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Research Article

Controllability Criteria for Linear Fractional Differential Systems with State Delay and Impulses

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This paper is concerned with the controllability of linear fractional differential systems with delay in state and impulses. The factors of such systems including fractional derivative, impulses, and delay are taken into account synchronously. The expression of state response for such systems is derived, and the sufficient and necessary conditions of controllability criteria are established. Both the proposed criteria and illustrative examples show that the controllability property of the linear systems is dependent neither on the order of fractional derivative, on delay nor on impulses.

1. Introduction

In this paper, we consider the controllability of linear fractional differential systems with state delay and impulses as follows:

$$\begin{gathered} D^{\alpha} x(t) = Ax(t) + Bx(t-\tau) + Cu(t), \\ t \in [0,T] \setminus \{t_1, t_2, \ldots, t_k\}, \\ \Delta x(t_i) = x(t_i^+) - x(t_i^-) = I_i(x(t_i)), \quad i = 1, 2, \ldots, k, \\ x(t) = \varphi(t), \quad t \in [-\tau, 0], \end{gathered} \tag{1}$$

where $D^{\alpha}x(t)$ denotes an α order Caputo's fractional derivative of $x(t)$, $0 < \alpha < 1$, A, B, and C are the known constant matrices and satisfy $A, B \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{n\times m}$, τ is a positive constant, $x \in \mathbb{R}^n$ is the state variable, $u \in \mathbb{R}^m$ is the control input, $\varphi \in \mathbf{C}([-\tau, 0], \mathbb{R}^n)$ is the initial state function, where $\mathbf{C}([-\tau, 0], \mathbb{R}^n)$ denotes the space of all continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}^n$, $I_i : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is continuous for $i = 1, 2, \ldots, k$, and

$$x(t_i^+) = \lim_{\varepsilon\to 0^+} x(t_i+\varepsilon), \qquad x(t_i^-) = \lim_{\varepsilon\to 0^-} x(t_i+\varepsilon) \tag{2}$$

represent the right and left limits of $x(t)$ at $t = t_i$ and the discontinuous points

$$t_1 < t_2 < \cdots < t_i < \cdots < t_k, \tag{3}$$

where $0 = t_0 < \tau < t_1, t_k < t_{k+1} = T < +\infty$, and $x(t_i) = x(t_i^-)$ which implies that the solution of system (1) is left continuous at t_i.

The subject of fractional differential equations is gaining much importance and attention (see [1–11] and references therein). Fractional differential equations have been proved to be an excellent tool in the modelling of many phenomena in various fields of engineering, physics, and economics. In fact, fractional differential equations are considered as an alternative model to nonlinear differential equations. At the same time, time delay is one of the inevitable problems in practical engineering applications, which has an important effect on the stability and performance of system. In the last few years, the results with regard to the fractional delay differential systems have been presented in [12–15].

Although most dynamical systems are analyzed in either the continuous or discrete-time domain, many real systems in physics, chemistry, biology, engineering, and information science may experience abrupt changes as certain instants during the continuous dynamical processes. This kind of impulsive behaviors can be modeled by impulsive systems. The basic theory of impulsive differential equations can be found in the monographs of Baĭnov and Simeonov [16], Benchohra et al. [17], and the paper of Fečkan et al. [18].

On the other hand, controllability is the most fundamental concept in modern control theory, which has close connections to pole assignment, structural decomposition, quadratic optimal control, and so forth. Some important results concerning the control theory for various kinds of systems have been obtained in [19–36] and references therein. Kalman et al. [19] have investigated the controllability of linear dynamical systems based on the algebraic approach. Wonham and Morse [20] have discussed the pole assignment problems of linear systems based on the geometric approach. In [21–24], the authors have discussed the controllability of integer derivative delay systems. In [25, 26], the controllability of the descriptor (singular) systems has been considered. Impulsive control systems with integer derivative have been investigated in [27–29]. For integer derivative control systems with state delay and impulses, Zhang et al. [27] have derived the sufficient conditions for the controllability based on the fixed point theorem. It is worth pointing out that notable contributions have been made to fractional control systems in [30–36]. The different techniques have been developed to investigate the control problems of fractional differential systems, such as fractional sliding manifold approach [30], fixed point theorems [31–34], functional analysis method [33, 34], and algebraic method [35, 36]. To the best of our knowledge, there are no relevant reports on the controllability of fractional differential systems with state delay and impulses as treated in the current literature. In this paper, the factors of control systems including the Caputo's fractional derivative, impulses, and delay are taken into account synchronously. The purpose of this paper is to establish the sufficient and necessary conditions of controllability for system (1) based on the algebraic approach. The recent research surge in developing the theory of fractional control systems has motivated and inspired our present work.

