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Research Article

On the Estimations of the Small Periodic Eigenvalues

Seza Dinibütün and O. A. Veliev

Department of Mathematics, Dogus University, Acıbadem, Kadiköy, 81010 Istanbul, Turkey

Correspondence should be addressed to Seza Dinibütün; sdinibutun@dogus.edu.tr

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We estimate the small periodic and semiperiodic eigenvalues of Hill's operator with sufficiently differentiable potential by two different methods. Then using it we give the high precision approximations for the length of nth gap in the spectrum of Hill-Sehrodinger operator and for the length of nth instability interval of Hill's equation for small values of n. Finally we illustrate and compare the results obtained by two different ways for some examples.

1. Introduction

Let $P(q)$ and $S(q)$ be the operators generated in $L_2[0,\pi]$ by the differential expression

$$-y''(x)+q(x)y(x) \tag{1}$$

with the periodic

$$y(\pi)=y(0), \qquad y'(\pi)=y'(0) \tag{2}$$

and semiperiodic

$$y(\pi)=-y(0), \qquad y'(\pi)=-y'(0) \tag{3}$$

boundary conditions, respectively, where q is a real periodic function with period π. The eigenvalues of $P(q)$ and $S(q)$ for $q=0$ are $(2n)^2$ and $(2n+1)^2$ for $n\in\mathbb{Z}$, respectively. All eigenvalues of $P(0)$ and $S(0)$, except 0, are doubled. The eigenvalues of the operators $P(q)$ and $S(q)$, called periodic and semiperiodic eigenvalues, are denoted by λ_{2n} and λ_{2n+1} for $n\in\mathbb{Z}$, respectively, where

$$\begin{aligned}\lambda_0(q)<\lambda_{-1}(q)\le\lambda_1(q)<\lambda_{-2}(q)\le\lambda_2(q)\\ <\lambda_{-3}(q)\le\lambda_3(q)<\lambda_{-4}(q)\le\lambda_4(q)\cdots\end{aligned} \tag{4}$$

[1, see page 27]. The spectrum $\sigma(T(q))$ of the operator $T(q)$ generated in $L_2[0,2\pi]$ by (1) and the boundary conditions

$$y(2\pi)=y(0), \qquad y'(2\pi)=y'(0) \tag{5}$$

is the union of the periodic and semiperiodic eigenvalues, that is,

$$\begin{gathered}\sigma(P)=\{\lambda_{2n}:n\in\mathbb{Z}\},\\ \sigma(S)=\{\lambda_{2n+1}:n\in\mathbb{Z}\}, \qquad \sigma(T)=\{\lambda_n:n\in\mathbb{Z}\},\end{gathered} \tag{6}$$

since (5) holds if and only if either (2) or (3) holds [1, see page 33].

The spectrum of the operator $L(q)$ generated in $L_2(-\infty,\infty)$ by (1) consists of the intervals $[\lambda_{n-1}(q),\lambda_{-n}(q)]$ for $n=1,2,\ldots$. Moreover, these intervals are the closure of the stable intervals of equation

$$-y''(x)+q(x)y(x)=\lambda y(x). \tag{7}$$

The intervals (λ_{-n},λ_n) for $n=1,2,\ldots$ are the gaps in the spectrum. These intervals with $(-\infty,\lambda_0)$ are the instable intervals of (7) [1, see pages 32 and 82]. The length of nth gap in the spectrum of $L(q)$ (the length of $(n+1)$th instability interval of (7)) is

$$\gamma_n(q)=:\lambda_n(q)-\lambda_{-n}(q). \tag{8}$$

Therefore the estimations of the periodic and semiperiodic eigenvalues are also the investigations of the spectrum of $L(q)$ and of the stable intervals of (7).

In this paper we gave the estimations for the small periodic and semiperiodic eigenvalues when the real periodic potential q belongs to the Sobolev space $W_1^k[0,\pi]$ with $k>1$. These assumptions on the potential q imply that

$$q(x)=\sum_{n\in\mathbb{Z}} q_n e^{i2nx}, \qquad q_{-n}=\overline{q_n}, \qquad |q_n|\le\frac{r}{(2n)^m}, \quad (9)$$

where

$$q_n=\left(q,e^{i2nx}\right)=\int_0^\pi q(x)e^{-i2nx}dx, \qquad r=\int_{[0,\pi]}\left|q^{(k)}(x)\right|dx. \quad (10)$$

Without loss of generality, it is assumed that $q_0=0$.

It is wellknown that (see [2])

$$\left|\lambda_n(q)-\lambda_n(0)\right|\le\sup|q(x)|, \qquad \lambda_n(0)=n^2, \quad \forall n\in\mathbb{Z}. \quad (11)$$

To give a subtle estimate for the eigenvalues $\lambda_n(q)$, we write the potential q in the form

$$q(x)=p(x)+\sum_{|n|>s} q_n e^{i2nx}, \quad (12)$$

where

$$p(x)=\sum_{n:|n|\le s} q_n e^{i2nx}. \quad (13)$$

The inequality in (9) implies that

$$\sup_{x\in[0,\pi]}\left|q(x)-p(x)\right|\le\sum_{|n|>s}|q_n|\le\frac{r}{(k-1)(2s)^{k-1}}. \quad (14)$$

Hence, by the perturbation theory (see [2]) we have

$$\left|\lambda_n(q)-\lambda_n(p)\right|\le\frac{r}{(k-1)(2s)^{k-1}}, \qquad \left|\gamma_n(q)-\gamma_n(p)\right|\le\frac{2r}{(k-1)(2s)^{k-1}}. \quad (15)$$

Therefore to estimate $\lambda_n(q)$ and $\gamma_n(q)$ we can investigate the eigenvalues $\lambda_n(p)$ of the operator $T(p)$ and then use (8) and (15).

In the literature, there are a lot of studies about numerical estimation of the periodic and semiperiodic eigenvalues by using the finite difference method, finite element method, Prüfer transformations, and shooting method. Let us recall some of them. Andrew considered the computations of the eigenvalues by using finite element method [3] and finite difference method [4]. Then these results have been extended by Condon [5] and by Vanden Berghe et al. [6]. Ji and Wong used Prüfer transformation and shooting method in their studies [7–9]. Malathi et al. [10] used shooting technique and direct integration method for computing eigenvalues of periodic Sturm-Liouville problems.

We consider the small periodic and semiperiodic eigenvalues by other methods. First, in Section 2, we obtain an approximation of the eigenvalues $\lambda_{\pm n}(p)$ for $n>ms$, where m is the positive integer for determination of the error in estimations, by using the method of the paper [11], where the asymptotic formulas for the eigenvalues and eigenfunctions of the t-periodic boundary value problems were obtained. Then, in Section 3, using it and considering the matrix form of $T(p)$ we give an approximation with very small errors for all small periodic and semiperiodic eigenvalues. Finally, we apply these investigations to get approximations order $10^{-18}, 10^{-15}$, and 10^{-12} for the first 201 eigenvalues of the operator T with potentials $p_1(x)=2\cos 2x$, $p_2(x)=2\cos 2x+2\cos 4x$, and $p_3(x)=2\cos 2x+2\cos 4x+2\cos 6x$, respectively, and give a comparison between the approximated eigenvalues obtained by the different ways.

