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Research Article

On the Stability of Heat Equation

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We prove the generalized Hyers-Ulam stability of the heat equation, $\Delta u = u_t$, in a class of twice continuously differentiable functions under certain conditions.

1. Introduction

Let X be a normed space and let I be an open interval. If for any function $f : I \to X$ satisfying the differential inequality

$$\begin{aligned}\Big\| a_n(x)\, y^{(n)}(x) + a_{n-1}(x)\, y^{(n-1)}(x) \\ + \cdots + a_1(x)\, y'(x) + a_0(x)\, y(x) + h(x) \Big\| \le \varepsilon,\end{aligned} \tag{1}$$

for all $x \in I$ and for some $\varepsilon \ge 0$, there exists a solution $f_0 : I \to X$ of the differential equation

$$\begin{aligned}a_n(x)\, y^{(n)}(x) + a_{n-1}(x)\, y^{(n-1)}(x) \\ + \cdots + a_1(x)\, y'(x) + a_0(x)\, y(x) + h(x) = 0\end{aligned} \tag{2}$$

such that $\| f(x) - f_0(x)\| \le K(\varepsilon)$ for any $x \in I$, where $K(\varepsilon)$ is an expression of ε only, then we say that the above differential equation has the Hyers-Ulam stability.

If the above statement is also true when we replace ε and $K(\varepsilon)$ by $\varphi(x)$ and $\Phi(x)$, where $\varphi, \Phi : I \to [0, \infty)$ are functions not depending on f and f_0 explicitly, then we say that the corresponding differential equation has the generalized Hyers-Ulam stability. (This type of stability is sometimes called the Hyers-Ulam-Rassias stability.)

We may apply these terminologies for other differential equations and partial differential equations. For more detailed definitions of the Hyers-Ulam stability and the generalized Hyers-Ulam stability, refer to [1–7].

Obloza seems to be the first author who has investigated the Hyers-Ulam stability of linear differential equations (see [8, 9]). Here, we will introduce a result of Alsina and Ger (see [1]). If a differentiable function $f : I \to \mathbb{R}$ is a solution of the differential inequality $|y'(x) - y(x)| \le \varepsilon$, where I is an open subinterval of $\mathbb{R}$, then there exists a solution $f_0 : I \to \mathbb{R}$ of the differential equation $y'(x) = y(x)$ such that $|f(x) - f_0(x)| \le 3\varepsilon$ for any $x \in I$. This result was generalized by Miura et al. (see [10, 11]).

In 2007, Jung and Lee [12] proved the Hyers-Ulam stability of the first-order linear partial differential equation

$$a u_x(x, y) + b u_y(x, y) + c u(x, y) + d = 0, \tag{3}$$

where $a, b \in \mathbb{R}$ and $c, d \in \mathbb{C}$ are constants with $\mathfrak{R}(c) \ne 0$. It seems that the first paper dealing with Hyers-Ulam stability of partial differential equations was written by Prástaro and Rassias [13]. For a recent result on this subject, refer to [14].

In this paper, using an idea from the paper [15], we investigate the generalized Hyers-Ulam stability of the heat equation

$$\Delta u(x, t) - u_t(x, t) = 0 \tag{4}$$

in the class of radially symmetric functions, where Δ denotes the Laplace operator, $t > 0$, $x \in I$, and $I \subset \mathbb{R}^n$ is open. The heat equation plays an important role in a number of fields of science. It is strongly related to the Brownian motion in probability theory. The heat equation is also connected with chemical diffusion and it is sometimes called the diffusion equation.

2. Main Result

For a given integer $n \geq 2$, x_i denotes the ith coordinate of any point x in $\mathbb{R}^n$; that is, $x = (x_1, \ldots, x_i, \ldots, x_n)$. We assume that a, b, and t_1 are constants with $0 < a < b \leq \infty$ and $0 < t_1 \leq \infty$, and we define

$$D = \left\{x \in \mathbb{R}^n \mid a < |x| < b\right\}, \qquad T = \left\{t \in \mathbb{R} \mid 0 < t < t_1\right\}, \tag{5}$$

where $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$.

Due to an idea from [16, Section 2.3.1], we may search for a solution of (4) of the form $u(x,t) = (1/t^\alpha)v(|x|/t^\beta)$ for some twice continuously differentiable function v and constants α and $\beta > 0$. Based on this argument, we define

$$\begin{aligned} U = \Big\{ & u : D \times T \longrightarrow \mathbb{R} \mid u(x,t) = \frac{1}{t^\alpha} w(r) \quad \forall x \in D, t \in T \\ & \text{and for some function } w : (r_0, \infty) \longrightarrow \mathbb{R} \\ & \text{with } r = \frac{|x|}{t^\beta} \\ & \text{and } \lim_{r\to\infty} r^n w(r) = \lim_{r\to\infty} r^{n-1} w'(r) = 0 \Big\}, \end{aligned} \tag{6}$$

where we set

$$r_0 := \begin{cases} \dfrac{a}{t_1^\beta} & (\text{for } 0 < t_1 < \infty), \\ 0 & (\text{for } t_1 = \infty) \end{cases} \tag{7}$$

and the constants α and β will be chosen appropriately.