This paper is organized as follows. In Section 2, we recall some definitions and preliminary facts, and the expression of state response for system (1) is derived. In Section 3, the sufficient and necessary conditions of controllability criteria are established. In Section 4, some examples are given to illustrate the effectiveness and applicability of controllability criteria. Finally, some concluding remarks are drawn in Section 5.

2. Preliminaries

Throughout this paper, denote by $\mathbf{C}_p([0,T],\mathbb{R}^n)$ the space of all piecewise left continuous functions mapping the interval $[0,T]$ into $\mathbb{R}^n$.

Let us recall some definitions and preliminary facts. For more details, one can see [1–4].

Definition 1. The Riemann-Liouville's fractional integral of order $\alpha > 0$ with the lower limit zero for a function $f : R^+ \to R^n$ is defined as

$$D^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds, \quad t > 0, \tag{4}$$

provided the right side is pointwise defined on $[0,+\infty)$, where $\Gamma(\cdot)$ is the Gamma function.

Definition 2. The Caputo's fractional derivative of order α for a function $f : R^+ \to R^n$ is defined as

$$D^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha+1)} \int_0^t (t-s)^{m-\alpha} f^{(m+1)}(s)\, ds, \quad 0 \le m \le \alpha < m+1. \tag{5}$$

Definition 3. The Mittag-Leffler function in two parameters is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(k\alpha+\beta)}, \tag{6}$$

where $\alpha > 0$, $\beta > 0$, and $z \in \mathbb{C}$, $\mathbb{C}$ denotes the complex plane.

Definition 4. The Laplace transform of a function $f(t)$ is defined as

$$F(s) = \pounds[f(t)] = \int_0^{+\infty} e^{-st} f(t)\, dt, \quad s \in \mathbb{C}, \tag{7}$$

where $f(t)$ is n-dimensional vector-valued function.

Remark 5. If $\alpha \in (0,1)$, then

$$\pounds[(D^{\alpha} f)(t)] = s^{\alpha} \pounds[f(t)] - s^{\alpha-1} f(0). \tag{8}$$

Lemma 6 (see [2]). *Let $\mathbb{C}$ be complex plane, for any $\alpha > 0$, $\beta > 0$, and $A \in \mathbb{C}^{n\times n}$; then*

$$\pounds\left[t^{\beta-1} E_{\alpha,\beta}(At^{\alpha})\right] = s^{\alpha-\beta}(s^{\alpha} I - A)^{-1}, \qquad \Re(s) > \|A\|^{1/\alpha} \tag{9}$$

holds, where $\Re(s)$ represents the real part of the complex number s and I denotes the identity matrix.

In order to obtain the state response of system (1), we firstly consider the representation of solution for linear fractional delay differential systems without impulses as follows:

$$\begin{aligned} D^{\alpha} x(t) &= Ax(t) + Bx(t-\tau) + f(t), \quad t \in [0,T], \\ x(t) &= \varphi(t), \quad t \in [-\tau, 0]. \end{aligned} \tag{10}$$

Lemma 7. *Let $0 < \alpha < 1$; if $f : [0,T] \to \mathbb{R}^n$ is continuous and exponentially bounded, then the solution of system* (10) *can be represented as*

$$x(t) = \varphi(0) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}[A(t-s)^{\alpha}] \times [A\varphi(0) + Bx(s-\tau) + f(s)]\, ds, \quad t \in [0,T] \tag{11}$$

and $x(t) = \varphi(t)$, $t \in [-\tau, 0]$.

Proof. Applying the method of steps which has been presented in [12], then there exists a unique solution to system (10).