2. On Applications of the Asymptotic Methods

In this and next sections, for simplicity of the notation, $\lambda_n(p)$ is denoted by λ_n. By (11)–(13)

$$\left|\lambda_n-n^2\right|\le\sup|p(x)|\le\sum_{n=-s}^{s}|q_n|. \quad (16)$$

To get the subtle estimations for λ_n, that is, to observe the influence of the trigonometric polynomial $p(x)$ to the eigenvalue n^2 of $T(0)$, we use the formula

$$\left(\lambda_N-n^2\right)\left(\Psi_N,e^{inx}\right)=\left(p\Psi_N,e^{inx}\right) \quad (17)$$

obtained from the equation

$$-\Psi_N''(x)+p(x)\Psi_N(x)=\lambda_N\Psi_N(x) \quad (18)$$

by multiplying e^{inx}, where Ψ_N is the eigenfunction corresponding to the eigenvalue λ_n; $\|\Psi_n\|=1/\sqrt{\pi}$, $(\cdot,\cdot)$ and $\|\cdot\|$ denote inner product and norm in $L_2[0,\pi]$.

Introduce the notation

$$M=\sup_x|p(x)|, \qquad c=\sum_{n=-s}^{s}|q_n|, \qquad Q=\sup_n|q_n|, \qquad X_{N,n}=\left(\Psi_N,e^{inx}\right). \quad (19)$$

Using this notation and (13) in (17) we get

$$\left(\lambda_N-n^2\right)X_{N,n}=\sum_{k=-s}^{s}q_k X_{N,n-2k}. \quad (20)$$

In (20) replacing N by n and then iterating it m times, as in the paper [11], were done; we obtain

$$\left(\lambda_n-n^2\right)X_{n,n}=A_m\left(\lambda_n,n\right)X_{n,n}+R_{m+1}\left(\lambda_n,n\right), \quad (21)$$

where

$$A_m(\lambda_n, n) = \sum_{k=1}^{m} a_k(\lambda_n, n), \tag{22}$$

$$a_k(\lambda_n, n) = \sum_{n_1,n_2,\ldots,n_k=-s}^{s} \frac{q_{n_1}q_{n_2}\cdots q_{n_k}q_{-n_1-n_2-\cdots-n_k}}{\prod_{i=1,2,\ldots,k}\left[\lambda_n-(n-2n_1-2n_2\cdots-2n_i)^2\right]},$$

$$R_{m+1}(\lambda_n, n) = \sum_{n_1,n_2,\ldots,n_{m+1}=-s}^{s} \times \frac{q_{n_1}q_{n_2}\cdots q_{n_m}q_{n_{m+1}}X_{n,n-2n_1-2n_2-\cdots-2n_{m+1}}}{\prod_{i=1,2,\ldots m}\left[\lambda_n-(n-2n_1-2n_2\cdots-2n_i)^2\right]}, \tag{23}$$

$$n_j \neq 0, \quad \forall j, \sum_{j=1}^{k} n_j \neq 0, \quad \forall k = 1, 2, \ldots, m \tag{24}$$

under assumption that

$$\lambda_n - (n-2n_1\cdots-2n_i)^2 \neq 0 \tag{25}$$

for $i = 1, 2, \ldots, m$. Now using (21), estimating $X_{n,n}$ and R_{m+1}, we prove the following,

Theorem 1. *Let m be a positive integer. If the conditions*

$$|n| > ms, \quad 4(|n|-1) \geq 3M \tag{26}$$

hold, then the eigenvalue λ_n of the operator $T(p)$ satisfies

$$\lambda_n = n^2 + \sum_{k=1}^{m} \sum_{n_1,n_2,\ldots,n_k=-s}^{s} \times \frac{q_{n_1}q_{n_2}\cdots q_{n_k}q_{-n_1-n_2-\cdots-n_k}}{\prod_{i=1,2,\ldots k}\left[\lambda_n-(n-2n_1-2n_2\cdots-2n_i)^2\right]} + \alpha_{n,m}, \tag{27}$$

where

$$|\alpha_{n,m}| \leq \frac{2c^{m+1}}{(4(|n|-1)-M)^m}; \tag{28}$$

c, M, and $p(x)$ are defined in (19) *and* (13).

Proof. Since $q_0 = 0$ we have $0 < |n_i| \leq s$. This with (16), (19), and (26) implies that

$$\left|\lambda_n-(n-2n_1-2n_2-\cdots-2n_i)^2\right| \geq \left|n^2-(|n|-2)^2\right| - M = 4(|n|-1) - M \geq 2M > 0 \tag{29}$$

for $i = 1, 2, \ldots, m$; that is, assumption (25) holds. Therefore we can use (21).

Now we estimate $X_{n,n}$ and R_{m+1}. First let us estimate R_{m+1}. Since $\|\Psi_n\| = 1/\sqrt{\pi}$ by Schwarz inequality we have

$$\left|\left(\Psi_n(x), e^{i(n-2n_1-2n_2-\cdots-2n_{m+1})x}\right)\right| \leq 1. \tag{30}$$

This with (23) and (29) implies that

$$|R_{m+1}| \leq \frac{1}{(4(|n|-1)-M)^m} \times \sum_{n_1,n_2,\ldots,n_{m+1}=-s}^{s} \left|q_{n_1}q_{n_2}\cdots q_{n_m}q_{n_{m+1}}\right|. \tag{31}$$

Hence by definition of c (see (19)) we have

$$|R_{m+1}| \leq \frac{c^{m+1}}{(4(|n|-1)-M)^m}. \tag{32}$$

Now we estimate $X_{n,n}$. Arguing as in the proof of (29) we get

$$\left|\lambda_n-(n-2k)^2\right| \geq 2M, \quad \forall k \neq 0, n. \tag{33}$$

Therefore using (17) we get

$$\sum_{k\in\mathbb{Z},k\neq 0,n} \left|X_{n,n-2k}\right|^2 = \sum_{k\in\mathbb{Z},k\neq 0,n} \frac{\left|\left(\Psi_n, pe^{i((n-2k))x}\right)\right|^2}{\left|\lambda_n-(n-2k)^2\right|^2} \leq \frac{M^2}{(2M)^2} = \frac{1}{4}. \tag{34}$$