Theorem 1. *Let $\varphi : (r_0, \infty) \to [0, \infty)$ and $\psi : T \to [0, \infty)$ be functions such that*

$$\int_{r_0}^{\infty} \frac{e^{u^2/4}}{u^{n-1}} \int_u^\infty s^{n-1} \varphi(s)\, ds\, du < \infty, \tag{8}$$

$$c := \inf_{t\in T} t^{n/2+1} \psi(t) > 0. \tag{9}$$

If a twice continuously differentiable function $u \in U$ satisfies

$$\left|\Delta u(x,t) - u_t(x,t)\right| \leq \varphi\left(\frac{|x|}{\sqrt{t}}\right) \psi(t) \tag{10}$$

for all $x \in D$ and $t \in T$, then there exists a solution $u_0 : D \times T \to \mathbb{R}$ of the heat equation (4) such that $u_0 \in U$ and

$$\begin{aligned} \left|u(x,t) - u_0(x,t)\right| \leq{} & \frac{c}{t^{n/2}} e^{-(|x|/2\sqrt{t})^2} \\ & \times \int_{|x|/\sqrt{t}}^\infty \frac{e^{u^2/4}}{u^{n-1}} \int_u^\infty s^{n-1}\varphi(s)\, ds\, du \end{aligned} \tag{11}$$

for all $x \in D$ and $t \in T$.

Proof. Since $u(x,t)$ belongs to U, there exists a function $w : (r_0, \infty) \to \mathbb{R}$ such that

$$u(x,t) = \frac{1}{t^\alpha} w(r) \tag{12}$$

for any $x \in D$ and $t \in T$, where we set $r = |x|/t^\beta$. Using this notation, we calculate u_t and Δu:

$$\begin{aligned} u_t(x,t) &= -\frac{\alpha}{t^{\alpha+1}} w(r) - \frac{\beta}{t^{\alpha+1}} r w'(r), \\ u_{x_i}(x,t) &= \frac{1}{t^{\alpha+\beta}} w'(r) \frac{x_i}{|x|}, \\ u_{x_i x_i}(x,t) &= \frac{1}{t^{\alpha+\beta}} \left(\frac{1}{t^\beta} w''(r) \frac{x_i^2}{|x|^2} + w'(r)\left(\frac{1}{|x|} - \frac{x_i^2}{|x|^3}\right)\right). \end{aligned} \tag{13}$$

So we have

$$\begin{aligned} \Delta u(x,t) - u_t(x,t) = {}& \frac{1}{t^{\alpha+2\beta}} \\ & \times \left(w''(r) + \frac{n-1}{r} w'(r)\right) \\ & + \frac{1}{t^{\alpha+1}} \left(\beta r w'(r) + \alpha w(r)\right) \end{aligned} \tag{14}$$

for any $x \in D$, $t \in T$ and $r > r_0$.

If we set $\alpha = n/2$ and $\beta = 1/2$ in the previous equality, then we have

$$\begin{aligned} & \Delta u(x,t) - u_t(x,t) \\ &\quad = \frac{1}{t^{n/2+1}} \left(w''(r) + \frac{n-1}{r} w'(r) + \frac{r}{2} w'(r) + \frac{n}{2} w(r)\right) \\ &\quad = \frac{1}{r^{n-1}} \frac{1}{t^{n/2+1}} \Big(\left(r^{n-1} w''(r) + (n-1) r^{n-2} w'(r)\right) \\ &\qquad + \frac{1}{2}\left(r^n w'(r) + n r^{n-1} w(r)\right)\Big) \\ &\quad = \frac{1}{r^{n-1}} \frac{1}{t^{n/2+1}} \left(r^{n-1} w'(r) + \frac{r^n}{2} w(r)\right)' \end{aligned} \tag{15}$$

for all $x \in D$, $t \in T$ and $r > r_0$. Moreover, from the last equality and (10), it follows that

$$\begin{aligned} \left|\Delta u(x,t) - u_t(x,t)\right| &= \frac{1}{r^{n-1}} \frac{1}{t^{n/2+1}} \\ & \quad \times \left|\left(r^{n-1} w'(r) + \frac{r^n}{2} w(r)\right)'\right| \\ & \leq \varphi(r) \psi(t) \end{aligned} \tag{16}$$

or

$$\left|\left(r^{n-1} w'(r) + \frac{r^n}{2} w(r)\right)'\right| \leq r^{n-1} \varphi(r) t^{n/2+1} \psi(t) \tag{17}$$

for all $r > r_0$ and $t \in T$. In view of (9), we have

$$-cr^{n-1}\varphi(r) \leq \left(r^{n-1}w'(r) + \frac{r^n}{2}w(r)\right)' \leq cr^{n-1}\varphi(r) \quad (18)$$

for any $r > r_0$.