For $t \in [0, T]$, taking the Laplace transform with respect to t in both sides of system (10), we obtain

$$s^{\alpha}£[x(t)] - s^{\alpha-1}\varphi(0) = A£[x(t)] + £[Bx(t-\tau) + f(t)]. \quad (12)$$

Then (12) can be written as

$$£[x(t)] = (s^{\alpha}I - A)^{-1}s^{\alpha-1}\varphi(0) + (s^{\alpha}I - A)^{-1}£[Bx(t-\tau) + f(t)]. \quad (13)$$

From Definition 4 and Lemma 6, then (13) is equivalent to

$$\begin{aligned} £[x(t)] &= (s^{\alpha}I - A)^{-1}s^{\alpha-1}\varphi(0) + (s^{\alpha}I - A)^{-1}£[Bx(t-\tau) + f(t)] \\ &= (s^{\alpha}I - A)^{-1}s^{\alpha}£[\varphi(0)] + (s^{\alpha}I - A)^{-1}£[Bx(t-\tau) + f(t)] \\ &= £[\varphi(0)] + (s^{\alpha}I - A)^{-1} \times £[A\varphi(0) + Bx(t-\tau) + f(t)] \\ &= £[\varphi(0)] + £[t^{\alpha-1}E_{\alpha,\alpha}(At^{\alpha})] \times £[A\varphi(0) + Bx(t-\tau) + f(t)]. \end{aligned} \quad (14)$$

The convolution theorem of the Laplace transform applied to (14) yields the form

$$£[x(t)] = £[\varphi(0)] + £\left\{\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}[A(t-s)^{\alpha}] \times [A\varphi(0) + Bx(s-\tau) + f(s)]\,ds\right\}. \quad (15)$$

Applying the inverse Laplace transform, we obtain

$$x(t) = \varphi(0) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}[A(t-s)^{\alpha}] \times [A\varphi(0) + Bx(s-\tau) + f(s)]\,ds, \quad t \in [0, T]. \quad (16)$$

Therefore, we have the stated result. □

Lemma 8. *Let* $0 < \alpha < 1$ *and* $u \in \mathbf{C}_p([0, T], \mathbb{R}^m)$*; then state response of system* (1) *can be represented as follows.*

For $t \in [-\tau, 0]$,

$$x(t) = \varphi(t). \quad (17)$$

For $t \in [0, t_1]$,

$$x(t) = \varphi(0) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}[A(t-s)^{\alpha}] \times [A\varphi(0) + Bx(s-\tau) + Cu(s)]\,ds. \quad (18)$$

For $t \in (t_1, t_2]$,

$$x(t) = \varphi(0) + I_1(x(t_1^-)) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}[A(t-s)^{\alpha}] \times [A\varphi(0) + Bx(s-\tau) + Cu(s)]\,ds. \quad (19)$$

For $t \in (t_i, t_{i+1}]$, $i = 1, 2, \ldots, k$,

$$x(t) = \varphi(0) + \sum_{j=1}^{i} I_j(x(t_j^-)) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}[A(t-s)^{\alpha}] \times [A\varphi(0) + Bx(s-\tau) + Cu(s)]\,ds. \quad (20)$$

Proof. If $t \in [-\tau, 0]$, then the conclusion obviously holds. If $t \in [0, t_1]$, then, from Lemma 7,

$$\begin{aligned} x(t) &= \varphi(0) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}[A(t-s)^{\alpha}] \times [A\varphi(0) + Bx(s-\tau) + Cu(s)]\,ds, \\ x(t_1) &= \varphi(0) + \int_0^{t_1} (t_1-s)^{\alpha-1}E_{\alpha,\alpha}[A(t_1-s)^{\alpha}] \times [A\varphi(0) + Bx(s-\tau) + Cu(s)]\,ds. \end{aligned} \quad (21)$$

If $t \in (t_1, t_2]$, applying the idea used in [18], we have

$$\begin{aligned} x(t) = x(t_1^+) &- \int_0^{t_1} (t_1-s)^{\alpha-1}E_{\alpha,\alpha}[A(t_1-s)^{\alpha}] \times [A\varphi(0) + Bx(s-\tau) + Cu(s)]\,ds \\ &+ \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}[A(t-s)^{\alpha}] \times [A\varphi(0) + Bx(s-\tau) + Cu(s)]\,ds \end{aligned}$$

$$\begin{aligned}
&= x(t_1^-) + I_1(x(t_1^-)) \\
&\quad - \int_0^{t_1} (t_1 - s)^{\alpha-1} E_{\alpha,\alpha}\left[A(t_1 - s)^{\alpha}\right] \\
&\qquad \times \left[A\varphi(0) + Bx(s-\tau) + Cu(s)\right] ds \\
&\quad + \int_0^{t} (t - s)^{\alpha-1} E_{\alpha,\alpha}\left[A(t - s)^{\alpha}\right] \\
&\qquad \times \left[A\varphi(0) + Bx(s-\tau) + Cu(s)\right] ds \\
&= \varphi(0) + I_1(x(t_1^-)) \\
&\quad + \int_0^{t} (t - s)^{\alpha-1} E_{\alpha,\alpha}\left[A(t - s)^{\alpha}\right] \\
&\qquad \times \left[A\varphi(0) + Bx(s-\tau) + Cu(s)\right] ds.
\end{aligned} \tag{22}$$