This with Parseval's equality

$$\sum_{k\in\mathbb{Z}} \left|X_{n,n-2k}\right|^2 = \sum_{k\in\mathbb{Z},} \left|\left(\Psi_n, e^{i((n-2k))x}\right)\right|^2 = 1 \tag{35}$$

implies that

$$\left|X_{n,n}\right|^2 + \left|X_{n,-n}\right|^2 \geq \frac{3}{4}. \tag{36}$$

Hence at least one of the inequalities

$$\left|X_{n,n}\right| \geq \frac{1}{2}, \qquad \left|X_{n,-n}\right| \geq \frac{1}{2} \tag{37}$$

holds. If the first inequality holds, then dividing both sides of (21) by $X_{n,n}$ and using (23), (32) we obtain the proof of (27) and (28). If the second inequality holds, then instead of (21) using

$$\left(\lambda_n-(-n)^2\right)X_{n,-n} = A_m(\lambda_n, -n)X_{n,-n} + R_{m+1}(\lambda_n, -n), \tag{38}$$

taking into account that $A_m(\lambda_n, -n) = A_m(\lambda_n, n)$ and arguing as in the first case we get the proof in the second case. Theorem is proved. □

Now using (27) let us show that $\lambda_{\pm n}$ is close to the root of the equation

$$x = n^2 + f(x), \tag{39}$$

where

$$\begin{aligned} f(x) = & \sum_{n_1=-s}^{s} \frac{q_{n_1} q_{-n_1}}{x - (n - 2n_1)^2} \\ & + \sum_{n_1, n_2 = -s}^{s} \frac{q_{n_1} q_{n_2} q_{-n_1-n_2}}{\left(x - (n - 2n_1)^2\right)\left(x - (n - 2n_1 - 2n_2)^2\right)} \\ & + \dots + \sum_{n_1, n_2, \dots n_m = -s} \\ & \times \left(\left(q_{n_1} q_{n_2} \cdots q_{n_m} q_{-n_1-n_2-\dots-n_m} \right) \right. \\ & \quad \times \left(\left[x - (n - 2n_1)^2 \right] \left[x - (n - 2n_1 - 2n_2)^2 \right] \right. \\ & \quad \left. \left. \cdots \left[x - (n - 2n_1 - 2n_2 - \dots - 2n_m)^2 \right] \right)^{-1} \right). \end{aligned} \tag{40}$$

Theorem 2. *Let n be a positive integer satisfying*

$$n > ms, \quad 4(n-1) > M + 2c. \tag{41}$$

Then for all x and y from $[n^2 - M, n^2 + M]$ the inequality

$$|f(x) - f(y)| < K_n |x - y|, \tag{42}$$

where

$$K_n = \frac{Qc}{(4(n-1) - M)(4(n-1) - M - c)} < \frac{1}{2}, \tag{43}$$

holds, and (39) has a unique solution r_n on $[n^2 - M, n^2 + M]$. Moreover

$$|\lambda_{\pm n} - r_n| < \frac{2c^{m+1}}{(1 - K_n)(4(n-1) - M)^m} \tag{44}$$

and the length γ_n of nth gap in the spectrum of $L(p)$ (the length γ_n of $(n+1)$th instability interval of (7)) satisfies

$$\gamma_n = \lambda_n - \lambda_{-n} < \frac{4c^{m+1}}{(1 - K_n)(4(n-1) - M)^m}. \tag{45}$$

Proof. Let $f_1(x), f_2(x), \dots, f_m(x)$ be the first, second, and mth summations in the right-hand side of (40). Then

$$f_1'(x) = -\sum_{k=-s}^{s} \frac{|q_k^2|}{\left(x - (n-2k)^2\right)^2}. \tag{46}$$

For $x \in [n^2 - M, n^2 + M]$, using (29) and (41), we get

$$\left|x - (n-2k)^2\right| \geq 4(n-1) - M > 2c. \tag{47}$$

On the other hand

$$\sum_{k=-s}^{s} |q_k^2| \leq Qc. \tag{48}$$

This inequality with (47) and the inequality $Q \leq c$ (see (19)) imply that

$$|f_1'(x)| \leq \frac{Qc}{(4(n-1) - M)^2} < \frac{1}{4}. \tag{49}$$

In the same way we obtain

$$|f_k'(x)| \leq \frac{Qc^k}{(4(n-1) - M)^{k+1}} < \frac{1}{2^{k+1}} \tag{50}$$

for $k = 2, 3, \dots$. Thus by the geometric series formula we have

$$|f'(x)| \leq K_n < \frac{1}{2}, \quad \forall x \in [n^2 - M, n^2 + M], \tag{51}$$

where K_n is defined in (43), and by mean-value theorem (42) holds. Therefore by contraction mapping theorem (39) has a unique solution r_n on $[n^2 - M, n^2 + M]$.

Now let us prove (44). Let $F(x) = x - n^2 - f(x)$. Using the definition of r_n and $F(x)$ and then (40) we obtain $F(r_n) = 0$ and

$$|F(\lambda_n) - F(r_n)| \leq |\alpha_{n,m}|. \tag{52}$$

On the other hand by (51) we have $|F'(x)| \geq 1 - K_n$ for all $x \in [n^2 - M, n^2 + M]$. Therefore using the mean-value formula

$$|F(\lambda_n) - F(r_n)| = |F'(\zeta)| |\lambda_n - r_n|, \tag{53}$$

$\zeta \in [n^2 - M, n^2 + M]$, and (52) we obtain

$$|\lambda_n - r_n| \leq \frac{|\alpha_{n,m}|}{1 - K_n}. \tag{54}$$

This with (28) implies (44) for λ_n. In the same way we prove (44) for λ_{-n}. Therefore (45) follows from (44). The theorem is proved. □

Now let us approximate r_n by fixed-point iteration

$$x_{n,0} = n^2, \quad x_{n,1} = n^2 + f(x_{n,0}), \dots, x_{n,i} = n^2 + f(x_{n,i-1}). \tag{55}$$

Note that repeating the proof of (51) one can readily see that

$$|f(\lambda_n)| \leq \frac{Qc}{4(n-1) - M - c}, \qquad |f(n^2)| \leq \frac{Qc}{4(n-1) - c} \tag{56}$$

for all n satisfying (41).

