_n}T(s)S_nu_n ds-\frac{1}{t_n}\int_0^{t_n}T(s)S_np\,ds\right)\right\|^2\\
&+2\alpha_n\left\langle\left(\sum_{i=1}^{N}\delta_i A_i\right)\right.\\
&\times\left(\frac{1}{t_n}\int_0^{t_n}T(s)S_nu_n ds-\frac{1}{t_n}\int_0^{t_n}T(s)S_np\,ds\right),\\
&\left.\left.\frac{1}{t_n}\int_0^{t_n}T(s)S_nu_n ds-z_n\right\rangle\right\},
\end{aligned}
\tag{106}
$$

which means that

$$
\begin{aligned}
\|z_n-p\|^2\leq{}&\|x_n-p\|^2-\left\|\frac{1}{t_n}\int_0^{t_n}T(s)S_nu_n ds-z_n\right\|^2\\
&+2\alpha_n\left\|\left(\sum_{i=1}^{N}\delta_i A_i\right)\right.\\
&\times\left(\frac{1}{t_n}\int_0^{t_n}T(s)S_nu_n ds\right.\\
&\left.\left.-\frac{1}{t_n}\int_0^{t_n}T(s)S_np\,ds\right)\right\|\\
&\times\left\|\frac{1}{t_n}\int_0^{t_n}T(s)S_nu_n ds-z_n\right\|.
\end{aligned}
\tag{107}
$$

It follows that

$$
\begin{aligned}
&\|y_n-p\|^2\\
&\leq(1-\varepsilon_n\overline{\gamma})(1-\beta_n-\varepsilon_n\overline{\gamma})\|z_n-p\|^2\\
&\quad+(1-\varepsilon_n\overline{\gamma})\beta_n\|x_n-p\|^2\\
&\quad+\varepsilon_n^2\|rf(x_n)-Ap\|^2\\
&\quad+2\varepsilon_n\langle((1-\beta_n)I-\varepsilon_nA)(z_n-p),\\
&\qquad rf(x_n)-Ap\rangle\\
&\quad+2\beta_n\varepsilon_n\langle x_n-p,rf(x_n)-Ap\rangle\\
&\leq(1-\varepsilon_n\overline{\gamma})(1-\beta_n-\varepsilon_n\overline{\gamma})\\
&\quad\times\left\{\|x_n-p\|^2-\left\|\frac{1}{t_n}\int_0^{t_n}T(s)S_nu_n ds-z_n\right\|^2\right.\\
&\qquad+2\alpha_n\left\|\left(\sum_{i=1}^{N}\delta_i A_i\right)\right.\\
&\qquad\times\left(\frac{1}{t_n}\int_0^{t_n}T(s)S_nu_n ds\right.\\
&\qquad\left.\left.-\frac{1}{t_n}\int_0^{t_n}T(s)S_np\,ds\right)\right\|\\
&\qquad\left.\times\|S_nu_n-z_n\|\right\}\\
&\quad+(1-\varepsilon_n\overline{\gamma})\beta_n\|x_n-p\|^2+\varepsilon_n^2\|rf(x_n)-Ap\|^2\\
&\quad+2\varepsilon_n\langle((1-\beta_n)I-\varepsilon_nA)(z_n-p),rf(x_n)-Ap\rangle\\
&\quad+2\beta_n\varepsilon_n\langle x_n-p,rf(x_n)-Ap\rangle\\
&=(1-\varepsilon_n\overline{\gamma})^2\|x_n-p\|^2-(1-\varepsilon_n\overline{\gamma})\\
&\quad\times(1-\beta_n-\varepsilon_n\overline{\gamma})\left\|\frac{1}{t_n}\int_0^{t_n}T(s)S_nu_n ds-z_n\right\|^2\\
&\quad+2\alpha_n(1-\varepsilon_n\overline{\gamma})(1-\beta_n-\varepsilon_n\overline{\gamma})
\end{aligned}
$$

$$\begin{aligned}
&\times \left\| \left(\sum_{i=1}^{N}\delta_i A_i\right) \left(\frac{1}{t_n}\int_0^{t_n} T(s) S_n u_n ds - \frac{1}{t_n}\int_0^{t_n} T(s) S_n p\, ds\right)\right\| \left\|S_n u_n - z_n\right\| \\
&+ \varepsilon_n^2 \left\|rf(x_n) - Ap\right\|^2 \\
&+ 2\varepsilon_n \left\langle ((1-\beta_n) I - \varepsilon_n A)(z_n - p), rf(x_n) - Ap \right\rangle \\
&+ 2\beta_n\varepsilon_n \left\langle x_n - p, rf(x_n) - Ap\right\rangle \\
\le{}& \left\|x_n - p\right\|^2 - (1-\varepsilon_n\overline{\gamma})(1-\beta_n-\varepsilon_n\overline{\gamma}) \times \left\|\frac{1}{t_n}\int_0^{t_n} T(s) S_n u_n ds - z_n\right\|^2 \\
&+ 2\alpha_n (1-\varepsilon_n\overline{\gamma})(1-\beta_n-\varepsilon_n\overline{\gamma}) \times \left\| \left(\sum_{i=1}^{N}\delta_i A_i\right)\left(\frac{1}{t_n}\int_0^{t_n} T(s) S_n u_n ds - \frac{1}{t_n}\int_0^{t_n} T(s) S_n p\, ds\right)\right\| \\
&\times \left\|S_n u_n - z_n\right\| + \varepsilon_n^2 \left\|rf(x_n) - Ap\right\|^2 + 2\varepsilon_n \\
&\times \left\langle ((1-\beta_n) I - \varepsilon_n A)(z_n - p), rf(x_n) - Ap\right\rangle \\
&+ 2\beta_n\varepsilon_n \left\langle x_n - p, rf(x_n) - Ap\right\rangle .
\end{aligned} \tag{108}$$