If $t \in (t_2, t_3]$, then

$$\begin{aligned}
x(t) &= x(t_2^+) - \int_0^{t_2} (t_2 - s)^{\alpha-1} E_{\alpha,\alpha}\left[A(t_2 - s)^{\alpha}\right] \\
&\qquad \times \left[A\varphi(0) + Bx(s-\tau) + Cu(s)\right] ds \\
&\quad + \int_0^{t} (t - s)^{\alpha-1} E_{\alpha,\alpha}\left[A(t - s)^{\alpha}\right] \\
&\qquad \times \left[A\varphi(0) + Bx(s-\tau) + Cu(s)\right] ds \\
&= x(t_2^-) + I_2(x(t_2^-)) \\
&\quad - \int_0^{t_2} (t_2 - s)^{\alpha-1} E_{\alpha,\alpha}\left[A(t_2 - s)^{\alpha}\right] \\
&\qquad \times \left[A\varphi(0) + Bx(s-\tau) + Cu(s)\right] ds \\
&\quad + \int_0^{t} (t - s)^{\alpha-1} E_{\alpha,\alpha}\left[A(t - s)^{\alpha}\right] \\
&\qquad \times \left[A\varphi(0) + Bx(s-\tau) + Cu(s)\right] ds \\
&= \varphi(0) + \sum_{j=1}^{2} I_j\left(x\left(t_j^-\right)\right) \\
&\quad + \int_0^{t} (t - s)^{\alpha-1} E_{\alpha,\alpha}\left[A(t - s)^{\alpha}\right] \\
&\qquad \times \left[A\varphi(0) + Bx(s-\tau) + Cu(s)\right] ds.
\end{aligned} \tag{23}$$

If $t \in (t_i, t_{i+1}]$ $(i = 1, 2, \ldots, k)$, then the same argument implies the following expression:

$$\begin{aligned}
x(t) &= \varphi(0) + \sum_{j=1}^{i} I_j\left(x\left(t_j^-\right)\right) \\
&\quad + \int_0^{t} (t - s)^{\alpha-1} E_{\alpha,\alpha}\left[A(t - s)^{\alpha}\right] \\
&\qquad \times \left[A\varphi(0) + Bx(s-\tau) + Cu(s)\right] ds.
\end{aligned} \tag{24}$$

Thus, the proof is completed. □

3. Controllability Criteria for System (1)

In this section, we establish the sufficient and necessary conditions of controllability criteria for system (1) based on the algebraic approach.

Definition 9. System (1) is called controllable on $[0, \omega]$ ($\omega \in (0, T]$); for any initial function $\varphi \in \mathbf{C}([-\tau, 0], \mathbb{R}^n)$ and any state $x_\omega \in \mathbb{R}^n$, there exists a control input $u(t) \in \mathbf{C}_p([0, \omega], \mathbb{R}^m)$, such that the corresponding solution of (1) satisfies $x(\omega) = x_\omega$.

Theorem 10. *System* (1) *is controllable on* $[0, \omega]$ *if and only if the Gramian matrix*

$$\begin{aligned}
W_c[0, \omega] &= \int_0^{\omega} (\omega - s)^{\alpha-1}\left[E_{\alpha,\alpha}\left(A(\omega - s)^{\alpha}\right)\right] \\
&\qquad \times CC^*\left[E_{\alpha,\alpha}\left(A^*(\omega - s)^{\alpha}\right)\right] ds
\end{aligned} \tag{25}$$

is nonsingular for some $\omega \in [0, T]$, *where* $E_{\alpha,\alpha}(\cdot)$ *is the Mittag-Leffler function and* $*$ *denotes the matrix transpose.*