Theorem 3. *For the sequence $\{x_{n,i}\}$ defined by (55) the estimations*

$$|x_{n,i} - r_n| \leq K_n^i B \tag{57}$$

for $i = 1, 2, 3, \dots$ hold, where n satisfies (41), K_n is defined in Theorem 2, and

$$B = \frac{|f(n^2)|}{1 - K_n} \leq \frac{Qc}{(1 - K_n)(4(n-1) - c)}. \tag{58}$$

Proof. It is clear and well known that if f satisfies (42) then

$$\left|x_{n,i} - r_n\right| \leq K_n^i \left|x_{n,0} - r_n\right|. \tag{59}$$

Therefore to prove (57) it is enough to show that

$$\left|x_{n,0} - r_n\right| \leq B, \tag{60}$$

where B is defined in (58). By definition of r_n and $x_{n,0}$ we have

$$r_n - x_{n,0} = f(r_n) = f(r_n) - f(x_{n,0}) + f\left(n^2\right), \tag{61}$$

and by the mean-value theorem there exists $x \in [n^2 - M, n^2 + M]$ such that

$$f(r_n) - f(x_{n,0}) = f'(x)(r_n - x_{n,0}). \tag{62}$$

These two equalities imply that

$$(r_n - x_{n,0})\left(1 - f'(x)\right) = f\left(n^2\right). \tag{63}$$

This formula with (56) and (51) implies (60). □

Thus by (44) and (57) we have the approximation $x_{n,i}$ for $\lambda_{\pm n}$ with the error

$$E_{n,i} =: \left|\lambda_{\pm n} - x_{n,i}\right| < \frac{2c^{m+1}}{(1 - K_n)(4(n-1) - M)^m} + K_n^i B. \tag{64}$$

3. Estimation of the Small Eigenvalues

In this section we estimate the eigenvalues λ_N of the operator $T(p)$, for $|N| \leq l$, by investigating the system of $2S + 1$ equations

$$\begin{aligned}\left(\lambda_N - n^2\right) X_{N,n} - \sum_{k:|k|\leq s, |n-2k|\leq S} q_k X_{N,n-2k} \\ = \sum_{k:|k|\leq s, |n-2k|>S} q_k X_{N,n-2k}\end{aligned} \tag{65}$$

for $n = -S, -S+1, -S+2, \ldots, S$, where $S = l + 2rs$ and r is the positive integer for determination of the error in estimation,

$$4(l-1) - M - c > \max\left\{c, 2c^2\right\}; \tag{66}$$

the numbers M and c are defined in (19). The first, second, and jth equations of (65) are obtained from (20) by taking $n = -S, n = -S+1$, and $n = -S - 1 + j$, respectively, and by writing the terms with multiplicand $X_{N,n-2k}$ for $|n - 2k| \leq S$ on the left-hand side and the terms with multiplicand $X_{N,n-2k}$ for $|n - 2k| > S$ on the right-hand side.

To write (65) in the matrix form let us introduce the notations. Let A be $(2S+1)$ by $(2S+1)$ matrix $(a_{i,j})$ defined by

$$a_{i,i} = (-S - 1 + i)^2, \quad a_{i,i\mp 2k} = q_{\pm k} \tag{67}$$

for $i = 1, 2, \ldots, 2S+1$ and $k = 1, 2, \ldots s$ if $|i \mp 2k| \leq S$ and all other entries of A are zero. Since $q_{-n} = \overline{q_n}$ (see (9)), A is a Hermitian (self-adjoint) matrix and its eigenvalues are real numbers. Denote the eigenvalues of A by $\mu_0, \mu_{-1}, \mu_1, \mu_{-2}, \mu_2, \ldots, \mu_{-S}, \mu_S$, where

$$\mu_0 \leq \mu_{-1} \leq \mu_1 \leq \mu_{-2} \leq \mu_2 \leq \cdots \leq \mu_{-S} \leq \mu_S. \tag{68}$$

It is clear that

$$\left|\mu_{\pm n} - n^2\right| \leq c, \tag{69}$$

since the diagonal elements of A are n^2 for $n = -S, -S+1, -S+2, \ldots, S$ and the sum of the absolute values of the nondiagonal elements of each row is not greater than c (see (19)). Let $X_N = (X_{N,-S}, X_{N,-S+1}, \ldots, X_{N,S})$ and $R(\lambda_N) = (R_{-S}, R_{-S+1}, \ldots, R_S)$ be vectors of $\mathbb{C}^{2S+1}$, where $R_n = 0$ for $|n| \leq S - 2s$ and

$$R_n(\lambda_N) = \sum_{k:|k|\leq s, |n-2k|>S} q_k X_{N,n-2k} \tag{70}$$

for $S - 2s < |n| \leq S$. In this notation the system of (65) can be written in the matrix form

$$(\lambda_N I - A) X_N^T = R^T(\lambda_N). \tag{71}$$

First we prove that $X_{N,n}$ for $n = \pm(S+1), \pm(S+2), \ldots, \pm(S+2s)$, that is, the right-hand side $R^T(\lambda_N)$ of (71), is small (see Lemma 4). Then using it we prove that the nth eigenvalue λ_n of the operator $T(p)$ is close to the nth eigenvalue μ_n of the matrix A (see Theorem 6).

Lemma 4. *If $|N| \leq l$ and $l + 2rs < |n| \leq l + 2(r+1)s$, then*

$$\left|X_{N,n}\right| \leq \frac{c^{r+1}}{(2l)^{r+1}} =: \varepsilon, \tag{72}$$

$$\sum_{n:|n|>S} \left|X_{N,n}\right|^2 \leq \frac{4s\varepsilon^2 (2l)^2}{\left((2l)^2 - c^2\right)} = \frac{4sc^{2r+2}}{(2l)^{2r}\left((2l)^2 - c^2\right)} =: \delta. \tag{73}$$

Proof. First we prove (72) for positive n. The proof for negative n is similar. One can readily see from the estimations (27), (28) for $m = 2$, (56), and (66) that if $k \geq l$, then

$$\begin{aligned}\left|\lambda_k - k^2\right| &\leq \left|f(\lambda_k)\right| + \left|\alpha_{k,2}\right| \\ &\leq \frac{Qc}{4(k-1) - M - c} + \frac{2c^3}{(4(k-1) - M)^2} < 1.\end{aligned} \tag{74}$$

Using (74) and taking into account the condition on N and n we obtain

$$\begin{aligned}&\left|\lambda_N - (n - 2n_1 - \cdots - 2n_i)^2\right| \\ &\quad \geq \left|\lambda_N - (l+1)^2\right| \\ &\quad \geq \left|\lambda_l - (l+1)^2\right| > \left|l^2 - (l+1)^2\right| - 1 \geq 2l\end{aligned} \tag{75}$$

for $|n_i| \le s, i = 0, 1, \ldots, r$. On the other hand iterating (20) r times we get

$$X_{N,n} = \sum_{n_1,n_2,\ldots,n_r=-s}^{s} \times \left(\left(q_{n_1} q_{n_2} \cdots q_{n_{r+1}} \left(\Psi_N, e^{i(n-2n_1-\cdots-2n_{2r+1})x} \right) \right) \times \left(\left[\lambda_N - n^2 \right] \left[\lambda_N - (n-2n_1)^2 \right] \cdots \left[\lambda_N - (n - 2n_1 - \cdots - 2n_r)^2 \right] \right)^{-1} \right). \tag{76}$$

Therefore arguing as in the proof of (32) we get

$$|X_{N,n}| \le \frac{c^{r+1}}{(2l)^{r+1}} \tag{77}$$

for $l + 2rs < |n| \le l + 2(r+1)s$; that is, (72) is proved.