Therefore, from (108) and (102), one has

$$\begin{aligned}
&\left\|x_{n+1} - p\right\|^2 \\
&\le \gamma_n \left\|x_n - p\right\|^2 + (1-\gamma_n)\left\|y_n - p\right\|^2 \\
&\le \gamma_n \left\|x_n - p\right\|^2 + (1-\gamma_n) \\
&\quad \times \left\{ \left\|x_n - p\right\|^2 - (1-\varepsilon_n\overline{\gamma}) \times (1-\beta_n-\varepsilon_n\overline{\gamma}) \times \left\|\frac{1}{t_n}\int_0^{t_n} T(s)S_n u_n ds - z_n\right\|^2 \right\} \\
&\quad + 2\alpha_n (1-\varepsilon_n\overline{\gamma})(1-\beta_n-\varepsilon_n\overline{\gamma}) \\
&\quad \times \left\{ \left\| \left(\sum_{i=1}^{N}\delta_i A_i\right) \times \left(\frac{1}{t_n}\int_0^{t_n} T(s) S_n u_n ds - \frac{1}{t_n}\int_0^{t_n} T(s) S_n p\, ds\right)\right\| \right. \\
&\quad \times \left\|S_n u_n - z_n\right\| + \varepsilon_n^2 \left\|rf(x_n) - Ap\right\|^2 \\
&\quad + 2\varepsilon_n \left\langle ((1-\beta_n) I - \varepsilon_n A)(z_n - p), rf(x_n) - Ap\right\rangle \\
&\quad \left. + 2\beta_n\varepsilon_n \left\langle x_n - p, rf(x_n) - Ap\right\rangle \right\} \\
&= \left\|x_n - p\right\|^2 - (1-\gamma_n)(1-\varepsilon_n\overline{\gamma}) \\
&\quad \times (1-\beta_n-\varepsilon_n\overline{\gamma}) \left\|\frac{1}{t_n}\int_0^{t_n} T(s) S_n u_n ds - z_n\right\|^2 \\
&\quad + 2\alpha_n (1-\gamma_n)(1-\varepsilon_n\overline{\gamma})(1-\beta_n-\varepsilon_n\overline{\gamma}) \\
&\quad \times \left\{ \left\| \left(\sum_{i=1}^{N}\delta_i A_i\right) \times \left(\frac{1}{t_n}\int_0^{t_n} T(s) S_n u_n ds - \frac{1}{t_n}\int_0^{t_n} T(s) S_n p\, ds\right)\right\| \right. \\
&\quad \times \left\|S_n u_n - z_n\right\| + (1-\gamma_n)\varepsilon_n^2 \\
&\quad \times \left\|rf(x_n) - Ap\right\|^2 + 2(1-\gamma_n)\varepsilon_n \\
&\quad \times \left\langle ((1-\beta_n) I - \varepsilon_n A)(z_n - p), rf(x_n) - Ap\right\rangle \\
&\quad + 2(1-\gamma_n)\beta_n\varepsilon_n \\
&\quad \left. \times \left\langle x_n - p, rf(x_n) - Ap\right\rangle \right\} .
\end{aligned} \tag{109}$$

Then,

$$\begin{aligned}
&(1-\gamma_n)(1-\varepsilon_n\overline{\gamma})(1-\beta_n-\varepsilon_n\overline{\gamma}) \times \left\|\frac{1}{t_n}\int_0^{t_n} T(s)S_n u_n ds - z_n\right\|^2 \\
&\le \left\|x_n - p\right\|^2 - \left\|x_{n+1} - p\right\|^2 \\
&\quad + 2\alpha_n (1-\gamma_n)(1-\varepsilon_n\overline{\gamma})(1-\beta_n-\varepsilon_n\overline{\gamma}) \\
&\quad \times \left\{ \left\| \left(\sum_{i=1}^{N}\delta_i A_i\right)\left(\frac{1}{t_n}\int_0^{t_n} T(s) S_n u_n ds - \frac{1}{t_n}\int_0^{t_n} T(s) S_n p\, ds\right)\right\| \right. \\
&\quad \times \left\|S_n u_n - z_n\right\| + (1-\gamma_n)\varepsilon_n^2
\end{aligned}$$

$$\times \|rf(x_n) - Ap\|^2 + 2(1-\gamma_n)\varepsilon_n \times \langle ((1-\beta_n) I - \varepsilon_n A)(z_n - p), rf(x_n) - Ap\rangle + 2(1-\gamma_n)\beta_n\varepsilon_n \times \langle x_n - p, rf(x_n) - Ap\rangle \Big\}. \quad (110)$$

From (94), (105), and condition (i), one has

$$\lim_{n\to\infty}\left\|\frac{1}{t_n}\int_0^{t_n} T(s) S_n u_n ds - z_n\right\| = 0. \quad (111)$$

Let $p \in \Theta$ and $k \in \{1,2,\ldots,M\}$. Since $J^{F_k}_{r_{k,n}}$ is firmly nonexpansive, we obtain

$$\begin{aligned}
\left\|\mathfrak{J}^k_n x_n - p\right\|^2 &= \left\|J^{F_k}_{r_{k,n}}\mathfrak{J}^{k-1}_n x_n - J^{F_k}_{r_{k,n}} p\right\|^2 \\
&= \langle J^{F_k}_{r_{k,n}}\mathfrak{J}^{k-1}_n x_n - p, \mathfrak{J}^{k-1}_n x_n - p\rangle \\
&= \frac{1}{2}\left(\left\|\mathfrak{J}^k_n x_n - p\right\|^2 + \left\|\mathfrak{J}^{k-1}_n x_n - p\right\|^2 - \left\|\mathfrak{J}^k_n x_n - \mathfrak{J}^{k-1}_n x_n\right\|^2\right).
\end{aligned} \quad (112)$$

It follows that

$$\left\|\mathfrak{J}^k_n x_n - p\right\|^2 \le \|x_n - p\|^2 - \left\|\mathfrak{J}^k_n x_n - \mathfrak{J}^{k-1}_n x_n\right\|^2. \quad (113)$$