Proof. We firstly prove sufficiency of Theorem 10. If $W_c[0, \omega]$ is nonsingular, then $W_c^{-1}[0, \omega]$ is well defined. For any initial state $\varphi \in \mathbf{C}([-\tau, 0], \mathbb{R}^n)$, when $\omega \in [0, t_1]$, we take the control function as

$$\begin{aligned}
u(t) &= C^*\left[E_{\alpha,\alpha}\left(A^*(\omega - t)^{\alpha}\right)\right] W_c^{-1}[0, \omega] \\
&\quad \times \left[x_\omega - \varphi(0) - \int_0^{\omega} (\omega - \theta)^{\alpha-1} E_{\alpha,\alpha}\left(A(\omega - \theta)^{\alpha}\right)\right. \\
&\qquad \left. \times \left(A\varphi(0) + Bx(\theta - \tau)\right) d\theta\right].
\end{aligned} \tag{26}$$

Substituting $t = \omega$ in (18) and inserting (26) yield

$$\begin{aligned}
x(\omega) = \varphi(0) &+ \int_0^{\omega} (\omega - s)^{\alpha-1} E_{\alpha,\alpha}\left[A(\omega - s)^{\alpha}\right] \\
&\times \Bigg\{A\varphi(0) + Bx(s-\tau) \\
&\quad + CC^*\left[E_{\alpha,\alpha}\left(A^*(\omega - s)^{\alpha}\right)\right] \\
&\quad \times W_c^{-1}[0, \omega] \\
&\quad \times \Bigg[x_\omega - \varphi(0) \\
&\qquad - \int_0^{\omega} (\omega - \theta)^{\alpha-1} \\
&\qquad \times E_{\alpha,\alpha}\left(A(\omega - \theta)^{\alpha}\right) \\
&\qquad \times \left(A\varphi(0)\right. \\
&\qquad \left. + Bx(\theta - \tau)\right) d\theta\Bigg]\Bigg\} ds
\end{aligned}$$

$$\begin{aligned}&= \varphi(0) + \int_0^\omega (\omega - s)^{\alpha-1} E_{\alpha,\alpha}[A(\omega - s)^\alpha] \\ &\qquad \times [A\varphi(0) + Bx(s-\tau)]\,ds \\ &\quad + \left[x_\omega - \varphi(0) - \int_0^\omega (\omega-\theta)^{\alpha-1} E_{\alpha,\alpha}(A(\omega-\theta)^\alpha) \right. \\ &\qquad \left. \times (A\varphi(0) + Bx(\theta-\tau))\,d\theta \right] \\ &= x_\omega. \end{aligned} \tag{27}$$

Thus system (1) is controllable on $[0,\omega]$, $\omega \in [0,t_1]$.

For $\omega \in (t_1,t_2]$, we take the control function as

$$\begin{aligned} u(t) &= C^*[E_{\alpha,\alpha}(A^*(\omega-t)^\alpha)]W_c^{-1}[0,\omega] \\ &\quad \times \left[x_\omega - \varphi(0) - I_1(x(t_1^-)) \right. \\ &\quad - \int_0^\omega (\omega-\theta)^{\alpha-1} E_{\alpha,\alpha}(A(\omega-\theta)^\alpha) \\ &\quad \left. \times (A\varphi(0) + Bx(\theta-\tau))\,d\theta \right]. \end{aligned} \tag{28}$$

Substituting $t = \omega$ in (19) and inserting (28) yield

$$\begin{aligned} x(\omega) &= \varphi(0) + I_1(x(t_1^-)) \\ &\quad + \int_0^\omega (\omega-s)^{\alpha-1} E_{\alpha,\alpha}[A(\omega-s)^\alpha] \\ &\quad \times \left\{ A\varphi(0) + Bx(s-\tau) \right. \\ &\quad + CC^*[E_{\alpha,\alpha}(A^*(\omega-s)^\alpha)]W_c^{-1}[0,\omega] \\ &\quad \times \left[x_\omega - \varphi(0) - I_1(x(t_1^-)) \right. \\ &\quad - \int_0^\omega (\omega-\theta)^{\alpha-1} E_{\alpha,\alpha}(A(\omega-\theta)^\alpha) \\ &\quad \left.\left. \times (A\varphi(0) + Bx(\theta-\tau))\,d\theta \right] \right\} ds \\ &= \varphi(0) + I_1(x(t_1^-)) \\ &\quad + \int_0^\omega (\omega-s)^{\alpha-1} E_{\alpha,\alpha}[A(\omega-s)^\alpha] \\ &\quad \times [A\varphi(0) + Bx(s-\tau)]\,ds \\ &\quad + \left[x_\omega - \varphi(0) - I_1(x(t_1^-)) \right. \\ &\quad - \int_0^\omega (\omega-\theta)^{\alpha-1} E_{\alpha,\alpha}(A(\omega-\theta)^\alpha) \\ &\quad \left. \times (A\varphi(0) + Bx(\theta-\tau))\,d\theta \right] = x_\omega. \end{aligned} \tag{29}$$

Thus system (1) is controllable on $[0,\omega]$, $\omega \in (t_1,t_2]$.