Now we prove (73). By definition of S the left-hand side of (73) can be written in the form

$$\sum_{n:|n|>S} |X_{N,n}|^2 = \sum_{k=r}^{\infty} H_{N,k}, \tag{78}$$

where

$$H_{N,k} = \sum_{l+2ks<|n|\le l+2(k+1)s,} |X_{N,n}|^2. \tag{79}$$

In (72) replacing r by k one can readily see that

$$H_{N,k} \le \frac{4sc^{2k+2}}{(2l)^{2k+2}}. \tag{80}$$

Using this in (78) we obtain

$$\sum_{n:|n|>S} |X_{N,n}|^2 \le \sum_{k=r}^{\infty} \frac{4sc^{2k+2}}{(2l)^{2k+2}} \tag{81}$$

which implies (73), since the series in the right-hand side of (81) is a geometric series with first term $4s\varepsilon^2$ and factor $c^2/(2l)^2$. □

Note that (72) and (73) imply the following inequalities. By (70) and (72)

$$|R_n(\lambda_N)| < c\varepsilon, \quad \forall n : S - 2s < |n| \le S, \quad \forall |N| \le l, \tag{82}$$

and by the definition of $R(\lambda_N)$ we have

$$\|R(\lambda_N)\| \le 2c\varepsilon\sqrt{s}, \quad \forall |N| \le l. \tag{83}$$

Besides using (73) and Parseval's equality (35) we obtain

$$1 - \delta \le \sum_{n=-S}^{S} |X_{N,n}|^2 \le 1, \qquad \sqrt{1-\delta} \le \|X_N\| \le 1, \quad \forall |N| \le l. \tag{84}$$

Let $\{V_n^T : n = 0, \pm 1, \pm 2, \ldots, \pm S\}$ be orthonormal system of eigenvectors of the matrix A:

$$AV_n^T = \mu_n V_n^T, \tag{85}$$

where $\langle V_n, V_k \rangle = \delta_{n,k}, V_n = (V_{n,-S}, V_{n,-S+1}, \ldots, V_{n,S}) \in \mathbb{C}^{2S+1}$, and $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{C}^{2S+1}$ as well as in l_2. Denote by D the $(2S+1) \times (2S+1)$ diagonal matrix with diagonal elements

$$d_i = a_{i,i} = (-S - 1 + i)^2 \tag{86}$$

for $i = 1, 2, \ldots, 2S+1$. The eigenfunctions of D corresponding to the eigenvalues n^2 are e_{-n} and e_n, where $e_n = (e_{n,-S}, e_{n,-S+1}, \ldots, e_{n,S})^T, e_{n,n} = 1$, and $e_{n,k} = 0$ for all $k \ne n$. Multiplying both sides of (85) for $n = N$ by e_n we get

$$\left(\mu_N - n^2\right) V_{N,n} = \sum_{k=-s}^{s} q_k V_{N,n-2k}, \tag{87}$$

where $V_{N,n-2k} = 0$ if $|n - 2k| > S$. Instead of (20) using (87) and repeating the proof of (72) we obtain that if $|N| \le l$ and $|n| > S - 2s$, then

$$|V_{N,n}| \le \frac{c^r}{(2l)^r}. \tag{88}$$

To prove the main result of the paper we use the following.

Lemma 5. *Let* $c_{n,j} = \langle X_n^T, V_j^T \rangle$ *and* $n = 0, \pm 1, \pm 2, \ldots, \pm l$. *Then*

$$\left| c_{n,j} \left(\mu_j - \lambda_n \right) \right| \le 8sl\varepsilon^2 \tag{89}$$

for $j = 0, \pm 1, \pm 2, \ldots, \pm l$ *and*

$$\left| c_{n,j} \left(\mu_j - \lambda_n \right) \right| \le 2c\varepsilon\sqrt{s} \tag{90}$$

for $j = \pm(l+1), \pm(l+2), \ldots, \pm S$.

Proof. Since $\{V_j : j = 0, \pm 1, \pm 2, \ldots, \pm S\}$ is an orthonormal basis in $\mathbb{C}^{2S+1}$ we have

$$X_n^T = \sum_{j=-S}^{S} c_{n,j} V_j^T, \qquad |X_k|^2 = \sum_{j=-S}^{S} |c_{k,j}|^2. \tag{91}$$

Using this in (71) we get

$$\begin{aligned} R^T(\lambda_n) &= (\lambda_n I - A) X_n^T \\ &= \sum_{j=-S}^{S} \left(\lambda_n c_{n,j} V_j^T - A \left(c_{n,j} V_j^T \right) \right) \\ &= \sum_{j=-S}^{S} c_{n,j} \left(\lambda_n - \mu_j \right) V_j^T. \end{aligned} \tag{92}$$

Multiplying both sides by V_j^T we obtain

$$c_{n,j} \left(\lambda_n - \mu_j \right) = \left\langle R^T(\lambda_n), V_j^T \right\rangle. \tag{93}$$

On the other hand using the definition $R^T(\lambda_n)$, (82), and (88) we get

$$\left|\left\langle R^T(\lambda_n), V_j^T\right\rangle\right| \le 4sc\varepsilon\frac{c^r}{(2l)^r} = 8sl\varepsilon^2 \quad (94)$$

for all $n, j = 0, \pm1, \pm2, \dots, \pm l$. This with (93) implies (89).

By Schwarz inequality and (83) we have

$$\left|\left\langle R^T(\lambda_n), V_j^T\right\rangle\right| \le 2c\varepsilon\sqrt{s} \quad (95)$$

for all $n = 0, \pm1, \pm2, \dots, \pm l$ and $j = 0, \pm1, \pm2, \dots, \pm S$. Therefore (90) follows from (93). □

Introduce the notation

$$\begin{aligned} Y_n &= \left(\cdots X_{n,-S-1}, X_{n,-S}, X_{n,-S+1}, \dots, X_{n,S}, X_{n,S+1}, \dots\right), \\ U_j &= \left(\cdots 0, 0, V_{j,-S}, V_{j,-S+1}, \dots, V_{j,S}, 0, 0, \dots\right). \end{aligned} \quad (96)$$

Here Y_n and U_j are elements of l_2, and

$$\left\langle Y_n, U_j\right\rangle = \sum_{i=-\infty}^{\infty} X_{n,i}\overline{V_{j,i}} = \sum_{i=-S}^{S} X_{n,i}\overline{V_{j,i}} = \left\langle X_n^T, V_j^T\right\rangle = c_{n,j}. \quad (97)$$

Using equality (35) and the definition of Y_n and U_j one can easily verify that $\{Y_n : n = 0, \pm1, \pm2, \dots, \pm S\}$ and $\{U_n : n = 0, \pm1, \pm2, \dots, \pm S\}$ are the orthonormal systems in l_2.