Consequently, from (108), one has

$$\begin{aligned}
\|y_n - p\|^2 &\le (1-\varepsilon_n\overline{\gamma})(1-\beta_n-\varepsilon_n\overline{\gamma})\|z_n - p\|^2 + (1-\varepsilon_n\overline{\gamma})\beta_n\|x_n - p\|^2 \\
&\quad + \varepsilon_n^2\|rf(x_n) - Ap\|^2 + 2\varepsilon_n \times \langle ((1-\beta_n) I - \varepsilon_n A)(z_n - p), rf(x_n) - Ap\rangle \\
&\quad + 2\beta_n\varepsilon_n \langle x_n - p, rf(x_n) - Ap\rangle \\
&\le (1-\varepsilon_n\overline{\gamma})(1-\beta_n-\varepsilon_n\overline{\gamma}) \times \|u_n - p\|^2 + (1-\varepsilon_n\overline{\gamma})\beta_n\|x_n - p\|^2 \\
&\quad + \varepsilon_n^2\|rf(x_n) - Ap\|^2 + 2\varepsilon_n \times \langle ((1-\beta_n) I - \varepsilon_n A)(z_n - p), rf(x_n) - Ap\rangle \\
&\quad + 2\beta_n\varepsilon_n \langle x_n - p, rf(x_n) - Ap\rangle \\
&= (1-\varepsilon_n\overline{\gamma})(1-\beta_n-\varepsilon_n\overline{\gamma}) \times \left\|\mathfrak{J}^k_n x_n - p\right\|^2 + (1-\varepsilon_n\overline{\gamma})\beta_n \times \|x_n - p\|^2 + \varepsilon_n^2\|rf(x_n) - Ap\|^2 \\
&\quad + 2\varepsilon_n \langle ((1-\beta_n) I - \varepsilon_n A)(z_n - p), rf(x_n) - Ap\rangle \\
&\quad + 2\beta_n\varepsilon_n \langle x_n - p, rf(x_n) - Ap\rangle \\
&\le (1-\varepsilon_n\overline{\gamma})(1-\beta_n-\varepsilon_n\overline{\gamma}) \times \left(\|x_n - p\|^2 - \left\|\mathfrak{J}^k_n x_n - \mathfrak{J}^{k-1}_n x_n\right\|^2\right) \\
&\quad + (1-\varepsilon_n\overline{\gamma})\beta_n\|x_n - p\|^2 + \varepsilon_n^2\|rf(x_n) - Ap\|^2 \\
&\quad + 2\varepsilon_n \langle ((1-\beta_n) I - \varepsilon_n A)(z_n - p), rf(x_n) - Ap\rangle \\
&\quad + 2\beta_n\varepsilon_n \langle x_n - p, rf(x_n) - Ap\rangle \\
&\le \|x_n - p\|^2 - (1-\varepsilon_n\overline{\gamma})(1-\beta_n-\varepsilon_n\overline{\gamma}) \times \left\|\mathfrak{J}^k_n x_n - \mathfrak{J}^{k-1}_n x_n\right\|^2 + \varepsilon_n^2\|rf(x_n) - Ap\|^2 \\
&\quad + 2\varepsilon_n \langle ((1-\beta_n) I - \varepsilon_n A)(z_n - p), rf(x_n) - Ap\rangle \\
&\quad + 2\beta_n\varepsilon_n \langle x_n - p, rf(x_n) - Ap\rangle.
\end{aligned} \quad (114)$$

Then,

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\gamma_n x_n + (1-\gamma_n) y_n - p\|^2 \\
&\le \gamma_n\|x_n - p\|^2 + (1-\gamma_n)\|y_n - p\|^2 \\
&\le \|x_n - p\|^2 + (1-\gamma_n) \times \Big\{ -(1-\varepsilon_n\overline{\gamma})(1-\beta_n-\varepsilon_n\overline{\gamma}) \times \left\|\mathfrak{J}^k_n x_n - \mathfrak{J}^{k-1}_n x_n\right\|^2 \\
&\quad + \varepsilon_n^2\|rf(x_n) - Ap\|^2 + 2\varepsilon_n \langle ((1-\beta_n) I - \varepsilon_n A)(z_n - p), rf(x_n) - Ap\rangle \\
&\quad + 2\beta_n\varepsilon_n \langle x_n - p, rf(x_n) - Ap\rangle \Big\}.
\end{aligned} \quad (115)$$

That is,

$$\begin{aligned}
&(1-\gamma_n)-(1-\varepsilon_n\overline{\gamma})(1-\beta_n-\varepsilon_n\overline{\gamma}) \\
&\quad\times\left\|\mathfrak{T}_n^k x_n-\mathfrak{T}_n^{k-1}x_n\right\|^2 \\
&\leq\left\|x_n-p\right\|^2-\left\|x_{n+1}-p\right\|^2 \\
&\quad+(1-\gamma_n)\varepsilon_n^2\left\|rf(x_n)-Ap\right\|^2 \\
&\quad+2(1-\gamma_n)\varepsilon_n \\
&\quad\times\left\langle((1-\beta_n)I-\varepsilon_n A)(z_n-p),rf(x_n)-Ap\right\rangle \\
&\quad+2(1-\gamma_n)\beta_n\varepsilon_n\left\langle x_n-p,rf(x_n)-Ap\right\rangle.
\end{aligned}\tag{116}$$

By condition (i) and (94), for $k\in\{1,2,\ldots,M\}$, we obtain

$$\lim_{n\to\infty}\left\|\mathfrak{T}_n^k x_n-\mathfrak{T}_n^{k-1}x_n\right\|=0.\tag{117}$$

Therefore, we have

$$\begin{aligned}
&\left\|u_n-x_n\right\| \\
&=\left\|\mathfrak{T}_n^k x_n-\mathfrak{T}_n^0 x_n\right\|\leq\left\|\mathfrak{T}_n^k x_n-\mathfrak{T}_n^{k-1}x_n\right\| \\
&\quad+\left\|\mathfrak{T}_n^{k-1}x_n-\mathfrak{T}_n^{k-2}x_n\right\|+\cdots+\left\|\mathfrak{T}_n^1 x_n-\mathfrak{T}_n^0 x_n\right\|.
\end{aligned}\tag{118}$$

From (117), one has

$$\lim_{n\to\infty}\left\|u_n-x_n\right\|=0.\tag{119}$$

Notice that

$$\left\|u_n-y_n\right\|\leq\left\|u_n-x_n\right\|+\left\|x_n-y_n\right\|,\tag{120}$$

Applying (119) and (93), we have

$$\lim_{n\to\infty}\left\|u_n-y_n\right\|=0.\tag{121}$$

Since

$$\left\|u_n-z_n\right\|\leq\left\|u_n-y_n\right\|+\left\|y_n-z_n\right\|,\tag{122}$$

this together with (94) yields that

$$\lim_{n\to\infty}\left\|u_n-z_n\right\|=0.\tag{123}$$

Consequently, we obtain

$$\lim_{n\to\infty}\left\|\frac{1}{t_n}\int_0^{t_n}T(s)S_n u_n ds-u_n\right\|=0.\tag{124}$$