For $\omega \in (t_i,t_{i+1}]$, $i = 1,2,\ldots,k$, we take the control function as

$$\begin{aligned} u(t) &= C^*[E_{\alpha,\alpha}(A^*(\omega-t)^\alpha)]W_c^{-1}[0,\omega] \\ &\quad \times \left[x_\omega - \varphi(0) - \sum_{j=1}^i I_j(x(t_j^-)) \right. \\ &\quad - \int_0^\omega (\omega-\theta)^{\alpha-1} E_{\alpha,\alpha}(A(\omega-\theta)^\alpha) \\ &\quad \left. \times (A\varphi(0) + Bx(\theta-\tau))\,d\theta \right]. \end{aligned} \tag{30}$$

Substituting $t = \omega$ in (20) and inserting (30), then the same argument implies $x(\omega) = x_\omega$. Therefore system (1) is controllable on $[0,\omega]$.

Next, we prove necessity of Theorem 10. Suppose $W_c[0,\omega]$ is singular, without loss of generality; for $\omega \in (t_i,t_{i+1}]$, $i = 1,2,\ldots,k$, there exists a nonzero vector z_0 such that

$$z_0^* W_c[0,\omega] z_0 = 0. \tag{31}$$

That is,

$$\begin{aligned} &\int_0^\omega z_0^*(\omega-s)^{\alpha-1}[E_{\alpha,\alpha}(A(\omega-s)^\alpha)] \\ &\quad \times CC^*[E_{\alpha,\alpha}(A^*(\omega-s)^\alpha)]z_0\,ds = 0. \end{aligned} \tag{32}$$

Then it follows

$$z_0^* E_{\alpha,\alpha}[A(\omega-s)^\alpha] C = 0 \tag{33}$$

on $s \in [0,\omega]$. Since system (1) is controllable, there exist control inputs $u_1(t)$ and $u_2(t)$ such that

$$\begin{aligned} x(\omega) &= \varphi(0) + \sum_{j=1}^i I_j(x(t_j^-)) \\ &\quad + \int_0^\omega (\omega-s)^{\alpha-1} E_{\alpha,\alpha}[A(\omega-s)^\alpha] \\ &\quad \times [A\varphi(0) + Bx(s-\tau) + Cu_1(s)]\,ds = 0, \end{aligned} \tag{34}$$

$$\begin{aligned} z_0 &= \varphi(0) + \sum_{j=1}^i I_j(x(t_j^-)) \\ &\quad + \int_0^\omega (\omega-s)^{\alpha-1} E_{\alpha,\alpha}[A(\omega-s)^\alpha] \\ &\quad \times [A\varphi(0) + Bx(s-\tau) + Cu_2(s)]\,ds. \end{aligned} \tag{35}$$

Combining (34) and (35) yields

$$z_0 - \int_0^\omega (\omega-s)^{\alpha-1} E_{\alpha,\alpha}[A(\omega-s)^\alpha] C[u_2(s) - u_1(s)]\,ds = 0. \tag{36}$$

Multiplying z_0^* on both sides of (36), we get

$$z_0^* z_0 - \int_0^{\omega} (\omega - s)^{\alpha-1} z_0^* E_{\alpha,\alpha} \left[A(\omega - s)^{\alpha}\right] \times C \left[u_2(s) - u_1(s)\right] ds = 0. \tag{37}$$

According to $z_0^* E_{\alpha,\alpha}[A(\omega-s)^{\alpha}]C = 0$, we have $z_0^* z_0 = 0$. Thus $z_0 = 0$. This contradiction therefore completes the proof. □

Theorem 10 presents a geometric type criterion. By the algebraic transform and computation, we can obtain an algebraic criterion which is similar to the famous Kalman's rank condition [19].