Now we are ready to prove the following main result.

Theorem 6. *If* $l > \max\{c^2, 2c, 3s\}$ *then the inequality*

$$\left|\lambda_n - \mu_n\right| \le \frac{8Ssc^{2r+2}}{(2l)^{2r+1}} \quad (98)$$

holds for all $n = 0, \pm1, \pm2, \dots, \pm l$, *where* S, r, l *and* c, s *are defined in* (65) *and* (19).

Proof. Suppose to the contrary and without loss of generality that (98) does not hold for some $0 \le n \le l$. Then either $\lambda_n < \mu_n - (8Ssc^{2r+2}/(2l)^{2r+1})$ or $\lambda_n > \mu_n + (8Ssc^{2r+2}/(2l)^{2r+1})$. Let us consider the case $\lambda_n < \mu_n - (8Ssc^{2r+2}/(2l)^{2r+1})$. Then

$$\lambda_k < \mu_j - \frac{8Ssc^{2r+2}}{(2l)^{2r+1}}, \quad (99)$$

and hence by (89) $|c_{k,j}| < 1/2S$ for all $k = 0, \pm1, \pm2, \dots, \pm n$, $j = n, \pm(n+1), \pm(n+2) \dots, \pm l$. It implies that

$$\left|c_{k,n}\right|^2 + \sum_{j:n<|j|\le l}\left|c_{k,j}\right|^2 \le \frac{2l+1-2n}{4S^2} \quad (100)$$

for $k = 0, \pm1, \pm2, \dots, \pm n$. On the other hand from Parseval's equality (91) we have

$$\sum_{k=-n}^{n}\left|X_k\right|^2 = \sum_{k=-n}^{n}\sum_{j=-S}^{S}\left|c_{k,j}\right|^2. \quad (101)$$

Now we are going to get a contradiction by proving that the left-hand side of (101) is greater than the right-hand side of (101). Using (84), the definition of δ, and the conditions on l one can easily verify that

$$\sum_{k=-n}^{n}\left|X_k\right|^2 \ge 2n+1-(2n+1)\delta > 2n+\frac{3}{4}. \quad (102)$$

To estimate the right-hand side of (101) we write it as $S_1 + S_2 + S_3$, where

$$\begin{aligned} S_1 &= \sum_{k=-n}^{n}\left(\left|c_{k,n}\right|^2 + \sum_{j:n<|j|\le l}\left|c_{k,j}\right|^2\right), \\ S_2 &= \sum_{k=-n}^{n}\sum_{j=-n}^{n-1}\left|c_{k,j}\right|^2, \qquad S_3 = \sum_{k=-n}^{n}\left(\sum_{j:|j|>l}\left|c_{k,j}\right|^2\right). \end{aligned} \quad (103)$$

Using (100) and taking into account that $(2l+1-2n)+(2n+1) \le 2S$ and hence $(2l+1-2n)(2n+1) \le S^2$ we obtain

$$S_1 \le \frac{(2l+1-2n)(2n+1)}{4S^2} < \frac{1}{4}. \quad (104)$$

Now let us estimate S_3. Using (99), (69), and then the inequality $l > 2c$ we obtain

$$\left|\lambda_k - \mu_j\right| > \left|\mu_l - \mu_j\right| > |j| \quad (105)$$

for $k = 0, \pm1, \pm2, \dots, \pm n$ and $|j| > l$. Therefore this, (90), and the definition ε imply that

$$\begin{aligned} S_3 &= \sum_{k=-n}^{n}\left(\sum_{j:|j|>l}\left|c_{k,j}\right|^2\right) \\ &\le (2n+1)\sum_{j:|j|>l}\left(\frac{2c\varepsilon\sqrt{s}}{j}\right)^2 \\ &< (2n+1)\frac{4sc^2\varepsilon^2}{l} < \frac{1}{4}. \end{aligned} \quad (106)$$

Now let us estimate S_2. Using (97) and the Bessel inequality for the elements U_i for $i = -n, -n+1, \dots, n-1$ with respect to the orthonormal systems $\{Y_n : n = 0, \pm1, \pm2, \dots, \pm n\}$ of l_2 we obtain

$$\sum_{k=-n}^{n}\left|c_{k,i}\right|^2 \le \left|U_i\right|^2 = 1, \qquad S_2 = \sum_{i=-n}^{n-1}\sum_{k=-n}^{n}\left|c_{k,i}\right|^2 \le 2n. \quad (107)$$

The inequalities (104)–(107) show that the right side of (101) is less than $2n + (1/2)$, which contradicts (102). In the same way we investigate the case $\lambda_n > \mu_n + (8Ssc^{2r+2}/(2l)^{2r+1})$. The theorem is proved. □

4. Examples and Conclusion

In this section we illustrate the results of Sections 2 and 3 for the following examples. Let the potential $p_s(x)$ for $s = 1, 2, 3$ of the operator $T(p_s)$ have the form

$$p_s(x) = \sum_{n=1}^{s}\left(e^{i2nx} + e^{-i2nx}\right) = \sum_{n=1}^{s} 2\cos 2nx; \quad (108)$$

Table 1: Estimations for $T(p_1)$.

	$x_{n,3}$	$E_{n,3}$	γ_n
$n = 7$	49.0119073043627	0.00401827341683563	0.00803652968036530
$n = 8$	64.0090356900908	0.00232226049016466	0.00464451589853519
$n = 9$	81.0070967373201	0.00146120590904089	0.00292241001412498
$n = 10$	100.005724155838	0.00097836132370372	0.00195672191528545
$n = 20$	400.001412301984	$8.57660779334148 \times 10^{-5}$	0.00017153215300668
$n = 30$	900.000626190365	$2.27805363772165 \times 10^{-5}$	$4.55610726195539 \times 10^{-5}$
$n = 40$	1600.00035193858	$9.11289409047171 \times 10^{-6}$	$1.82257881647412 \times 10^{-5}$
$n = 50$	2500.00022515394	$4.5213654576927 \times 10^{-6}$	$9.04273091219341 \times 10^{-6}$
$n = 60$	3600.00015632421	$2.56272510680566 \times 10^{-6}$	$5.12545021275656 \times 10^{-6}$
$n = 70$	4900.00011483597	$1.59021161389524 \times 10^{-6}$	$3.18042322750835 \times 10^{-6}$
$n = 80$	6400.0000879141	$1.05364682405463 \times 10^{-6}$	$2.10729364800086 \times 10^{-6}$
$n = 90$	8100.0000694591	$7.33717636691826 \times 10^{-7}$	$1.46743527333693 \times 10^{-6}$
$n = 100$	10000.0000562596	$5.31248113844258 \times 10^{-7}$	$1.06249622766647 \times 10^{-6}$

Table 2: Estimations for $T(p_2)$.