Step 4. Letting $z=P_\Theta(I-A+rf)z$, we show

$$\limsup_{n\to\infty}\left\langle(rf-A)z,x_n-z\right\rangle\leq 0.\tag{125}$$

We know that $P_\Theta(I-A+rf)$ is a contraction. Indeed, for any $x,y\in H$, we have

$$\begin{aligned}
&\left\|P_\Theta(I-A+rf)x-P_\Theta(I-A+rf)y\right\| \\
&\quad\leq\left\|(I-A+rf)x-(I-A+rf)y\right\| \\
&\quad\leq(1-(\overline{\gamma}-r\eta))\left\|x-y\right\|,
\end{aligned}\tag{126}$$

and hence $P_\Theta(I-A+rf)$ is a contraction due to $(1-(\overline{\gamma}-r\eta))\in(0,1)$. Thus, Banach's Contraction Mapping Principle guarantees that $P_\Theta(I-A+rf)$ has a unique fixed point, which implies $z=P_\Theta(I-A+rf)z$.

We claim that $z\in F(\Gamma)$. Since $\{u_{n_i}\}\subset\{u_n\}$ is bounded in C, without loss of generality, we can assume that $\{u_{n_i}\}\rightharpoonup z$. Since C is closed and convex, C is weakly closed. Thus we have $z\in C$. For $0\leq s<\infty$, notice that

$$\begin{aligned}
&\left\|u_{n_i}-T(h)u_{n_i}\right\| \\
&\leq\left\|u_{n_i}-\frac{1}{t_{n_i}}\int_0^{t_{n_i}}T(s)S_{n_i}u_{n_i}ds\right\| \\
&\quad+\left\|\frac{1}{t_{n_i}}\int_0^{t_{n_i}}T(s)S_{n_i}u_{n_i}ds\right. \\
&\qquad\left.-T(h)\frac{1}{t_{n_i}}\int_0^{t_{n_i}}T(s)S_{n_i}u_{n_i}ds\right\| \\
&\quad+\left\|T(h)\frac{1}{t_{n_i}}\int_0^{t_{n_i}}T(s)S_{n_i}u_{n_i}ds-T(h)u_{n_i}\right\| \\
&\leq 2\left\|u_{n_i}-\frac{1}{t_{n_i}}\int_0^{t_{n_i}}T(s)S_{n_i}u_{n_i}ds\right\| \\
&\quad+\left\|\frac{1}{t_{n_i}}\int_0^{t_{n_i}}T(s)S_{n_i}u_{n_i}ds\right. \\
&\qquad\left.-T(h)\frac{1}{t_{n_i}}\int_0^{t_{n_i}}T(s)S_{n_i}u_{n_i}ds\right\|.
\end{aligned}\tag{127}$$

It follows from (124) and Lemma 4 that

$$\lim_{n\to\infty}\left\|u_{n_i}-T(h)u_{n_i}\right\|=0.\tag{128}$$

Thus, (128) and Lemma 5 assert that $z\in F(\Gamma)$. Since $\{x_{n_i}\}\subset\{x_n\}$ is bounded in C, without loss of generality, we can assume that $\{x_{n_i}\}\rightharpoonup\omega$. It follows from (94) that $z_{n_i}\rightharpoonup\omega$. Since C is closed and convex, C is weakly closed. Thus we have $\omega\in C$.

Let us show $\omega\in F(S)$. For the sake of contradiction, suppose that $\omega\notin F(S)$, that is, $S\omega\neq\omega$. Since $z\in F(\Gamma)$, by our assumption, we have $T_i\omega\in F(\Gamma)$ and then $S_n\omega\in F(\Gamma)$. Hence $(1/t_n)\int_0^{t_n}T(s)S_n\omega\,ds=S_n\omega$. Therefore, by (124) and Opial condition, we have

$$\begin{aligned}
&\liminf_{n\to\infty}\left\|u_{n_i}-\omega\right\| \\
&\quad<\liminf_{n\to\infty}\left\|u_{n_i}-S\omega\right\| \\
&\quad\leq\liminf_{n\to\infty}\left\{\left\|u_{n_i}-\frac{1}{t_{n_i}}\int_0^{t_{n_i}}T(s)S_{n_i}u_{n_i}ds\right\|\right.
\end{aligned}$$

$$+\left\|\frac{1}{t_{n_i}}\int_0^{t_{n_i}} T(s) S_{n_i} u_{n_i} ds - \frac{1}{t_{n_i}}\int_0^{t_{n_i}} T(s) S_{n_i}\omega\, ds\right\| + \left\|S_{n_i}\omega - S\omega\right\|\Bigg\}$$
$$\leq \liminf_{n\to\infty} \left\|u_{n_i} - \omega\right\|, \tag{129}$$

which derives a contradiction. Thus, we obtain $\omega \in F(S) = \cap_{i=1}^{\infty} F(T_i)$.

Next, we claim that $\omega \in \cap_{i=1}^{M} \mathrm{SEP}(F_i)$. Since $u_n = \mathfrak{I}_n^k x_n$ for $k = 1, 2, \ldots, M$, we obtain

$$F_k\left(\mathfrak{I}_n^k x_n, y\right) + \frac{1}{r_n}\left\langle y - \mathfrak{I}_n^k x_n, \mathfrak{I}_n^k x_n - \mathfrak{I}_n^{k-1} x_n\right\rangle \geq 0, \quad \forall y \in C. \tag{130}$$

From (A2), one has

$$\frac{1}{r_n}\left\langle y - \mathfrak{I}_n^k x_n, \mathfrak{I}_n^k x_n - \mathfrak{I}_n^{k-1} x_n\right\rangle \geq F\left(y, \mathfrak{I}_n^k x_n\right). \tag{131}$$

Replacing n by n_i, we have

$$\left\langle y - \mathfrak{I}_{n_i}^k x_{n_i}, \frac{1}{r_{n_i}}\left(\mathfrak{I}_{n_i}^k x_{n_i} - \mathfrak{I}_{n_i}^{k-1} x_{n_i}\right)\right\rangle \geq F_k\left(y, \mathfrak{I}_{n_i}^k x_{n_i}\right). \tag{132}$$

It follows from $(1/r_{n_i})(\mathfrak{I}_{n_i}^k x_{n_i} - \mathfrak{I}_{n_i}^{k-1} x_{n_i}) \to 0$ and $\mathfrak{I}_{n_i}^k x_{n_i} \rightharpoonup \omega$ that

$$F_k(y, \omega) \leq 0, \quad y \in C, \tag{133}$$

for $k = 1, 2, \ldots, M$.