Theorem 11. *System* (1) *is controllable on* $[0, \omega]$ *if and only if*

$$\operatorname{rank}\left[C \mid AC \mid \cdots \mid A^{n-1}C\right] = n. \tag{38}$$

Proof. According to Cayley-Hamilton theorem, $t^{\alpha-1}E_{\alpha,\alpha}(At^{\alpha})$ can be represented as

$$t^{\alpha-1} E_{\alpha,\alpha}\left(At^{\alpha}\right) = \sum_{k=0}^{+\infty} \frac{t^{k\alpha+\alpha-1}}{\Gamma(k\alpha+\alpha)} A^k = \sum_{k=0}^{n-1} G_k(t) A^k. \tag{39}$$

For $\omega \in [0, t_1]$,

$$\begin{aligned} x(\omega) &= \varphi(0) + \int_0^{\omega} (\omega - s)^{\alpha-1} E_{\alpha,\alpha}\left[A(\omega-s)^{\alpha}\right] \times \left[A\varphi(0) + Bx(s-\tau) + Cu(s)\right] ds \\ &= \varphi(0) + \sum_{k=0}^{n-1} \int_0^{\omega} G_k(\omega - s) A^k \times \left[A\varphi(0) + Bx(s-\tau) + Cu(s)\right] ds. \end{aligned} \tag{40}$$

Let

$$\Psi = \varphi(0) + \sum_{k=0}^{n-1} \int_0^{\omega} G_k(\omega - s) A^k \left[A\varphi(0) + Bx(s-\tau)\right] ds. \tag{41}$$

Then combining (40) with (41) yields

$$\begin{aligned} x(\omega) - \Psi &= \sum_{k=0}^{n-1} A^k C \int_0^{\omega} G_k(\omega - s) u(s) ds \\ &= \left[C \mid AC \mid \cdots \mid A^{n-1}C\right] \begin{bmatrix} d_0 \\ d_1 \\ \vdots \\ d_{n-1} \end{bmatrix}, \end{aligned} \tag{42}$$

where $d_k = \int_0^{\omega} G_k(\omega - s)u(s)ds$, $k = 0, 1, \ldots, n-1$. Note that, for arbitrary $\varphi \in \mathbf{C}([-\tau, 0], \mathbb{R}^n)$ and $x(\omega) \in \mathbb{R}^n$, the sufficient and necessary condition to have a control input $u(t)$ satisfying (42) is that

$$\operatorname{rank}\left[C \mid AC \mid \cdots \mid A^{n-1}C\right] = n. \tag{43}$$

For $\omega \in (t_i, t_{i+1}]$, $i = 1, 2, \ldots, k$,

$$\begin{aligned} x(\omega) &= \varphi(0) + \sum_{j=1}^{i} I_j\left(x\left(t_j^-\right)\right) + \int_0^{\omega} (\omega - s)^{\alpha-1} E_{\alpha,\alpha}\left[A(\omega-s)^{\alpha}\right] \times \left[A\varphi(0) + Bx(s-\tau) + Cu(s)\right] ds \\ &= \varphi(0) + \sum_{j=1}^{i} I_j\left(x\left(t_j^-\right)\right) + \sum_{k=0}^{n-1} \int_0^{\omega} G_k(\omega - s) A^k \times \left[A\varphi(0) + Bx(s-\tau) + Cu(s)\right] ds. \end{aligned} \tag{44}$$

Combining (41) with (44) yields

$$\begin{aligned} x(\omega) - \Psi - \sum_{j=1}^{i} I_j\left(x\left(t_j^-\right)\right) &= \sum_{k=0}^{n-1} A^k C \int_0^{\omega} G_k(\omega - s) u(s) ds \\ &= \left[C \mid AC \mid \cdots \mid A^{n-1}C\right] \begin{bmatrix} d_0 \\ d_1 \\ \vdots \\ d_{n-1} \end{bmatrix}. \end{aligned} \tag{45}$$

Note that, for arbitrary $\varphi \in \mathbf{C}([-\tau, 0], \mathbb{R}^n)$ and $x(\omega) \in \mathbb{R}^n$, the sufficient and necessary condition to have a control input $u(t)$ satisfying (45) is that

$$\operatorname{rank}\left[C \mid AC \mid \cdots \mid A^{n-1}C\right] = n. \tag{46}$$

Thus, the proof is completed. □

Remark 12. System (1) is controllable if and only if the resolvent condition $\lambda(\lambda I + Q_{\omega})^{-1} \to 0$ as $\lambda \to 0$ holds (here Q_{ω} is the respective Gramian matrix in the nonfractional, nondelay, and nonimpulsive case) since this is equivalent to the rank condition in the finite dimensional case [19, 35, 36].