We integrate each term of the last inequality from r to ∞ and take account of the definition of U to get

$$\begin{aligned} -c\int_r^\infty s^{n-1}\varphi(s)\,ds &\leq -r^{n-1}w'(r) - \frac{r^n}{2}w(r) \\ &\leq c\int_r^\infty s^{n-1}\varphi(s)\,ds \end{aligned} \quad (19)$$

or

$$\left|w'(r) + \frac{r}{2}w(r)\right| \leq \frac{c}{r^{n-1}}\int_r^\infty s^{n-1}\varphi(s)\,ds \quad (20)$$

for all $r > r_0$.

According to [17, Theorem 1], together with (8), there exists a unique $\gamma \in \mathbb{R}$ such that

$$\left|w(r) - \gamma e^{-r^2/4}\right| \leq ce^{-r^2/4}\int_r^\infty \frac{e^{u^2/4}}{u^{n-1}}\int_u^\infty s^{n-1}\varphi(s)\,ds\,du \quad (21)$$

for all $r > r_0$, or equivalently

$$\begin{aligned} \left|u(x,t) - \frac{\gamma}{t^{n/2}}e^{-(|x|/2\sqrt{t})^2}\right| \leq \frac{c}{t^{n/2}} &\\ \times e^{-(|x|/2\sqrt{t})^2} &\\ \times \int_{|x|/\sqrt{t}}^\infty \frac{e^{u^2/4}}{u^{n-1}}\int_u^\infty s^{n-1}\varphi(s)\,ds\,du, & \end{aligned} \quad (22)$$

for all $x \in D$ and $t \in T$.

Now, we set

$$u_0(x,t) := \frac{\gamma}{t^{n/2}}e^{-(|x|/2\sqrt{t})^2} \quad (23)$$

for all $x \in D$ and $t \in T$. Then it is easy to show that $u_0 \in U$ and u_0 is a solution of the heat equation (4). Moreover, inequality (11) is an immediate consequence of (22). □

Corollary 2. *Let $\varphi : (r_0, \infty) \to [0, \infty)$ and $\psi : T \to [0, \infty)$ be functions. Assume that $n > 2$, $0 < t_1 < \infty$ and that there exist constants c and θ such that*

$$\varphi\left(\frac{|x|}{\sqrt{t}}\right) \leq \theta e^{-(|x|/2\sqrt{t})^2}\left(\frac{|x|}{\sqrt{t}}\right)^{2-n} \quad (\forall x \in D, t \in T), \quad (24)$$

$$c := \inf_{t\in T} t^{n/2+1}\psi(t) > 0. \quad (25)$$

If a twice continuously differentiable function $u \in U$ satisfies

$$\left|\Delta u(x,t) - u_t(x,t)\right| \leq \varphi\left(\frac{|x|}{\sqrt{t}}\right)\psi(t) \quad (26)$$

for all $x \in D$ and $t \in T$, then there exists a solution $u_0 : D \times T \to \mathbb{R}$ of the heat equation (4) such that $u_0 \in U$ and

$$\left|u(x,t) - u_0(x,t)\right| \leq \frac{2c\theta}{n-2}\frac{|x|^{2-n}}{t}e^{-(|x|/2\sqrt{t})^2}, \quad (27)$$

for all $x \in D$ and $t \in T$.

Proof. It follows from (24) that

$$\int_u^\infty s^{n-1}\varphi(s)\,ds \leq \int_u^\infty s^{n-1}\theta e^{-s^2/4}s^{2-n}ds = 2\theta e^{-u^2/4} \quad (28)$$

for all $u > r_0$. Moreover, by the previous inequality, it holds that

$$\begin{aligned} &\int_{r_0}^\infty \frac{e^{u^2/4}}{u^{n-1}}\int_u^\infty s^{n-1}\varphi(s)\,ds\,du \\ &\quad\leq \int_{r_0}^\infty \frac{e^{u^2/4}}{u^{n-1}}2\theta e^{-u^2/4}du = \frac{2\theta}{n-2}r_0^{2-n} < \infty, \end{aligned} \quad (29)$$

since the assumption, $0 < t_1 < \infty$, implies that $r_0 > 0$.

According to Theorem 1, there exists a solution $u_0 \in U$ of the heat equation (4) such that inequality (27) holds, for all $x \in D$ and $t \in T$. □

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