Put $z_t = ty + (1-t)\omega$ for all $t \in (0,1]$ and $y \in C$. Then, we have $z_t \in C$ and then $F(z_t, \omega) \leq 0$. Hence, from (A1) and (A4), we have

$$\begin{aligned} 0 &= F_k(z_t, z_t) \leq tF_k(z_t, y) + (1-t) F_k(z_t, y) \\ &\leq tF_k(z_t, y), \end{aligned} \tag{134}$$

which means $F_k(z_t, y) \geq 0$. From (A3), we obtain $F_k(\omega, y) \geq 0$ for $y \in C$ and then $\omega \in \mathrm{SEP}(F_k)$ for $k = 1, 2, \ldots, M$, that is, $\omega \in \cap_{i=1}^{M} \mathrm{SEP}(F_k)$.

Finally, we claim that $\omega \in \cap_{i=1}^{N} \mathrm{VI}(C, A_i)$.

We define the maximal monotone operator

$$Qq_1 = \begin{cases} \left(\sum_{i=1}^{N}\delta_i A_i\right) q_1 + N_C q_1, & \omega_1 \in C, \\ \emptyset, & \omega_1 \notin C. \end{cases} \tag{135}$$

Since A_i is relaxed (μ_i, ν_i)-cocoercive for $i = 1, 2$, we have

$$\begin{aligned} &\left\langle \left(\sum_{i=1}^{N}\delta_i A_i\right) x - \left(\sum_{i=1}^{N}\delta_i A_i\right) y, x - y\right\rangle \\ &= \sum_{i=1}^{N}\delta_i \left\langle A_i x - A_i y, x - y\right\rangle \\ &\geq \sum_{i=1}^{N}\delta_i \left(-\mu_i \left\|A_i x - A_i y\right\|^2 + \nu_i \left\|x - y\right\|^2\right) \\ &\geq \sum_{i=1}^{N}\delta_i \left(\nu_i - \mu_i \omega_1^i\right) \left\|x - y\right\|^2 \\ &\geq 0, \end{aligned} \tag{136}$$

which yields that $\sum_{i=1}^{N}\delta_i A_i$ is monotone. Thus, Q is maximal monotone. Let $(q_1, q_2) \in G(Q)$. Since $q_2 - (\sum_{i=1}^{N}\delta_i A_i) q_1 \in N_C q_1$ and $z_n \in C$, we have

$$\left\langle q_1 - z_n, q_2 - \left(\sum_{i=1}^{N}\delta_i A_i\right) q_1\right\rangle \geq 0. \tag{137}$$

On the other hand, it follows from $z_n = P_C((1/t_n)\int_0^{t_n} T(s) S_n u_n ds - \alpha_n(\sum_{i=1}^{N}\delta_i A_i)(1/t_n)\int_0^{t_n} T(s) S_n u_n ds)$ that

$$\left\langle q_1 - z_n, z_n - \left(\frac{1}{t_n}\int_0^{t_n} T(s) S_n u_n ds - \alpha_n \left(\sum_{i=1}^{N}\delta_i A_i\right) \times \frac{1}{t_n}\int_0^{t_n} T(s) S_n u_n ds\right)\right\rangle \geq 0, \tag{138}$$

and hence

$$\left\langle q_1 - z_n, \frac{z_n - (1/t_n)\int_0^{t_n} T(s) S_n u_n ds}{\alpha_n} + \left(\sum_{i=1}^{N}\delta_i A_i\right)\frac{1}{t_n}\int_0^{t_n} T(s) S_n u_n ds\right\rangle \geq 0. \tag{139}$$

It follows that

$$\begin{aligned}
&\langle q_1 - z_{n_i}, q_2\rangle \\
&\geq \left\langle q_1 - z_{n_i}, \left(\sum_{i=1}^{N}\delta_i A_i\right) q_1 \right\rangle \\
&\geq \left\langle q_1 - z_{n_i}, \left(\sum_{i=1}^{N}\delta_i A_i\right) q_1 \right\rangle \\
&\quad - \left\langle q_1 - z_{n_i}, \frac{z_{n_i} - S_{n_i}u_{n_i}}{\alpha_{n_i}} \right. \\
&\qquad \left. + \left(\sum_{i=1}^{N}\delta_i A_i\right) S_{n_i}u_{n_i} \right\rangle \\
&= \left\langle q_1 - z_{n_i}, \left(\sum_{i=1}^{N}\delta_i A_i\right) \right. \\
&\qquad \times \left(q_1 - \frac{1}{t_{n_i}}\int_0^{t_{n_i}} T(s) S_{n_i}u_{n_i} ds\right) \\
&\qquad \left. - \frac{z_{n_i} - (1/t_{n_i})\int_0^{t_{n_i}} T(s) S_{n_i}u_{n_i} ds}{\alpha_{n_i}} \right\rangle \\
&= \left\langle q_1 - z_{n_i}, \left(\sum_{i=1}^{N}\delta_i A_i\right)(q_1 - z_{n_i}) \right\rangle \\
&\quad + \left\langle q_1 - z_{n_i}, \left(\sum_{i=1}^{N}\delta_i A_i\right)(z_{n_i} - S_{n_i}u_{n_i}) \right\rangle \\
&\quad - \left\langle q_1 - z_{n_i}, \frac{z_{n_i} - S_{n_i}u_{n_i}}{\alpha_{n_i}} \right\rangle \\
&\geq \left\langle q_1 - z_{n_i}, \left(\sum_{i=1}^{N}\delta_i A_i\right) \right. \\
&\qquad \left. \times \left(z_{n_i} - \frac{1}{t_{n_i}}\int_0^{t_{n_i}} T(s) S_{n_i}u_{n_i} ds\right)\right\rangle \\
&\quad - \left\langle q_1 - z_{n_i}, \frac{z_{n_i} - (1/t_{n_i})\int_0^{t_{n_i}} T(s) S_{n_i}u_{n_i} ds}{\alpha_{n_i}} \right\rangle,
\end{aligned} \tag{140}$$

which implies that

$$\langle q_1 - \omega, q_2\rangle \geq 0. \tag{141}$$

Since Q is maximal monotone, we obtain that $\omega \in Q^{-1}0$. From Lemma 8, we obtain $\omega \in \mathrm{VI}(C, \sum_{i=1}^{N}\delta_i A_i)$, that is, $\omega \in (\cap_{i=1}^{N}\mathrm{VI}(C, A_i))$. Thus, $\omega \in \Theta$.