	$x_{n,3}$	$E_{n,3}$	γ_n
$n = 13$	169.006553875546	0.00801822430426367	0.01603644646924830
$n = 14$	196.005629484083	0.00602192413268590	0.01204384713096120
$n = 15$	225.004888933687	0.00463706113393842	0.00927412163367219
$n = 16$	256.004286247051	0.00364633797625785	0.00729267558174552
$n = 17$	289.003789043447	0.00291892692052321	0.00583785361582058
$n = 18$	324.003373962035	0.00237284215158748	0.00474568416174256
$n = 19$	361.003023794203	0.00195492184884340	0.00390984360625575
$n = 20$	400.002725629827	0.00162966444959707	0.00325932883855288
$n = 30$	900.001203843083	0.00040657655861401	0.00081315311464415
$n = 40$	1600.00067569654	0.00015796542624476	0.00031593085219360
$n = 50$	2500.00043201403	$7.70627570578593 \times 10^{-5}$	0.00015412551405901
$n = 60$	3600.00029984729	$4.32014439412344 \times 10^{-5}$	$8.64028878675565 \times 10^{-5}$
$n = 70$	4900.00022022411	$2.66004761436181 \times 10^{-5}$	$5.32009522823765 \times 10^{-5}$
$n = 80$	6400.00016857341	$1.75241067230997 \times 10^{-5}$	$3.50482134443501 \times 10^{-5}$
$n = 90$	8100.00013317449	$1.21491644560633 \times 10^{-5}$	$2.42983289113352 \times 10^{-5}$
$n = 100$	10000.0001078601	$8.76569198293195 \times 10^{-6}$	$1.75313839654927 \times 10^{-5}$

Table 3: Estimations for $T(p_3)$.

	$x_{n,3}$	$E_{n,3}$	γ_n
$n = 19$	361.004488989457	0.012018209724884	0.024036418816389
$n = 20$	400.004042369632	0.00990095572697077	0.0198019110415735
$n = 30$	900.001776635742	0.00230548774143664	0.00461097546712649
$n = 40$	1600.00099552468	0.00086829674995455	0.00173659349819394
$n = 50$	2500.00063601104	0.00041615591363515	0.00083231182695096
$n = 60$	3600.00044125198	0.00023064366100756	0.00046128732193269
$n = 70$	4900.00032399850	0.00014088338405301	0.00028176676807951
$n = 80$	6400.00024796876	$9.22660434495073 \times 10^{-5}$	0.00018453208688902
$n = 90$	8100.00019587583	$6.36768120642807 \times 10^{-5}$	0.00012735362412432
$n = 100$	10000.0001586304	$4.57782012510395 \times 10^{-5}$	$9.15564025001002 \times 10^{-5}$

Table 4: Approximation of eigenvalues.

	p_1	p_2	p_3
λ_0	−0.455138604105	−0.451676027152	−0.4539320948685
λ_{-1}	−0.110248816992	−0.040158274572	−0.0204737818081
λ_1	1.859108072514	1.4456177812459	1.3907354889190
λ_{-2}	3.917024772994	2.8976658743702	2.9541319115098
λ_2	4.371300982731	5.1886431499537	4.8580498527548
λ_{-3}	9.047739259808	8.9161585304864	7.9082824512658
λ_3	9.078368847202	9.4153327308285	10.2941738497520
λ_{-4}	16.032970081406	16.0004107071615	15.9213717462580
λ_4	16.033832340360	16.1585649096071	16.3957158213096
λ_{-5}	25.020840823290	25.0389311983095	24.9848629686203
λ_5	25.020854345449	25.0538295076160	25.1789211080558
λ_{-6}	36.014289910633	36.0293767228453	36.0144251509371
λ_6	36.014290046045	36.0319035321757	36.0877507661928
λ_{-7}	49.010418249424	49.0218195042565	49.0311600838136
λ_7	49.010418250365	49.0219701639618	49.0394601884444
λ_{-8}	64.007937189247	64.0164674336750	64.0248999242659
λ_8	64.007937189258	64.0164851040169	64.0271961781896
λ_{-9}	81.006250326633	81.0128685694864	81.0198291868669
λ_9	81.006250326634	81.0128693419217	81.0203601164022
λ_{-10}	100.005050675157	100.010339593273	100.015987594137
λ_{10}	100.005050675158	100.010339662550	100.016034084442
λ_{-20}	400.001253135321	400.002520531313	400.003809046181
λ_{20}	400.001253135326	400.002520531318	400.003809046182
λ_{-30}	900.000556173742	900.001115142518	900.001678193187
λ_{30}	900.000556173751	900.001115142519	900.001678193192
λ_{-40}	1600.00031269547	1600.00062627292	1600.00094113218
λ_{40}	1600.00031269548	1600.00062627292	1600.00094113219
λ_{-50}	2500.00020008004	2500.00040052089	2500.00060148494
λ_{50}	2500.00020008004	2500.00040052089	2500.00060148495
λ_{-60}	3600.00013892748	3600.00027802883	3600.00041738205
λ_{60}	3600.00013892749	3600.00027802885	3600.00041738205
λ_{-70}	4900.00010206165	4900.00020421710	4900.00030650836
λ_{70}	4900.00010206165	4900.00020421711	4900.00030650836
λ_{-80}	6400.00007813720	6400.00015632939	6400.00023460110
λ_{80}	6400.00007813721	6400.00015632940	6400.00023460110
λ_{-90}	8100.00006173602	8100.00012350634	8100.00018532630
λ_{90}	8100.00006173602	8100.00012350635	8100.00018532634
λ_{-100}	10000.00005000500	10000.00010003250	10000.00015009260
λ_{100}	10000.00005000500	10000.00010003260	10000.00015009260

that is, $q_n = q_{-n} = 1$ for $1 \le n \le s$ and $q_n = q_{-n} = 0$ for $n > s$, where q_n is defined in (9). Note that the operator $T(p_1)$ is a famous Mathieu operator. By (19) and (108), $Q = 1$ and $M = c$. For $s = 1, 2, 3$ the constant M or c has the values of 2, 4, 6, respectively. The fixed point approximations $x_{n,3}$ determined in (55), where $f(x)$ is defined by (40) with $m = 3$, of the eigenvalues $\lambda_{\pm n}$ of the operators $T(p_s)$ for $s = 1, 2, 3$ are given in Tables 1, 2, and 3, respectively. Moreover, the estimations of the error $E_{n,3} = |\lambda_{\pm n} - x_{n,3}|$ (see (64)) and the length γ_n of the nth gap (see (45)) are also given in Tables 1, 2, and 3.