4. Illustrative Examples

In this section, we give two examples to illustrate the presented criteria.

Example 13. Consider the controllability of linear fractional differential systems with state delay and impulses as follows:

$$D^{1/2}x(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix} x\left(t - \frac{1}{3}\right) + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u(t), \quad t \in [0,4] \setminus \{1,2,3\},$$
$$\Delta x(t_i) = \frac{1}{2} x(t_i^-), \quad t_i = i, i = 1,2,3,$$
$$x(t) = e^t, \quad t \in \left[-\frac{1}{3}, 0\right]. \tag{47}$$

Now, we apply Theorem 10 to prove that system (47) is controllable on $[0,4]$. Let us take

$$\alpha = \frac{1}{2}, \qquad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 2 \\ 1 \end{bmatrix}. \tag{48}$$

By computation, we have

$$CC^* = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}, \tag{49}$$

$$E_{1/2,1/2}\left(A(4-s)^{1/2}\right) = \sum_{k=0}^{1} \frac{A^k (4-s)^{k/2}}{\Gamma(k/2 + 1/2)} = \begin{bmatrix} \frac{1}{\sqrt{\pi}} & (4-s)^{\frac{1}{2}} \\ 0 & \frac{1}{\sqrt{\pi}} \end{bmatrix}, \tag{50}$$

$$E_{1/2,1/2}\left(A^*(4-s)^{1/2}\right) = \sum_{k=0}^{1} \frac{A^k (4-s)^{k/2}}{\Gamma(k/2 + 1/2)} = \begin{bmatrix} \frac{1}{\sqrt{\pi}} & 0 \\ (4-s)^{1/2} & \frac{1}{\sqrt{\pi}} \end{bmatrix}. \tag{51}$$

Substituting $\omega = 4$ in (25) and combining (25) with (49)–(51) yield

$$W_c[0,4] = \int_0^4 (4-s)^{-1/2} \left[E_{1/2,1/2}\left(A(4-s)^{1/2}\right)\right] \times CC^* \left[E_{1/2,1/2}\left(A^*(4-s)^{1/2}\right)\right] ds = \begin{bmatrix} \frac{16}{\pi} + \frac{16}{\sqrt{\pi}} + \frac{16}{3} & \frac{8}{\pi} + \frac{4}{\sqrt{\pi}} \\ \frac{8}{\pi} + \frac{4}{\sqrt{\pi}} & \frac{4}{\pi} \end{bmatrix}. \tag{52}$$

Obviously, $W_c[0,4]$ is nonsingular. Thus by Theorem 10, system (47) is controllable on $[0,4]$.

Example 14. Consider the controllability of linear fractional differential systems with state delay and impulses as follows:

$$D^{1/3}x(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -4 & -3 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 2 & -1 & 0 \end{bmatrix} x\left(t - \frac{\pi}{3}\right) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} u(t),$$
$$t \in \left[0, \frac{5}{2}\pi\right] \setminus \left\{\frac{\pi}{2}, \pi, \frac{3}{2}\pi, 2\pi\right\},$$
$$\Delta x(t_i) = \frac{1}{3} x(t_i^-), \quad t_i = \frac{i\pi}{2}, \quad i = 1,2,3,4,$$
$$x(t) = \sin t, \quad t \in \left[-\frac{\pi}{3}, 0\right]. \tag{53}$$

Now, we apply Theorem 11 to prove that system (53) is controllable on $[0,(5/2)\pi]$. Let us take

$$\alpha = \frac{1}{3}, \qquad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -4 & -3 \end{bmatrix},$$
$$B = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 2 & -1 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}. \tag{54}$$

Then one can obtain

$$\operatorname{rank}\left[C \mid AC \mid \cdots \mid A^{n-1}C\right] = \operatorname{rank} \begin{bmatrix} 1 & 0 & 0 & \star & \star & \star \\ 0 & 1 & -1 & \star & \star & \star \\ 0 & 0 & 2 & \star & \star & \star \end{bmatrix} = 3. \tag{55}$$

Thus by Theorem 11, system (53) is controllable on $[0,(5/2)\pi]$.

5. Conclusions

In this paper, the controllability criteria for linear fractional differential systems with delay in the state and impulses have been investigated. The sufficient and necessary conditions for the controllability of such systems have been established. Furthermore, both the proposed criteria and illustrative examples have shown that the controllability property of the linear systems is dependent neither on the order of fractional derivative, on delay nor on impulses.

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