Since $z = P_\Theta(I - A + rf)z$, one has

$$\begin{aligned}
&\limsup_{n\to\infty} \langle (rf - A) z, x_n - z\rangle \\
&\quad = \lim_{i\to\infty} \langle (rf - A) z, x_{n_i} - z\rangle \\
&\quad = \langle (rf - A) z, \omega - z\rangle \\
&\quad \leq 0.
\end{aligned} \tag{142}$$

Furthermore,

$$\begin{aligned}
\langle (rf - A) z, y_n - z\rangle = {} & \langle (rf - A) z, y_n - x_n\rangle \\
& + \langle (rf - A) z, x_n - z\rangle.
\end{aligned} \tag{143}$$

From (93) and (142), we have

$$\limsup_{n\to\infty} \langle (rf - A) z, y_n - z\rangle \leq 0. \tag{144}$$

Step 5. Finally, we show that x_n converges strongly to $z = P_\Theta(I - A + rf)z$. Indeed, from (61) and (70), we obtain

$$\begin{aligned}
&\|y_n - z\|^2 \\
&\quad = \|\varepsilon_n rf(x_n) + \beta_n x_n \\
&\qquad + ((1-\beta_n) I - \varepsilon_n A) z_n - z\|^2 \\
&\quad = \|((1-\beta_n) I - \varepsilon_n A)(z_n - z) \\
&\qquad + \beta_n (x_n - z) + \varepsilon_n (rf(x_n) - Az)\|^2 \\
&\quad \leq \|((1-\beta_n) I - \varepsilon_n A)(z_n - z) + \beta_n (x_n - p)\|^2 \\
&\qquad + 2\varepsilon_n \langle ((1-\beta_n) I - \varepsilon_n A)(z_n - z) \\
&\qquad\qquad + \beta_n (x_n - z) + \varepsilon_n (rf(x_n) - Az), \\
&\qquad\qquad rf(x_n) - Ap\rangle \\
&\quad = \|((1-\beta_n) I - \varepsilon_n A)(z_n - z) + \beta_n (x_n - z)\|^2 \\
&\qquad + 2\varepsilon_n \langle (y_n - z, rf(x_n) - Az\rangle \\
&\quad \leq (1-\beta_n) \left\| \frac{((1-\beta_n) I - \varepsilon_n A)}{1-\beta_n}(z_n - z)\right\|^2 \\
&\qquad + \beta_n \|x_n - z\|^2 \\
&\qquad + 2r\varepsilon_n \langle y_n - z, f(x_n) - f(z)\rangle \\
&\qquad + 2\varepsilon_n \langle y_n - z, f(z) - Az\rangle
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{(1-\beta_n-\varepsilon_n\overline{\gamma})^2}{1-\beta_n}\|x_n-z\|^2 \\
&\quad+\beta_n\|x_n-z\|^2 \\
&\quad+r\eta\varepsilon_n\left(\|x_n-z\|^2+\|y_n-z\|^2\right) \\
&\quad+2\varepsilon_n\langle y_n-z, f(z)-Az\rangle \\
&=\left(1-(2\overline{\gamma}-r\eta)+\frac{(\varepsilon_n\overline{\gamma})^2}{1-\beta_n}\right)\|x_n-z\|^2 \\
&\quad+r\eta\varepsilon_n\|y_n-z\|^2 \\
&\quad+2\varepsilon_n\langle y_n-z, f(z)-Az\rangle,
\end{aligned} \tag{145}$$

which implies that

$$\begin{aligned}
&\|y_n-z\|^2 \\
&\quad\leq\left(1-\frac{2(\overline{\gamma}-r\eta)\varepsilon_n}{1-r\eta\varepsilon_n}\right)\|x_n-z\|^2 \\
&\quad\quad+\frac{\varepsilon_n}{1-r\eta\varepsilon_n} \\
&\quad\quad\times\left\{\frac{\overline{\gamma}^2\varepsilon_n}{1-\beta_n}\|x_n-z\|^2\right. \\
&\quad\quad\quad\left.+2\langle y_n-z, f(z)-Az\rangle\right\}.
\end{aligned} \tag{146}$$

It follows from (146) that

$$\begin{aligned}
&\|x_{n+1}-z\|^2 \\
&\quad=\|\gamma_n x_n+(1-\gamma_n)y_n-p\|^2 \\
&\quad\leq\gamma_n\|x_n-z\|^2+(1-\gamma_n)\|y_n-z\|^2 \\
&\quad\leq\|x_n-z\|^2+(1-\gamma_n) \\
&\quad\quad\times\left\{\left(1-\frac{2(\overline{\gamma}-r\eta)\varepsilon_n}{1-r\eta\varepsilon_n}\right)\|x_n-z\|^2\right. \\
&\quad\quad\quad+\frac{\varepsilon_n}{1-r\eta\varepsilon_n} \\
&\quad\quad\quad\times\left(\frac{\overline{\gamma}^2\varepsilon_n}{1-\beta_n}\|x_n-z\|^2\right. \\
&\quad\quad\quad\quad\left.\left.+2\langle y_n-z, f(z)-Az\rangle\right)\right\} \\
&\quad=\left(1-\frac{2(1-\gamma_n)(\overline{\gamma}-r\eta)\varepsilon_n}{1-r\eta\varepsilon_n}\right)\|x_n-z\|^2 \\
&\quad\quad+\frac{(1-\gamma_n)\varepsilon_n}{1-r\eta\varepsilon_n} \\
&\quad\quad\times\left(\frac{\overline{\gamma}^2\varepsilon_n}{1-\beta_n}\|x_n-z\|^2\right. \\
&\quad\quad\quad\left.+2\langle y_n-z, f(z)-Az\rangle\right).
\end{aligned} \tag{147}$$