The method of Section 3 gives high precision results for the calculation of the small eigenvalues. Let us illustrate it by using formula (98) for the first 201 eigenvalues $\lambda_0, \lambda_{-1}, \lambda_1, \lambda_{-2}, \lambda_2, \ldots, \lambda_{-100}, \lambda_{100}$ of the operators $T(p_s)$ for $s = 1, 2, 3$. It means that the number l in (98) is 100 (see the first sentence of Section 3). To find an approximation with error of order 10^{-18} for the eigenvalues of $T(p_1)$ we take $r = 5$. Therefore for the potential $p_s(x)$, where $s = 1, 2, 3$, the number S is $l + 2rs = 100 + 10s$ and the number of equations in (65) is $2S + 1 = 200 + 20s + 1$. The matrices of (65) corresponding to the potentials $p_1(x), p_2(x), p_3(x)$ and denoted by A_1, A_2, A_3 are of order 221, 241, and 261, respectively. The approximate eigenvalues $\mu_0, \mu_{-1}, \mu_1, \mu_{-2}, \mu_2, \ldots, \mu_{-100}, \mu_{100}$ of the matrices A_1, A_2, A_3 are given in Table 4. By (98) the eigenvalues μ_n are very close to the eigenvalues λ_n of the operator $T(p_s)$. One can readily see from (98) that the approximation $|\lambda_n - \mu_n|$ of λ_n by the eigenvalues μ_n is arbitrary small if r is a large number and c is

Table 5: Approximation of the lengths of the gaps.

	p_1	p_2	p_3
γ_1	1.96935688950626	1.48577605581811	1.41120927072708
γ_2	0.45427620973738	2.29097727558352	1.90391794124493
γ_3	0.03062958739405	0.49917420034206	2.38589139848613
γ_4	0.00086225895372	0.15815420244566	0.47434407505152
γ_5	$1.35221586674561 \times 10^{-5}$	0.01489830930653	0.19405813943552
γ_6	$1.35412271617952 \times 10^{-7}$	0.00252680933036	0.07332561525572
γ_7	$9.41085431804822 \times 10^{-10}$	0.00015065970523	0.00830010463081
γ_8	$1.09992015495664 \times 10^{-11}$	$1.76703419043633 \times 10^{-5}$	0.00229625392370
γ_9	$5.82645043323282 \times 10^{-13}$	$7.72435271301219 \times 10^{-7}$	0.00053092953526
γ_{10}	$1.22213350550737 \times 10^{-12}$	$6.92769646093439 \times 10^{-8}$	$4.64903052659338 \times 10^{-5}$
γ_{20}	$5.11590769747272 \times 10^{-12}$	$4.88853402202949 \times 10^{-12}$	$9.09494701772928 \times 10^{-13}$
γ_{30}	$8.64019966684282 \times 10^{-12}$	$4.54747350886464 \times 10^{-13}$	$4.43378667114303 \times 10^{-12}$
γ_{40}	$3.41060513164848 \times 10^{-12}$	$3.63797880709171 \times 10^{-12}$	$1.02318153949454 \times 10^{-11}$
γ_{50}	$3.18323145620525 \times 10^{-12}$	$6.82121026329696 \times 10^{-12}$	$5.45696821063757 \times 10^{-12}$
γ_{60}	$7.27595761418343 \times 10^{-12}$	$1.90993887372315 \times 10^{-11}$	$4.09272615797818 \times 10^{-12}$
γ_{70}	$3.63797880709171 \times 10^{-12}$	$5.45696821063757 \times 10^{-12}$	$2.72848410531878 \times 10^{-12}$
γ_{80}	$6.3664629124105 \times 10^{-12}$	$5.45696821063757 \times 10^{-12}$	$1.81898940354586 \times 10^{-12}$
γ_{90}	$1.81898940354586 \times 10^{-12}$	$6.3664629124105 \times 10^{-12}$	$4.09272615797818 \times 10^{-11}$
γ_{100}	$1.09139364212751 \times 10^{-11}$	$2.91038304567337 \times 10^{-11}$	0

a small number. If the potential q is smooth function, then the number c is a small number (see (13) and (19)), and hence (98) gives better approximations for smooth potentials. Moreover if s is a small number, that is, the number of summand of p_s (see (108)) is small, then we can choose r so that the order of the matrix A_s is not a large number while the approximation (98) is a very small number. By formula (98) $|\lambda_n - \mu_n|$, where $n = 0, \pm1, \pm2, \ldots, \pm100$, for the potentials $p_1(x)$, $p_2(x)$, and $p_3(x)$ is not greater than

$$\frac{8 \times 110 \times 2^{12}}{(200)^{11}} = \frac{11}{625} 10^{-17},$$
$$\frac{8 \times 120 \times 2 \times 4^{12}}{(200)^{11}} = \frac{3}{1907\ 348\ 632\ 812\ 500}, \quad (109)$$
$$\frac{8 \times 130 \times 3 \times 6^{12}}{(200)^{11}} = \frac{20\ 726\ 199}{62\ 500\ 000\ 000\ 000\ 000\ 000},$$

respectively. Thus in Section 3 there are the following observations to be considered. Instead of the matrices of order 201 investigating a little big matrices, namely, matrices of order 221, 241, and 261, we find an approximation of order $10^{-18}, 10^{-15}$, and 10^{-12} for the first 201 eigenvalues of $T(p_1), T(p_2)$, and $T(p_3)$, respectively. Moreover this approach is applicable for the trigonometric polynomial potentials and for the sufficiently differentiable periodic potentials.

The estimations of the lengths $\gamma_1, \gamma_2, \ldots, \gamma_{100}$ of the gaps are given in Table 5. It is known that [12] for large n the behavior of γ_n is sensitive to smoothness properties of the potential q. If q is m times differentiable, then $\gamma_n = O(n^{-m})$. If q is analytic function, then $\gamma_n = O(e^{-an})$ for some positive a. For the Mathieu operator $T(p_1)$ the following asymptotic formula holds: $\gamma_n = O(4^n/((n-1)!)^2)$. Thus for large n the length γ_n of the nth gap is a very small number. Table 5 confirms this result for large n (see γ_n for $n \geq 10$). Moreover Table 5 shows that these results continue to hold for $n > 5$. Since for the small values of n ($n \leq 5$) the asymptotic formulas do not give any information, we cannot compare the theoretical results with the results in Table 5. Note that in Tables 4 and 5 the eigenvalues and the lengths of the gaps are computed using Matlab. In Table 4 this program transects to 14 figures, because this accuracy is acceptable for estimations of the eigenvalues. However, we compute the lengths of the gaps without transaction, since (as it is noted above) for large n the theoretical results give the estimations of γ_n with very high accuracy.

It is natural and well known that for large eigenvalues the asymptotic method gives us approximations with smaller errors. Since the method of Section 3 gives high precision results for the small eigenvalues and gaps (see Tables 4 and 5), the comparison of the Tables 1–5, where we estimate the eigenvalues and gaps by the methods of Sections 2 and 3, respectively, for the potential (108), shows that the results of the asymptotic method given in Tables 1–3 are not precise for the small eigenvalues.

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