From condition (i) and (142), we know that

$$\begin{aligned}
&\sum_{i=1}^{\infty}\frac{2(1-\gamma_n)(\overline{\gamma}-r\eta)\varepsilon_n}{1-r\eta\varepsilon_n}=\infty, \\
&\lim_{n\to\infty}\sup\frac{(1-\gamma_n)\varepsilon_n}{1-r\eta\varepsilon_n}\left(\frac{\overline{\gamma}^2\varepsilon_n}{1-\beta_n}\|x_n-z\|^2\right. \\
&\quad\left.+2\langle y_n-z, f(z)-Az\rangle\right)\leq 0.
\end{aligned} \tag{148}$$

we can conclude from Lemma 10 that $x_n \to z$ as $n \to \infty$. This completes the proof of Theorem 16. □

Theorem 17. *Let C be a nonempty closed convex subset of a real Hilbert space H, and let F_k, $k \in \{1,2,\ldots,M\}$ be bifunction from $C\times C \to R$ satisfying (A1)–(A4). Let $\{T_i\}_{i=1}^{\infty}$ be k_i-strict pseudocontractive mappings of C into self with $\kappa = \sup_i k_i$ and let $\rho_j = (\alpha_1^j, \alpha_2^j, \alpha_2^j) \in I\times I\times I$, where $I = [0,1]$, $\alpha_1^j+\alpha_2^j+\alpha_2^j = 1$, $\alpha_1^j+\alpha_2^j \leq b < 1$, and $\alpha_1^j, \alpha_2^j, \alpha_2^j \in (\kappa, 1)$ for all $j = 1,2,\ldots$. For every $n \in N$, let S_n and S be S-mapping generated by $T_n, T_{n-1},\ldots,T_1$ and $\rho_n, \rho_{n-1},\ldots,\rho_1$ and $T_n, T_{n-1},\ldots$, and $\rho_n, \rho_{n-1},\ldots$, respectively. Let $B: C \to H$ be ω-Lipschitz continuous and relaxed (μ,ν)-cocoercive mappings with $\nu-\mu\omega^2 > 0$, let f be a contraction of H into itself with $\eta \in (0,1)$, and let A be a strongly positive linear bounded self-adjoint operator with the coefficients $\overline{\gamma} > 0$ and $0 < r < \overline{\gamma}/\eta$. Assume that*

$$\Theta := \left(\bigcap_{i=1}^{\infty}\{F(T_i)\}\right)\cap\left(\bigcap_{k=1}^{M}\mathrm{SEP}(F_k)\right)\cap \mathrm{VI}(C,B). \tag{149}$$

Let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and

$$\begin{aligned}
u_n &= J_{r_{M,n}}^{F_M}J_{r_{M-1,n}}^{F_{M-1}}\cdots J_{r_{2,n}}^{F_2}J_{r_{1,n}}^{F_1}x_n, \\
z_n &= P_C(S_n u_n-\alpha_n B S_n u_n), \\
y_n &= \varepsilon_n r f(x_n)+\beta_n x_n \\
&\quad+((1-\beta_n)I-\varepsilon_n A)z_n, \\
x_{n+1} &= \gamma_n x_n+(1-\gamma_n)y_n, \quad \forall n\in N,
\end{aligned} \tag{150}$$

where $\{S_n : C \to C\}$ is the sequence defined by (37). If $\{\varepsilon_n\}$, $\{\beta_n\}$ are two sequences in $(0,1)$ and $\{\gamma_n\} \subset [c_1, c_2] \subset (0,1)$ and

$\{r_{k,n}\}$, for $k \in \{1, 2, \ldots, M\}$ is a real sequence in $(0, \infty)$ satisfing the following conditions:

(i) $\lim_{n\to\infty}\varepsilon_n = 0$, $\sum_{i=1}^{\infty} \varepsilon_n = \infty$,

(ii) $0 < \lim_{n\to\infty} \inf \beta_n \le \lim_{n\to\infty} \sup \beta_n < 1$ *and* $\lim_{n\to\infty}|\beta_{n+1} - \beta_n| = 0$,

(iii) $\lim_{n\to\infty} \inf r_{k,n} > 0$ *and* $\lim_{n\to\infty}|r_{k,n+1} - r_{k,n}| = 0$, *for* $k \in \{1, 2, \ldots, M\}$,

(iv) $\{\alpha_n\} \subset [g_1, g_2] \subset (0, (2(\nu - \mu\omega^2))/\omega^2)$ *and* $\lim_{n\to\infty}|\alpha_{n+1} - \alpha_n| = 0$.

Then $\{x_n\}$ converges strongly to $z \in \Theta$, where z is the unique solution of variational inequality

$$\lim_{n\to\infty} \sup \langle (rf - A) z, p - z\rangle \le 0, \quad \forall p \in \Theta, \tag{151}$$

which is the optimality condition for the minimization problem

$$\min_{z\in\Theta} \frac{1}{2} \langle Az, z\rangle - h(z), \tag{152}$$

where h is a potential function for rf (i.e., $h'(z) = rf(z)$ for $z \in H$).

Proof. By Theorem 16, for $i = 1, 2, \ldots, N$, letting $A_i = B$, we can obtain Theorem 17. □

Theorem 18. *Let C be a nonempty closed convex subset of a real Hilbert space H, and let F_k, $k \in \{1, 2, \ldots, M\}$ be a bifunction from $C \times C \to R$ satisfying (A1)–(A4). Let $\{T_i\}_{i=1}^{\infty}$ be k_i-strict pseudo-contractive mappings of C into self with $\kappa = \sup_i k_i$ and let $\rho_j = (\alpha_1^j, \alpha_2^j, \alpha_2^j) \in I \times I \times I$, where $I = [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_2^j = 1$, $\alpha_1^j + \alpha_2^j \le b < 1$, and $\alpha_1^j, \alpha_2^j, \alpha_2^j \in (\kappa, 1)$ for all $j = 1, 2, \ldots$. For every $n \in N$, let S_n and S be S-mapping generated by $T_n, T_{n-1}, \ldots, T_1$ and $\rho_n, \rho_{n-1}, \ldots, \rho_1$ and $T_n, T_{n-1}, \ldots$, and $\rho_n, \rho_{n-1}, \cdots$, respectively. Let $A_i : C \to H$ be ω_i-Lipschitz continuous and relaxed (μ_i, ν_i)-cocoercive mappings with $\nu_i - \mu_i\omega_i^2 > 0$, for $i = 1, 2, \ldots, N$, let f be a contraction of H into itself with $\eta \in (0, 1)$, and let A be is a strongly positive linear bounded self-adjoint operator with the coefficients $\overline{\gamma} > 0$ and $0 < r < \overline{\gamma}/\eta$. Assume that*

$$\Theta := \left(\bigcap_{i=1}^{\infty} \{F(T_i)\}\right) \cap \mathrm{EP}(F) \cap \left(\bigcap_{i=1}^{N} \mathrm{VI}(C, A_i)\right). \tag{153}$$

Let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and

$$\begin{gathered} F(u_n, y) + \frac{1}{r}(y - u_n, u_n - x_n) \ge 0, \\ z_n = P_C\left(S_n u_n - \alpha_n\left(\sum_{i=1}^{N}\delta_i A_i\right) S_n u_n\right), \\ y_n = \varepsilon_n rf(x_n) + \beta_n x_n + ((1 - \beta_n) I - \varepsilon_n A) z_n, \\ x_{n+1} = \gamma_n x_n + (1 - \gamma_n) y_n, \quad \forall n \in N, \end{gathered} \tag{154}$$

where $\{S_n : C \to C\}$ is the sequence defined by (37) *and $\delta_i \in (0, 1)$, $\sum_{i=1}^{N} \delta_i = 1$. If $\{\varepsilon_n\}$, $\{\beta_n\}$ are two sequences in $(0, 1)$ and $\{\gamma_n\} \subset [c_1, c_2] \subset (0, 1)$ and $\{r_{k,n}\}$, for $k \in \{1, 2, \ldots, M\}$ is a real sequence in $(0, \infty)$ satisfing the following conditions:*

(i) $\lim_{n\to\infty}\varepsilon_n = 0$, $\sum_{i=1}^{\infty} \varepsilon_n = \infty$,

(ii) $0 < \lim_{n\to\infty} \inf \beta_n \le \lim_{n\to\infty} \sup \beta_n < 1$ *and* $\lim_{n\to\infty}|\beta_{n+1} - \beta_n| = 0$,

(iii) $\lim_{n\to\infty} \inf r_n > 0$ *and* $\lim_{n\to\infty}|r_{n+1} - r_n| = 0$,

(iv) $\{\alpha_n\} \subset [g_1, g_2] \subset (0, (2\sum_{i=1}^{N} \delta_i(\nu_i - \mu_i\omega_i^2))/(\sum_{i=1}^{N} \delta_i\omega_i^2))$ *and* $\lim_{n\to\infty}|\alpha_{n+1} - \alpha_n| = 0$.

Then $\{x_n\}$ converges strongly to $z \in \Theta$, where z is the unique solution of variational inequality

$$\lim_{n\to\infty} \sup \langle (rf - A) z, p - z\rangle \le 0, \quad \forall p \in \Theta, \tag{155}$$

which is the optimality condition for the minimization problem

$$\min_{z\in\Theta} \frac{1}{2} \langle Az, z\rangle - h(z), \tag{156}$$

where h is a potential function for rf (i.e., $h'(z) = rf(z)$ for $z \in H$).

Proof. By Theorem 16, letting $M = 1$ for all $n \ge 1$, we can obtain Theorem 19. □

Theorem 19. *Let C be a nonempty closed convex subset of a real Hilbert space H, and let F_k, $k \in \{1, 2, \ldots, M\}$ be a bifunction from $C \times C \to R$ satisfying (A1)–(A4). Let $B : C \to H$ be ω-Lipschitz continuous and relaxed (μ, ν)-cocoercive mappings with $\nu - \mu\omega^2 > 0$, and let f be a contraction of H into itself with $\eta \in (0, 1)$, and let A be is a strongly positive linear bounded self-adjoint operator with the coefficients $\overline{\gamma} > 0$ and $0 < r < \overline{\gamma}/\eta$. Assume that*

$$\Theta := \left(\bigcap_{k=1}^{M} \mathrm{SEP}(F_k)\right) \cap \mathrm{VI}(C, B). \tag{157}$$

Let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and

$$\begin{gathered} u_n = J_{r_{M,n}}^{F_M} J_{r_{M-1,n}}^{F_{M-1}} \cdots J_{r_{2,n}}^{F_2} J_{r_{1,n}}^{F_1} x_n, \\ z_n = P_C(u_n - \alpha_n B u_n), \\ y_n = \varepsilon_n rf(x_n) + \beta_n x_n + ((1 - \beta_n) I - \varepsilon_n A) z_n, \\ x_{n+1} = \gamma_n x_n + (1 - \gamma_n) y_n, \quad \forall n \in N. \end{gathered} \tag{158}$$

If $\{\varepsilon_n\}$, $\{\beta_n\}$ are two sequences in $(0, 1)$ and $\{\gamma_n\} \subset [c_1, c_2] \subset (0, 1)$ and $\{r_{k,n}\}$, for $k \in \{1, 2, \ldots, M\}$ is a real sequence in $(0, \infty)$ satisfing the following conditions:

(i) $\lim_{n\to\infty}\varepsilon_n = 0$, $\sum_{i=1}^{\infty} \varepsilon_n = \infty$,

(ii) $0 < \lim_{n\to\infty} \inf \beta_n \le \lim_{n\to\infty} \sup \beta_n < 1$ *and* $\lim_{n\to\infty}|\beta_{n+1} - \beta_n| = 0$,

(iii) $\lim_{n\to\infty} \inf r_{k,n} > 0$ *and* $\lim_{n\to\infty}|r_{k,n+1} - r_{k,n}| = 0$, *for* $k \in \{1, 2, \ldots, M\}$.

(iv) $\{\alpha_n\} \subset [g_1, g_2] \subset (0, (2(\nu - \mu\omega^2))/\omega^2)$ *and* $\lim_{n\to\infty}|\alpha_{n+1} - \alpha_n| = 0$.

Then $\{x_n\}$ *converges strongly to* $z \in \Theta$*, where* z *is the unique solution of variational inequality*

$$\lim_{n \to \infty} \sup \langle (rf - A) z, p - z \rangle \leq 0, \quad \forall p \in \Theta, \tag{159}$$

which is the optimality condition for the minimization problem

$$\min_{z \in \Theta} \frac{1}{2} \langle Az, z \rangle - h(z), \tag{160}$$

where h *is a potential function for* rf *(i.e.,* $h'(z) = rf(z)$ *for* $z \in H$*).*

Proof. By Theorem 17, letting $T_n = I$ for all $n \leq 1$, we can obtain Theorem 19. □

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